THE IMPROVED EULER-JACOBI FORMULA AND THE PLANAR CUBIC POLYNOMIAL VECTOR FIELDS IN $\mathbb{R}^2$

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. The new Euler-Jacobi formula for points with multiplicity two provides an algebraic relation between the singular points of a polynomial vector field and their topological indices. Using this formula we obtain the configuration of the singular points together with their topological indices for the planar cubic polynomial differential systems when these systems have eight finite singular points, being one of them with multiplicity two. The case with nine finite singular points has already been solved using the classical Euler-Jacobi formula.

1. Introduction and statement of the main results

Consider the planar polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

in $\mathbb{R}^2$ where $P(x, y)$ and $Q(x, y)$ are real polynomials of degree n and m respectively. The vector field associated to the planar polynomial differential system (1) will be denoted by X.

When $n = m = 2$ and system (1) has four singular points the possible configurations of the (topological) indices of these points where characterized by Berlinskii in [2] in 1960. Later on in 1999 Gasull and Torregrosa [7] also for systems (1) with $n = m = 2$ but with three singular points one with multiplicity two extended the Berlinskii theorem to these systems. More recently, the authors in [9] with $n = m = 3$ (i.e. for cubic polynomial differential systems) classify all the possible configurations of the indices of these systems having nine singular points. The objective of this paper is to classify all the configurations of indices of systems (1) with $n = m = 3$ having eight singular points one of them with multiplicity two.

When system (1) has nm finite singular points the classical Euler-Jacobi formula (see [1] for its proof) gives an algebraic relation between the indices of these finite singular points. When system (1) has points with multiplicity two, the classical Euler-Jacobi formula is no longer valid. However in [7] the

---

2010 *Mathematics Subject Classification.* Primary 34A05. Secondary 34C05, 37C10.
Key words and phrases. Euler-Jacobi formula, singular points, topological index, polynomial differential systems.

authors (Gasull and Torregrosa) provided a generalization of the classical Euler-Jacobi formula when the system has points with multiplicity two. Using this new formula and the index theory we obtain the distribution of the singular points of cubic polynomial differential systems when this system has eight finite singularities, being one of them with multiplicity two.

We denote by $A_X = A$ the set of points $p \in \mathbb{R}^2$ such that $X(p) = 0$. Given a finite subset B of $\mathbb{R}^2$, we denote by $\hat{B}$, $\partial\hat{B}$ and $\#B$ its convex hull, the boundary of the convex hull, and its cardinal, respectively.

Set $A_0 = A$ and $A_i = A \cap \partial\big(\widehat{A \setminus A_0 \cup \cdots \cup A_{i-1}}\big)$ for $i \geq 1$. Note that there exists $q \in \mathbb{N}$ such that $A_q \neq \emptyset$ and $A_{q+1} = \emptyset$.

We say that A has the *configuration* $(K_0; K_1; K_2; \ldots; K_q)$ if $K_i = \#(A_i \cap \partial\hat{A}_i)$. We say that A has the configuration $(K_0; K_1; K_2; \ldots; K_r; *)$ if we do not specify for the values of K_i for i between $r+1$ and q. We also say that the singular points of X belonging to $A_i \cap \partial\hat{A}_i$ are on the *i-th level*.

We want to be more precise and study also the indices of the singular points of X. Then we substitute each K_i by the sign of the indices of the points of $A_i \cap \partial\hat{A}_i$, i.e. instead of K_i in the configuration we write the string $(s_1^i, s_2^i, \ldots, s_{K_i}^i)$ where $s_i^j \in \{+, -, 0\}$. When $A_i \cap \partial\hat{A}_i$ is a polygon, the starting s_1^i will be the point with multiplicity two (denoted by p_0) if such point is in the i-th level and the signs $s_2^i, \ldots, s_{K_i}^i$ are the signs of the list of positive or negative indices that follow point with multiplicity two in counterclockwise or clockwise sense according with the largest list of points between the two lists closest to the point with multiplicity two. In case that both closest lists have the same length, we choose the one with the second largest closest list, and so one. In fact when there are ℓ equal consecutive signs, for instance if they are $+$, then instead of $+ \cdots +$ ℓ-times we shall write $\ell+$. In the case that the i-th level does not contain the point with multiplicity two, then s_1^i is the length of the largest list of positive or negative indices of the singular point in the i-th level. The numbers $s_2^i, \ldots, s_{K_1}^i$ are chosen following the previous criteria changing the point with multiplicity two by s_1^i.

When $A_i \cap \partial\hat{A}_i$ is a segment, which does not contain the point with multiplicity two, the signs of the strings are ordered starting at one of the endpoints. Then we start for the endpoint having the larger list of signs independently if this list is formed by plus or minus signs. In case that the length of the list of signs of both endpoints are equal, then we choose to start with the endpoint whose second list is larger, and so on.

If $A_i \cap \partial\hat{A}_i$ contains the point with multiplicity two we identify all the list of signs of this segment cyclically, i.e. after one endpoint it follows the other endpoint. Then the starting sign in the list is the sign 0 of the point of multiplicity two, and after it we choose the largest list closest to p_0. In case that the two lists of signs separated by p_0 have the same length, then we choose to start with the list near p_0 whose second list is larger, and so on.

With this notation we can state the main result of the paper. We denote by $i_X(a)$ the index of a singular point $a \in A$ of a planar cubic polynomial vector field X.

It was proved in [6] and [8] that in the case of cubic polynomial differential systems the absolute value of the sum of the indices of the points is either 1 or 3. In the next theorem, which is our first main theorem we characterize all the possible configurations for cubic polynomial differential systems when the absolute value of the sum of the indices is three.

Theorem 1. *For planar cubic polynomial differential systems having* 8 *singular points with one of multiplicity two, and* $|\sum_{a\in A} i_X(a)| = 3$, *the only possible configurations for the eight topological indices of their singularities are*

(5;3) $(5+;0,2-)$, $(0,4+;2-,+)$, $(5-;0,2+)$, $(0,4-;2+,-)$.
(4;4) $(4+;0,2-,+)$, $(4+;0,2-,+)$, $(0,3+;+,-,+,-)$, $(0,3+;2+,2-)$, $(4-;0,2+,-)$, $(4-;0,+,-,+)$, $(0,3-;+,-,+,-)$, $(0,3-;2+,2-)$.
(4;3;1) $(4+;+,2-;0)$, $(4+;0,2-;+)$, $(0,3+;2-,+;+)$, $(0,3+;2+,-;-)$, $(4-;2+,-;0)$, $(4-;0,2+;-)$, $(0,3-;2+,-;+)$, $(0,3-;2-,+;+)$.
(3;5) $(3+;0,+,-,+,-)$, $(0,2+;2+,-,+,-)$, $(3-;0,-,+,-,+)$, $(0,2-;2-,+,-,+)$.
(3;4;1) $(3+;2+,2-;0)$, $(3+;+,-,+,-;0)$, $(0,2+;+,-,+,-;+)$, $(3-;2+,2-;0)$, $(3-;-,+,-,+;0)$, $(0,2-;-,+,-,+;-)$.
(3;3;2) $(3+;2+,-;0,-)$, $(3+;0,-,+;+,-)$, $(0,2+;2+,-;+,-)$, $(3-;2-,+;0,+)$, $(3-;0,+,-;+,-)$, $(0,2-;2-,+;+,-)$.

and there exist examples of such cubic polynomial differential systems with these configurations.

The proof of Theorem 1 is given in section 3. To complete the cubic case it is necessary to characterize all possible configurations when $|\sum_{a\in A} i_X(a)| = 1$ which is done in the following theorem.

Theorem 2. *For planar cubic polynomial differential systems having* 8 *singular points with one of multiplicity two, and* $|\sum_{a\in A} i_X(a)| = 1$, *the only possible configurations for the eight topological indices of their singularities are*

(8) $(0,+,-,+,-,+,-,+)$, $(0,2+,-,+,-,+,-)$, $(0,-,+,-,+,-,+,-)$ *and* $(0,2-,+,-,+,-,+)$.
(7;1) $(0,+,-,+,-,+,-;+)$, $(0,2+,-,+,-,+;-)$, $(0,3+,-,+,-;-)$, $(2+,-,+,-,+,-;0)$, $(0,-,+,-,+,-,+;-)$, $(0,2+,-,+,-,+;-)$, $(0,3-,+,-,+;+)$ *and* $(2-,+,-,+,-,+;0)$.

(6;2) $(0,+,-,+,-,+;+,-),\ (0,-,2+,-,+;+,-),\ (0,2+,-,+,-;+,-),$
$(0,2+,-,2+;2-),\ (2+,-,2+,-;0,-),\ (3+,-,+,-;0,-),$
$(+,-,+,-,+,-;0,+),\ (0,-,+,-,+,-;+,-),\ (0,+,2-,+,-;+,-),$
$(0,2-,+,-,+;+,-),\ (0,2-,+,2-;2+),\ (2-,+,2-,+;0,+),$
$(3-,+,-,+;0,+)$ *and* $(+,-,+,-,+,-;0,-).$

(5;3) $(0,-,2+,-;2+,-),\ (0,+,-,+,-;2+,-),\ (0,2+,-,+;2-,+),$
$(0,3+,-;2-,+),\ (2+,-,+,-;0,+,-),\ (4+,-;0,2-),$
$(0,+,2-,+;2-,+),\ (0,-,+,-,+;2-,+),\ (0,2-,+,-;2+,-),$
$(0,3-,+;2+,-),\ (2-,+,-,+;0,-,+)$ *and* $(4-,+;0,2+).$

(4;4) $(+,-,+,-;0,+,-,+,-),\ (+,-,+,-;0,2+,-),\ (0,-,+,-;3+,-),$
$(0,+,-,+;+,-,+,-),\ (0,+,-,+;2+,2-),\ (0,2+,-;+,-,+,-),$
$(0,2+,-;2+,2-),\ (0,3+;3-,+),\ (3+,-;0,-,+,-),$
$(3+,-;0,2-,+),\ (4+;0,3-),\ (+,-,+,-;0,-,+,-,+),$
$(+,-,+,-;0,2-,+),\ (0,+,-,+;3-,+),\ (0,-,+,-;+,-,+,-),$
$(0,-,+,-;2+,2-),\ (0,2-,+;-,+,-,+),\ (0,2-,+;2+,2-),$
$(0,3-;3+,-),\ (3-,+;0,+,-,+),\ (3-,+;0,2+,-)$ *and*
$(4-;0,3+).$

(4;3;1) $(0,+,-,+;2-,+;+),\ (+,-,+,-;0,+,-;+),\ (+,-,+,-;0,2+;-),$
$(+,-,+,-;2+,-;0),\ (0,-,+,-;2+,-;+),\ (0,-,+,-;3+;-),$
$(0,2+,-;2+,-;-),\ (0,3+;3-;+),\ (3+,-;2-,+;0),$
$(3+,-;0,+,-;-),\ (3+,-;0,2-;+),\ (4+;3-;0),\ (4+;0,2-;-),$
$(0,2+,-;2-,+;+),\ (0,-,+,-;2+,-;+),\ (0,-,+,-;2+,-;-),$
$(-,+,-,+;0,-,+;-),\ (-,+,-,+;0,2-;+),\ (-,+,-,+;2-,+;0),$
$(0,-,+,-;2-,+;-),\ (0,+,-,+;3-;+),\ (0,2-,+;2-,+;+),$
$(0,3-;3+;-),\ (3-,+;2+,-;0),\ (3-,+;0,-,+;+),\ (3-,+;0,2+;-),$
$(4-;3+;0),\ (4-;0,2+;+),\ (0,2-,+;2+,-;-)$ *and* $(0,+,-,+;2-,+;-).$

(3;5) $(0,+,-;2+,-,+,-),\ (0,2+;2-,+,-,+),\ (2+,-;0,-,2+,-),$
$(2+,-;0,+,-,+,-),\ (3+;0,2-,+,-),\ (0,+,-;2-,+,-,+),$
$(0,2-;2+,-,+,-),\ (2-,+;0,+,2-,+),\ (2-,+;0,-,+,-,+)$ *and*
$(3-;0,2+,-,+).$

(3;4;1) $(0,+,-;+,-,+,-;+),\ (0,+,-;3+,-;-),\ (0,2+;3-,+;-),$
$(2+,-;0,2-,+;+),\ (2+,-;0,2+,-;-),\ (2+,-;2+,2-;0),$
$(3+;0,2-,+;-),\ (3+;0,3-;+),\ (3+;3-,+;0),\ (0,-,+;-,+,-,+;-),$
$(0,-,+;3-,+;+),\ (0,2-;3+,-;+),\ (2-,+;0,2+,-;-),$
$(2-,+;0,2-,+;+),\ (2-,+;2-,2+;0),\ (3-;0,2+,-;+),$
$(3-;0,3+;-),$ *and* $(3-;3+,-;0),$

(3;3;2) $(0,+,-;2+,-;+,-),\ (0,2+;2-,+;+,-),\ (0,2+;3-;2+),$
$(2+,-;2-,+;0,+),\ (2+,-;0,+,-;+,-),\ (2+,-;2+,-;0,-),$
$(3+;0,2+;+,-),\ (2+,-;0,2-;2+),\ (3+;3-;0,+)\ (0,-,+;2-,+;+,-),$
$(0,2-;2+,-;-,+),\ (0,2-;3+;2-),\ (2-,+;2+,-;0,-),$
$(2-,+;0,-,+;-,+),\ (2-,+;2-,+;0,+),\ (3-;0,2-;+,-),$
$(2-,+;0,2+;2-)$ *and* $(3-;3+;0,-).$

and there exist examples of such cubic polynomial differential systems with these configurations.

The proof of Theorem 2 is given in section 4. As mentioned above with Theorems 1 and 2 we provide the classification of the configurations of the singular points and their topological indices for all planar polynomial differential systems with eight finite singular points being one (that we will translate it at the origin) double.

2. Preliminaries

First of all we observe that if a configuration exists for a cubic polynomial vector field X with $\#A_X = 8$, being one of them double then it is possible to construct the same configuration but interchanging points with index $+1$ with points with index -1. For doing that it is enough to take $Y = (-P, Q)$ instead of $X = (P, Q)$. So we can restrict ourselves to the cases in which $\sum_{a\in A} i_X(a) \geq 0$. Moreover it was proved in [6] and [8] that in this case the sum of the absolute value of the indices of the points is either 3 or 1. Since there is only one point with multiplicity two we must have five points with positive index and two points with negative index when the sum of the indices of the finite singular points is three, and four points with positive index and three points with negative index when the sum of the indices of the finite singular points is one.

It follows from index theory (see for instance [3]) that if two points collide then the indices add. Since the indices of the semi-hyperbolic points can only be $-1, 1, 0$ and the indices of the hyperbolic points when there are nine finite singular points can only be $-1, 1$ it follows that only points with indices -1 and $+1$ can collide. In short, the only possible configurations in Theorem 1 can be from colliding two hyperbolic points with index $+1$ and -1 in a configuration of a planar cubic polynomial differential system such that $\#A = 9$ and $|\sum_{a\in A} i_X(a)| = 3$. In [9] the authors obtained such configurations in the following theorem.

Theorem 3. *For planar cubic polynomial differential systems having* 9 *singular points and* $|\sum_{a\in A} i_X(a)| = 3$ *the only possible configurations for the nine topological indices of their singularities are*

(5;3;1) *with* $(5+; 3-; +)$, $(5-; 3+; -)$;
(4;5) *with* $(4+; +, 2-, +, -)$, $(4-; 2+, -, +, -)$;
(4;4;1) *with* $(4+; +, 3-; +)$, $(4-; 3+, -; -)$;
(4;3;2) *with* $(4+; 3-; 2+)$, $(4+; +, 2-; +, -)$, $(4-; 3+; 2-)$, $(4-; 2+, -; +, -)$;
(3;6) *with* $(3+; +, -, +, -, +, -)$, $(3-; +, -, +, -, +, -)$;
(3;5;1) *with* $(3+; +, 2-, +, -; +)$, $(3-; 2+, -, +, -; -)$;
(3;4;2) *with* $(3+; 2+, 2-; +, -)$, $(3-; 2+, 2-; +, -)$;
(3;3;3) *with* $(3+; 2+, -; +, 2-)$, $(3-; +, 2-; 2+, -)$;

and there exist examples of such cubic polynomial differential systems with these configurations.

Lemma 4. *For planar cubic polynomial differential systems with* 8 *singular points being one with multiplicity two, and* $|\sum_{a\in A} i_X(a)| = 3$*, the only possible configurations for the eight topological indices of their singularities are of the form* $(K+;*)$ *where* $K \leq 5$ *or* $(0, L+;*)$ *where* $L \leq 4$*.*

Proof. It follows directly from Theorem 3 and the fact that the point with multiplicity two can only formed by colliding two hyperbolic points in a configuration of planar cubic polynomial differential systems such that $\#A = 9$ and $|\sum_{a\in A} i_X(a)| = 3$. □

It follows that a singular point p of system (1) is *simple* if and only if the determinant of the Jacobian matrix of P and Q at p is different from zero, i.e.

$$J(P,Q)(p) := J(p) = \left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}\right)\Big|_p \neq 0.$$

A singular point p of system (1) is *double* if and only if $J(p) = 0$ and $I(P,Q)(p) := I(p) \neq 0$ where

$$\begin{aligned} I(P,Q)(p) := I(p) = & \left(\frac{\partial P}{\partial y}\right)^2 \left(\frac{\partial P}{\partial x}\frac{\partial^2 Q}{\partial x \partial x^2} - \frac{\partial Q}{\partial x}\frac{\partial^2 P}{\partial x^2}\right) \\ & - 2\frac{\partial P}{\partial x}\frac{\partial P}{\partial y}\left(\frac{\partial P}{\partial x}\frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial Q}{\partial x}\frac{\partial^2 P}{\partial x \partial y}\right) \\ & + \left(\frac{\partial P}{\partial x}\right)^2 \left(\frac{\partial P}{\partial x}\frac{\partial^2 Q}{\partial y^2} - \frac{\partial Q}{\partial x}\frac{\partial^2 P}{\partial y^2}\right)\Big|_p. \end{aligned}$$

For a proof see Lemma 2.2 of [7]. Moreover it is well-known that for planar polynomial differential systems (1), a simple point p has index 1 (if $J(p) > 0$) or -1 (if $J(p) < 0$) (see for instance [10]) and that points with multiplicity two of our system (1) has index zero.

In this paper we consider the case in which one of the 8 singular points, that we will denote by p_0, has multiplicity two. We will use the new Euler-Jacobi formula for points with multiplicity two proved by Gasull and Torregrosa in [7] which can be stated as follows. To state it we need the following notation. We write

$$\begin{aligned} P(x,y) &= P_{10}x + P_{01}y + P_{20}x^2 + P_{11}xy + P_{02}y^2 + \dots, \\ Q(x,y) &= Q_{10}x + Q_{01}y + Q_{20}x^2 + Q_{11}xy + Q_{02}y^2 + \dots \end{aligned}$$

and given a polynomial R we also write it as

$$R(x,y) = R_{00} + R_{10}x + R_{01}y + R_{20}x^2 + R_{11}xy + R_{02}y^2 + \dots$$

The next result is proved in Theorem 3.2 of [7] for two real polynomials of degrees n and m. We state it here for the case in which $n = m = 3$ and when the system has eight finite singular points one with multiplicity two.

Theorem 5. *Consider a cubic system of two real polynomials in the variables x and y. If the set of zeroes of that system (that we denote by A) contains exactly eight elements (seven being simple and one with multiplicity two p_0 that without loss of generality we can assume it is at the origin), then for any polynomial R of degree less than or equal to 3 we have*

$$\sum_{a\in A_S}\frac{R(a)}{J(a)}+S(0)=0 \tag{2}$$

where A_S denotes the set of simple zeroes of the system and $S(0)$ is equal to

$$S(0)=\frac{4P_{10}R_{00}N}{I(0)^2}+\frac{2P_{10}(P_{10}R_{01}-P_{01}R_{10})}{I(0)},$$

where

$$\begin{aligned}N={}&P_{10}^3(Q_{10}P_{03}-P_{10}Q_{03})-P_{10}^2P_{01}(Q_{10}P_{12}-P_{10}Q_{12})\\&+P_{10}P_{01}^2(Q_{10}P_{21}-P_{10}Q_{21})-P_{01}^3(Q_{10}P_{30}-P_{10}Q_{30})\\&+P_{10}^3(Q_{11}P_{02}-P_{11}Q_{02})-2P_{10}^2P_{01}(Q_{20}P_{02}-P_{20}Q_{02})\\&+P_{10}P_{01}^2(Q_{20}P_{11}-P_{20}Q_{11}).\end{aligned}$$

In the proof of Theorems 1 and 2 we will denote by L_{ij} the straight line through the points p_i and p_j where $i,j\in\{1,\dots,8\}$ and also $L_{ij}^2=L_{ij}L_{ij}$. Moreover during all the paper we will consider straight lines with only two singular points because if there are straight-lines with three singular points the arguments become even easier. Furthermore, we note that when proving that some configurations are not possible, we will take $p_0=(0,0)$ and a conic (generally formed by two straight lines) passing through p_0. Doing so, we have that $R_{00}=R_{10}=R_{01}=0$, and consequently $S(0)=0$ in formula (2).

Lemma 6. *For planar cubic polynomial differential systems such that $\#A_X=8$, and $|\sum_{a\in A}i_X(a)|=1$, there are no configurations of the form $(K*,2-,*;*)$, that is, there are no configurations with two consecutive points with negative index in the 0-level.*

Proof. Assume that there are two consecutive points in the 0-level with negative index and denote them by p_1,p_2. Since the absolute value of the sum of the indices is one and we can assume that the sum is positive, we have an additional point with negative index, that we denote by p_3. Hence, applying the Euler-Jacobi formula (2) with $R=L_{0,p_3}^2L_{p_1,p_2}$ we reach a contradiction. □

3. Proof of Theorem 1

It follows from Lemma 4 that the possible configurations are: $(5+;3)$, $(0,4+;3)$, $(4+;4)$, $(0,3+;4)$, $(4+;3;1)$, $(0,3+;4;1)$, $(3+;5)$, $(0,2+;5)$, $(3+;4;1)$,

$(0, 2+; 4; 1)$, $(3+; 3; 2)$ and $(0, 2+; 3; 2)$. We will study each of them separately.

Configuration $(5+; 3)$ Since there are five points with positive index and two points with negative index the unique possible configuration is $(5+; 0, 2-)$. The cubic system (1) with

$$\begin{aligned} P(x,y) = &\quad -6y + 5y^2 + 2x^2y - y^3, \\ Q(x,y) = &\quad \frac{1}{5}(48 + 32x - 40y - 12x^2 - 26xy + 8y^2 - 8x^3 - x^2y + 5xy^2), \end{aligned}$$

has the singular points

$$\begin{aligned} &(-1., 4.), \quad (0,3), \quad (-2,0), \quad (2,0), \quad (1,4), \\ &(-1,1), \quad (-3/2, 0), \quad (0,2), \quad (0,2) \end{aligned}$$

in the configuration $(5+; 0, 2-)$.

Configuration $(0, 4+; 3)$ Since there are five points with positive index and two points with negative index the unique possible configuration is $(0, 4+; 2-, +)$. The cubic system (1) with

$$\begin{aligned} P(x,y) = &\quad 6y - 5y^2 - 2x^2y + y^3, \\ Q(x,y) = &\quad -\frac{1}{10}(48x + 3y - y^2 - 46xy - 12x^3 - 2x^2y + 10xy^2), \end{aligned}$$

has the singular points

$$\begin{aligned} &(0,3), \quad (0,3), \quad (-2,0), \quad (2,0), \quad (1,4), \quad (-1,1), \\ &(0.428571, 1.71429), \quad (1,1), \quad (0,0) \end{aligned}$$

in the configuration $(0, 4+; 2-, +)$.

Configuration $(4+; 4)$ Since there are five points with positive index and two points with negative index the unique possible configurations of the form $(4+; 4)$ are $(4+; 0, -, +, -)$ and $(4+; 0, 2-, +)$. The cubic system (1) with

$$\begin{aligned} P(x,y) = &\ 1.4523 + 1.23689x + 3.08149y + 3.84729x^2 + 1.05231xy \\ &- 3.52134y^2 + 0.168176x^3 + 0.910772x^2y - y^3, \\ Q(x,y) = &\ 1.2082 + 11.867x + 14.1512y + 2.98455x^2 + 7.00776xy \\ &+ 3.81719y^2 + 0.0233874x^3 + 0.827415x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-3.9, -3.6), \quad (-3.92, -3.5), \quad (-3.84.., -3.5..), \\ &(-3.84.., -3.5..), \quad (16.24.., -13.16..), \quad (3.99.., -3.74..), \\ &(-2.99.., 2.82..), \quad (2.00., -1.73..), \quad (0.86.., -0.80..) \end{aligned}$$

in the configuration $(4+; 0, -, +, -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -2.14508 - 2.27088x - 0.61618y - 1.93249x^2 + 0.32733xy\\ &\quad + 2.46863y^2 + 0.26016x^3 + 1.06646x^2y - y^3,\\ Q(x,y) &= -5.23367 - 4.92581x - 0.81582y - 3.21993x^2 - 0.9972xy\\ &\quad + 2.79274y^2 - 0.78125x^3 - 0.24175x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(-4,3),\quad (3,-3),\quad (-3.29..,2.66..),\quad (-3,2.82..),\quad (4,5),\\ &(-0.95..,1.25..),\quad (-0.95..,1.25..),\quad (-3,-2.82..),\quad (-2,-1.73..)\end{aligned}$$

in the configuration $(4+;0,2-,+)$.

Configuration $(0,3+;4)$ Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,3+;+,-,+,-)$ and $(0,3+;2+,2-)$. The cubic system (1) with

$$\begin{aligned}P(x,y) &= -2.14507 - 2.27088x - 0.61617y - 1.93248x^2 + 0.32733xy\\ &\quad + 2.46863y^2 + 0.26016x^3 + 1.06646x^2y - y^3,\\ Q(x,y) &= -5.23368 - 4.92581x - 0.81582y - 3.21993x^2 - 0.99719xy\\ &\quad + 2.79274y^2 - 0.78125x^3 - 0.24175x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(-4,3),\quad (3,-3),\quad (-3.29..,2.66..),\quad (-3,2.82..),\quad (4,5),\\ &(-0.95..,1.25..),\quad (-0.95..,1.25..),(-3,-2.82),\quad (-2,-1.73)\end{aligned}$$

in the configuration $(0,3+;+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.00003 - 0.00002x + 0.19126y - 0.00002x^2 + 0.46652xy\\ &\quad + 0.11736y^2 + 0.98626x^2y - y^3,\\ Q(x,y) &= 0.00001 - 1.09067y + 0.16407xy + 2.18129y^2\\ &\quad + 0.43317x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(-3.00..,2.82..),\quad (-5.05..,0.00..),\quad (4.00..,0.00..),\\ &(4.00..,0.00..),\quad (-0.73..,0.67..),\quad (-1.50..,0),\\ &(0.99..,0),\quad (0,0.49..),\quad (-2.00..,-1.73..)\end{aligned}$$

in the configuration $(0,3+;2+,2-)$.

Configuration $(4+;3;1)$ Since there are five points with positive index and two points with negative index the unique possible configurations are $(4+;2-,+;0)$, $(4+;0,2-;+)$ and $(4+;0,+,-;-)$.

We will show that the configuration $(4+;0,+,-;-)$ is not possible. Denote by p_1,p_2,p_3,p_4 the four points in the 0-level (with positive index), by p_0,p_5,p_6 the points in the 1st-level (being p_0 the point with multiplicity two, p_5 the point with positive index and p_6 the singular point with negative index) and by p_7 the singular point in the 2nd level. There exists $k_0,k_1 \in \{1,2,3,4\}$ so that $L_{k_0,6}$ leaves p_{k_1} on one side and the other five points in the other side. We denote by p_{k_2} and p_{k_3} the two remaining points in the 0-level. Note that we can consider, without loss of generality, that L_{0,k_2} leaves k_3 on one side and the other five points in the other side. Applying the Euler-Jacobi formula to $S = L_{0,5}L_{0,k_2}L_{k_0,6}$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.01043 + 0.08997x - 0.00631y - 0.0381x^2 + 0.29991xy \\ &\quad + 0.16308y^2 - 0.0623x^3 - 0.69456x^2y + y^3, \\ Q(x,y) &= 0.03992 - 0.38324x - 0.55026y + 0.0967x^2 - 1.57747xy \\ &\quad - 2.27773y^2 + 0.24662x^3 - 0.41836x^2y - xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-3.,-2.82..), \quad (-1.5,0), \quad (1,0), \quad (-2,1.73..), \quad (4,-3), \\ &(0,-0.3), \quad (0,-0.3), \quad (0.44..,-0.41..), \quad (-0.1,0.12) \end{aligned}$$

in the configuration $(4+;2-,+;0)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.94486 - 1.31372x - 1.34391y - 2.35727x^2 + 0.14990xy \\ &\quad + 2.92903y^2 + 0.19308x^3 + 1.08818x^2y - y^3, \\ Q(x,y) &= -7.09812 - 5.92471x + 0.59531y - 1.43097x^2 - 0.98248xy \\ &\quad + 1.45502y^2 - 0.53935x^3 - 0.39364x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-4,3), \quad (-3,2.82..), \quad ,(-3,2.82..), \quad (3,-2.82..), \quad (3,5), \\ &(0.02..,2.03..), \quad (-0.95..,1.25..), \quad (-3,-2.82..), \quad (-2,-1.73..) \end{aligned}$$

in the configuration $(4+;0,2-;+)$.

Configuration $(0,3+;3;1)$ Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,3+;2-,+;+)$ and $(0,3+;2+,-;-)$. The cubic system (1) with

$$\begin{aligned} P(x,y) &= 8.01755 + 13.6686x + 13.329y + 8.63431x^2 + 18.4872xy \\ &\quad + 6.48002y^2 + 3.14207x^3 + 4.69624x^2y - y^3, \\ Q(x,y) &= -1.58427 - 1.41114x + 1.3319y - 0.934538x^2 - 0.607739xy \\ &\quad + 2.32775y^2 - 0.692771x^3 - 0.130663x^2y + xy^2, \end{aligned}$$

has the singular points

$$(-4,-4),\quad (-4,5),\quad (-4,5),\quad (-13.50..,11.16..),\quad (-1.5,-0.5),$$
$$(0.57..,-1.16..),\quad (-1,0.2),\quad (2,-1.73..),\quad (4,-3)$$

in the configuration $(0,3+;2-,+;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 13.31010 + 6.56921x + 10.49492y + 0.55371x^2 + 6.40611xy \\ &\quad + 4.24196y^2 + 0.94893x^3 + 1.993x^2y - y^3, \\ Q(x,y) &= -1.51411 + 0.83433x + 7.4396y + 1.98657x^2 + 0.93647xy \\ &\quad + 1.48236y^2 - 0.36592x^3 - 0.0735x^2y + xy^2, \end{aligned}$$

has the singular points

$$(-119.62..,69.71..),\quad (-4,-4),\quad (45.73..,-24.47..),\quad (-2.52..,-2.17..),$$
$$(-4,5),\quad (-4,5),\quad (-1.5,-0.5),\quad (1.85,-2),\quad (2,-1.73..)$$

in the configuration $(0,3+;2+,-;-)$.

Configuration $(3+;5)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(3+;0,+,-,+,-)$, $(3+;0,+,2-,+)$ and $(3+;0,2+,2-)$. We show that the configurations $(3+;0,+,2-,+)$ and $(3+;0,2+,2-)$ are not possible.

For the configuration $(3+;0,+,2-,+)$ denote by p_1,p_2,p_3 the points in the 0-level and by p_4,p_7 and p_5,p_6 the points in the 1st level with positive and negative indices, respectively. The curve $C = L_{0,5}L_{0,6}$ has different signs on p_7 and on p_4. Moreover there exits $k_0 \in \{1,2,3\}$ so that $C(p_{k_0})$ has either the same sign than $C(p_4)$, or the same sign than $C(p_7)$. Applying formula (2) to $L_{k_1,k_2}C$ (note that here $R_{0,0} = R_{1,0} = R_{0,1} = 0$ and so $S(0) = 0$) with $k_1,k_2 \in \{1,2,3\}$ and $k_i \neq k_0$ for $i = 1,2$ we reach to a contradiction. Hence this case is not possible.

For the configuration $(3+;0,2+;2-)$ denote by p_1,p_2,p_3 the points in the 0-level and by p_4,p_5 the points in the 1st level with negative index. Note that there exists $k_0 \in \{1,2,3\}$ so that the curve $L_{0,4}L_{0,5}$ evaluated on p_{k_0} has the same sign as C evaluated on two positive indices of the 1st level. Applying formula (2) with $C = L_{0,4}L_{0,5}L_{k_1,k_2}$ where $k_1,k_2 \in \{1,2,3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach to a contradiction.

In short the unique possible configuration is $(3+;0,+,-,+,-)$. The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.10892 - 0.06352x - 1.52889y + -0.06353x^2 - 2.2135xy \\ &\quad + 0.09531y^2 + 0.01816x^3 - 1.45684x^2y + y^3, \\ Q(x,y) &= -1.43214 + 0.83542x + 1.82603y + x0.83542x^2 + 0.31892xy \\ &\quad + 1.74687y^2 - 0.23869x^3 - 0.09658x^2y + xy^2, \end{aligned}$$

has the singular points

$$
\begin{aligned}
&(-3.00.., 2.83..), \quad (4.00.., 0.00..), \quad (0.15.., -1.45..),\\
&(-0.73.., 0.67..), \quad (-0.73.., 0.67..), \quad (-1.5.., 0),\\
&(-2.99.., -2.82..), \quad (0.99.., 0), \quad (-2.00.., -1.73..)
\end{aligned}
$$

in the configuration $(3+; 0, +, -, +, -)$.

Configuration $(0, 2+; 5)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(0, 2+; 3+, 2-)$ and $(0, 2+; 2+, -, +, -)$. We will show that the configuration $(0, 2+; 3+, 2-)$ is not possible. Indeed, denote by p_1, p_2 the points in the 0 level and by p_3, p_4, p_5 the points in the 1st level with positive index and by p_6, p_7 the points in the 1st level with negative index. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{6,7}$ we reach to a contradiction. In short only the configuration $(0, 2+; 2+, -, +, -)$ is possible. The cubic system (1) with

$$
\begin{aligned}
P(x, y) &= 0.24337 + 0.04054x - 0.28334y - 0.06592x^2 + 0.51321xy\\
&\quad + 0.09318y^2 + 0.01014x^3 + 1.09144x^2y - y^3,\\
Q(x, y) &= -1.69782 - 0.2829x + 2.22066y + 0.45983x^2 - 0.1617xy\\
&\quad + 2.35005y^2 - 0.07074x^3 - 0.300636x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{aligned}
&(-3.00.., 2.82..), \quad (4.00.., 0.00..), \quad (4.00.., 0.00..),\\
&(0.99.., -1.00..), \quad (-1.5.., 0), \quad (-0.49.., 0.46..),\\
&(-2.99.., -2.82..), \quad (0, 0.499..), \quad (-2.00.., -1.73..)
\end{aligned}
$$

in the configuration $(0, 2+; 2+, -, +, -)$.

Configuration $(3+; 4; 1)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(3+; 2+, 2-; 0)$, $(3+; +, -, +, -; 0)$, $(3+; 0, 2-, +; +)$, $(3+; 0, -, +, -; +)$, $(3+; 0, +, -, +; -)$ and $(3+; 0, 2+, -; -)$. We will show that the configuration $(3+; 0, +, -, +; -)$ is not possible. Take the curve $C = L_{0,4}L_{0,6}$. Note that there exists $k_0 \in \{1, 2, 3\}$ so that it is the unique singularity contained in a connected component of $\mathbb{R}^2 \setminus C$. Applying formula (2) with $C = L_{0,4}L_{0,5}L_{k_1,k_2}$ where $k_1, k_2 \in \{1, 2, 3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

The cubic system (1) with

$$
\begin{aligned}
P(x, y) &= 11.55589 - 4.10670x + 7.72417y + 1.08554x^2 - 1.88945xy\\
&\quad - 1.13607y^2 + 0.0411x^3 + 0.42839x^2y - y^3,\\
Q(x, y) &= 7.98585 + 5.44717x + 11.84708y + 0.7037x^2 + 4.54680xy\\
&\quad + 3.93911y^2 - 0.05877x^3 + 0.40006x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$(45.73.., -24.47..), \quad (-4,5), \quad (1.85,-2), \quad (2,-1.73..), \quad (1,-1.5),$$
$$(-4,-4), \quad (-1.42.., -2.11..), \quad (-2.52.., -2.17..), \quad (-2.52.., -2.17..)$$

in the configuration $(3+; 0, -, +, -; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 8.98459 + 6.6871x + 10.86426y + 2.87879x^2 + 6.1108xy \\ &\quad + 2.50325y^2 + 0.91190x^3 + 2.00473x^2y - y^3, \\ Q(x,y) &= 3.99511 + 1.65625x + 8.11454y - 0.114601x^2 + 1.88344xy \\ &\quad + 3.332y^2 - 0.27851x^3 + 0.00545x^2y + xy^2, \end{aligned}$$

has the singular points

$$(45.73.., -24.47..), \quad (14.69.., -8.08..), \quad (-4,5), \quad (2,-1.73..),$$
$$(1,-1.5), \quad (1,-1.5), \quad (-4,-4), \quad (-1.5,-0.5), \quad (-2.52.., -2.17..)$$

in the configuration $(3+; 2+, 2-; 0)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 18.01525 - 0.56436x + 12.6791y + 1.95534x^2 + 0.51759xy \\ &\quad - 0.9431y^2 + 0.19838x^3 + 0.7084x^2y - y^3, \\ Q(x,y) &= 7.63906 + 5.25699x + 11.58106y + 0.657x^2 + 4.41758xy \\ &\quad + 3.92875y^2 - 0.06722x^3 + 0.38503x^2y + xy^2, \end{aligned}$$

has the singular points

$$(45.73.., -24.47..), \quad (-4,5), \quad (4,-3), \quad (2,-1.73..), \quad (1,-1.5),$$
$$(-4,-4), \quad (-2.52.., -2.17..), \quad (-2.52.., -2.17..), \quad (-2.96.., -2.14..)$$

in the configuration $(3+; +, -, +, -; 0)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 12.53185 + 5.04636x + 15.68935y + 0.30004x^2 + 2.48145xy \\ &\quad + 2.9495y^2 + 0.03753x^3 + 0.34625x^2y - y^3, \\ Q(x,y) &= 7.74477 + 4.89906x + 11.2205y + 0.77834x^2 + 4.32362xy \\ &\quad + 3.69906y^2 - -0.04556x^3 + 0.4297x^2y + xy^2, \end{aligned}$$

has the singular points

$$(45.73..., -24.47...), \quad (-4,5), \quad (4,-3), \quad (1.85,-2), \quad (2,-1.73..),$$
$$(2,-1.73..), \quad (-1.5,-0.5), \quad (-2.52.., -2.17..), \quad (-3.51.., -2.36..)$$

in the configuration $(3+; 0, 2-, +; +)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 8.69530 + 2.74101x + 10.71702y + 0.56511x^2 + 1.26514xy \\
&\quad + 1.61948y^2 + 0.29941x^3 + 0.85775x^2y - y^3, \\
Q(x,y) &= 5.5855 + 4.1302x + 9.47293y + 0.9203x^2 + 4.29167xy \\
&\quad + 3.7649y^2 - 0.01426x^3 + 0.49555x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(-4,-4), \quad (45.73..,-24.47..), \quad (-4,5), \quad (-1.5,-0.5), \quad (0.8,-1.6), \\
(0,8,-1.6), \quad (1.044..,-1.59..), \quad (2,-1.73..), \quad (4.2,-2.9)
\end{gathered}
$$

in the configuration $(3+;0,2+,-;-)$.

Configuration $(0,2+;4;1)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,2+;2+,2-;+)$, $(0,2+;+,-,+,-;+)$ and $(0,2+;3+,-;-)$. We will show that the configuration $(0,2+;2+,2-;+)$ is not possible. Indeed, denote by p_1 and p_2 the points in the 0-level and by p_3,p_4 the points in the 1st level with negative index. Applying formula (2) with $C = L_{0,1}L_{0,2}L_{3,4}$ we reach a contradiction.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 9.39252 - 15.4142x - 2.77515y + 3.34204x^2 - 4.776xy \\
&\quad - 4.97594y^2 + 0.07817x^3 + 0.60294x^2y - y^3, \\
Q(x,y) &= 4.63046 - 7.21976x - 0.67095y + 3.4358x^2 + 1.63839xy \\
&\quad - 0.27316y^2 + 0.04718x^3 + 0.71861x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(-4,-4), \quad (-4,-4), \quad (-3.9,-4), \quad (-4,5), \quad (1,-1.5), \\
(45.73..,-24.47..), \quad (1.85,-2), \quad (2,-1.73..), \quad (1.03..,-0.71..)
\end{gathered}
$$

in the configuration $(0,2+;+,-,+,-;+)$

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 7.86131 - 22.585x - 10.2257y + 5.15825x^2 - 5.92109xy \\
&\quad - 7.34354y^2 + 0.21660x^3 + 0.9303x^2y - y^3, \\
Q(x,y) &= 2.50283 - 12.253x - 6.27431y + 4.96135x^2 + 0.88691xy \\
&\quad - 2.02498y^2 + 0.16245x^3 + 0.99043x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(-4,-4), \quad (-4,-4), \quad (-4.1,5.1), \quad (-4,5), \quad (45.73..,-24.47..), \\
(1.85,-3), \quad (1.88..,-2.65..), \quad (1,-1.85), \quad (2,-1.73..)
\end{gathered}
$$

in the configuration $(0,2+;3+,-;-)$.

Configuration $(3+;3;2)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(3+;0,2-;2+)$, $(3+;0,2+;2-)$, $(3+;0,+,-;+,-)$, $(3+;+,2-;0,+)$ and also $(3+;2+,-;0,-)$. We will show that the configurations $(3+;0,2-;2+)$ is not possible. Indeed, we denote by p_1,p_2,p_3 the points in the 0-level and by p_4,p_5 the points in the 1st level with negative index. Note that there exists $k_0 \in \{1,2,3\}$ so that $C = L_{0,4}L_{0,5}$ evaluated at p_{k_0} has the same sign than C evaluated to the two positive points in the 2nd index. Applying formula (2) with $C = L_{0,4}L_{0,5}L_{k_1,k_2}$ where $k_1,k_2 \in \{1,2,3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we have a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 7.80543 + 5.53466x + 11.72129y + 1.75211x^2 + 3.947459xy \\ &\quad + 2.83894y^2 + 0.4850x^3 + 1.23003x^2y - y^3, \\ Q(x,y) &= 3.70643 + 4.38387x + 6.9673y + 1.88746x^2 + 5.093xy \\ &\quad + 3.2218y^2 + 0.31337x^3 + 1.135x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-4, 5.20..), \quad (4.92.., -3.88..), \quad (4.00.., -2.99..), \\ &(3.82.., -2.91..), \quad (1-85, -2), \quad (1.85, -2), \\ &(1, -1.49..), \quad (-1.96.., -0.00..), \quad (-2.52.., -2.17..) \end{aligned}$$

in the configuration $(3+;0,2+;2-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 17.676 + 12.71329x + 23.36367y + 1.89862x^2 + 5.68740xy \\ &\quad + 3.79344y^2 + 0.00005x^3 + 0.45244x^2y - y^3, \\ Q(x,y) &= 10.45245 + 9.29009x + 14.92425y + 1.9876x^2 + 6.28217xy \\ &\quad + 3.87415y^2 - 0.01806x^3 + 0.60356x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-54.84.., -4.87..), \quad (-4, 5.2), \quad (4.92.., -3.88..) \\ &(4.00.., -2.99..), \quad (4.00.., -2.99..), \quad (1.85, -2), \\ &(1, -1.49..), \quad (-1.96.., -0.00..), \quad (-2.52.., -2.17..) \end{aligned}$$

in the configuration $(3+;0,+,-;+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 19.5475 + 16.6124x + 18.99y + 6.43719x^2 + 10.846xy \\ &\quad + 2.22562y^2 + 1.25946x^3 + 2.69803x^2y - y^3, \\ Q(x,y) &= 0.292006 - 5.41301x + 4.45102y - 2.412x^2 - 1.63724xy \\ &\quad + 2.91582y^2 - 0.598931x^3 - 0.619738x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned}&(-4,-4), \quad (45.73..,-24.47..), \quad (-2.52..,-2.17..),\\&(-4,5), \quad (-0.9..,-1.35..), \quad (-0.9,-1.35),\\&(-0.9,-1.35), \quad (-0.9,-1.34), \quad (4,-3)\end{aligned}$$

in the configuration $(3+;+,2-;0,+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 6.23278 + 4.39091x + 9.86634y + 1.72876x^2 + 3.67023xy\\&\quad + 2.68686y^2 + 0.56225x^3 + 1.35392x^2y - y^3,\\Q(x,y) &= 3.14549 + 3.97591x + 6.30566y + 1.87914x^2 + 4.99412xy\\&\quad + 3.16756y^2 + 0.34093x^3 + 1.1792x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(-4,5.20), \quad (4.92..,-3.88), \quad (4.00..,-2.99..),\\&(1.85,-2), \quad (1,-1.49..), \quad (1,-1.49..),\\&(-1.96..,-0.00..), \quad (-1.36..,-0.31..), \quad (-2.52..,-2.17..)\end{aligned}$$

in the configuration $(3+;2+,-;0,-)$.

Configuration $(0,2+;3;2)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,2+;2-,+;2+)$, $(0,2+;3+;2-)$ and $(0,2+;2+,-;+,-)$. We show that the configurations $(0,2+;+,2-;2+)$ and $(0,2+;3+;2-)$ are not possible.

For the configuration $(0,2+;2-,+;2+)$ denote by p_1,p_2 the points in the 0 level and by p_3,p_4 the two points with negative index in the 1st level. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{3,4}$ we reach a contradiction .

For the configuration $(0,2+;3+;2-)$ we denote by p_1,p_2 the points in the 0 level, by p_3,p_4,p_5 the points in the 1st level and by p_6,p_7 the points in the 2nd level. Note that there exists $k_0 \in \{1,2\}$ and $k_1 \in \{3,4,5\}$ so that $C = L_{0,k_1}L_{k_2,k_3}$ with $k_2,k_3 \in \{3,4,5\}$ such that $k_2 \neq k_1$ and $k_3 \neq k_1$ satisfies that $C(p_{k_0})$ has different sign than $C(p_6)$. Applying formula (2) with $R = CL_{0,k_4}$ with $k_4 \in \{1,2\}$ and $k_4 \neq k_0$, we have a contradiction.

In short only the configuration $(0,2+;2+,-;+,-)$ is possible. The cubic system (1) with

$$\begin{aligned}P(x,y) &= 6.80256 - 22.2257x + 10.2134y - 4.99883x^2 + 6.01111xy\\&\quad + 7.14855y^2 - 0.19868x^3 - 0.89485x^2y + y^3,\\Q(x,y) &= 1.09836 - 11.7764x - 6.25806y + 4.74987x^2 + 0.76750xy\\&\quad - 1.76632y^2 + 0.13868x^3 + 0.9434x^2y + xy^2,\end{aligned}$$

has the singular points

$$(-4., -4), \quad (-4, -4), \quad (-4, 5), \quad (-4.1, 5.1), \quad (45.73.., -24.47..),$$
$$(0.92.., -3.93..), \quad (1, -1.85), \quad (1.85, -2), \quad (2, -1.73..)$$

in the configuration $(0, 2+; 2+, -; +, -)$.

4. Proof of Theorem 2

The possible configurations are (8), $(7;1)$, $(6;2)$, $(5;3)$, $(4;4)$, $(4;3;1)$, $(3;5)$, $(3;4;1)$ and $(3;3;2)$. We will study each of them separately.

Configuration (8) Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -, +, -, +, -, +)$ and $(0, 2+, -, +, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 2 - y + 2x^2 - 2y^2 - x^2y + y^3, \\ Q(x,y) &= 2.64643 - 0.41589x + 0.91936y - 2.46187x^2 + 0.54440xy \\ &\quad - 1.7054y^2 + 0.58412x^3 - 0.22935x^2y + xy^2, \end{aligned}$$

has the singular points

$$(2,2), \quad (2,2), \quad (1,2), \quad (-0.58.., -0.80..), \quad (-0.70.., -0.70..),$$
$$(-0.86.., -0.5), \quad (-0.97.., -0.22..), \quad (0.58.., -0.80..), \quad (0.86.., -0.5)$$

in the configuration $(0, +, -, +, -, +, -, +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 2 - 2x^2 - y + x^2y - 2y^2 + y^3, \\ Q(x,y) &= 2.61768 - 0.68657x + 1.04665y - 2.44229x^2 + 0.30222xy \\ &\quad - 1.56953y^2 + 0.78357x^3 - 0.346x^2y + xy^2, \end{aligned}$$

has the singular points

$$(2,2), \quad (1,2), \quad (1,2), \quad (-0.58.., -0.80..), \quad (-0.70.., -0.70..),$$
$$(-0.14.., -0.98..), \quad (-0.86.., -0.5), \quad (0.42.., -0.90..), \quad (0.86.., -0.5)$$

in the configuration $(0, 2+, -, +, -, +, -)$.

Configuration $(7;1)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -, +, -, +, -; +)$, $(0, 2+, -, +, -, +; -)$, $(0, 3+, -, +, -; -)$ and $(2+, -, +, -, +, -; 0)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& \, 128.006 + 265.684x - 73.0216y - 93.2801x^2 + 260.74xy\\&- 188.824y^2 - 180.419x^3 + 125.935x^2y + y^3,\\Q(x,y) =& \, 151.198 + 312.596x - 83.878y - 110.076x^2 + 307.755xy\\&- 221.865y^2 - 211.959x^3 + 146.972x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(2,2),\quad (1,2),\quad (-0.58..,-0.80..),\quad (-0.70..,-0.70..),\quad (-0.86..,-0.5),\\&(-0.86..,-0.5),\quad (-0.84..,-0.49..),\quad (0.42..,-0.90..),\quad (0.86..,-0.5)\end{aligned}$$

in the configuration $(0,+,-,+,-,+,-;+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& 2 - y - 2x^2 - 2y^2 + x^2y + y^3,\\Q(x,y) =& - 8.08705 - 1.52929x - 3.76842y + 4.96361x^2 - 1.23536xy\\&+ 3.90598y^2 - 2.48180x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(4,2),\quad (2,2),\quad (1,2),\quad (1,2),\quad (-1.11..,2),\\&(-0.70..,-0.70..),\quad (-0.86..,-0.5),\quad (0.42..,-0.90..),\quad (0.58..,-0.80..)\end{aligned}$$

in the configuration $(0,2+,-,+,-,+;-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& 4.64583 + 5.30182x - 1.50138y - 1.3072x^2 + 0.45597xy\\&- 6.5235y^2 - 6.98704x^3 + 7.47839x^2y + y^3,\\Q(x,y) =& 1.21683 + 2.34265x - 1.07425y - 0.99325x^2 + 2.17057xy\\&- 2.65934y^2 - 2.10737x^3 + 1.08679x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.5,2),\quad (1.5,2),\quad (1,1.5),\quad (1.24..,0.57..),\quad (-0.86..,-0.5),\\&(-0.70..,-0.5),\quad (0.24..,-0.94),\quad 0.42..,-0.90..),\quad (0.58..,-0.80..)\end{aligned}$$

in the configuration $(0,3+,-,+,-;-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& - 5.45495 - 15.7188x + 3.26104y + 3.40044x^2 - 15.4262xy\\&+ 9.05317y^2 + 10.6742x^3 - 6.3916x^2y + y^3,\\Q(x,y) =& - 4.94356 - 16.6295x + 5.36845y + 3.03514x^2 - 15.344xy\\&+ 9.64123y^2 + 11.61x^3 - 7.84299x^2y + xy^2,\end{aligned}$$

has the singular points

$$(2,2),\quad (1,2),\quad (-0.58..,-0.80..),\quad (-0.71..,-0.71..),\quad (-0.70..,-0.70..),$$
$$(-0.70..,-0.70..),\quad (-0.86..,-0.5),\quad (0.42..,-0.90..),\quad (0.86..,-0.5)$$

in the configuration $(2+,-,+,-,+,-;0)$.

Configuration $(6;2)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,3+,-,+;2-)$, $(0,4+,-;2-)$, $(0,-,2+,-,+;+,-)$, $(0,+,-,+,-,+,-;+,-)$, $(0,2+,-,+,-;+,-)$, $(0,2+,-,2+;2-)$, $(2+,-,2+,-;0,-)$, $(3+,-,+,-;0,-)$ and $(+,-,+,-,+,-;0,+)$.

We will show that the configurations $(0,3+,-,+;2-)$ and $(0,4+,-;2-)$ are not possible.

For the configuration $(0,3+,-,+;2-)$ we denote by p_1,p_2,p_3 the three consecutive points with positive index being p_1 the closest point to p_0 and by p_5 the remaining point with positive index. Applying the Euler-Jacobi formula (2) to $R = L_{0,1}L_{0,5}L_{2,3}$ we reach a contradiction.

For the configuration $(0,4+,-;2-)$ we denote by p_1,p_2,p_3,p_4 the three consecutive points with positive index and by p_5,p_6,p_7 the points with negative index being p_5 the point in the 0-level. Consider the straight line $S = L_{6,7}$. Consider the straight line $L_{0,2}$. We have three possible cases. First the points p_6,p_7 are on the right-hand side of $L_{0,2}$, second the points p_6,p_7 are on the left-hand side of $L_{0,2}$, and third the points p_6,p_7 are one of the left-hand side, for instance p_6, and the other is on the right-hand side of $L_{0,2}$. In the first case applying (2) to $L_{0,1}L_{0,2}L_{3,4}$ we have a contradiction. In the second case applying (2) to $L_{0,1}L_{0,2}L_{4,5}$ we reach a contradiction, and finally in the third case applying (2) to $L_{0,1}L_{0,6}L_{5,7}$ we have a contradiction.

The cubic system (1) with

$$\begin{aligned}
P(x,y) &= -0.63728 - 1.01236x - 1.77583y + 0.35343x^2 - 1.55544xy \\
&\quad - 0.36373y^2 - 0.29452x^3 + 0.85409x^2y + y^3, \\
Q(x,y) &= 0.05170 - 0.95142x - 0.03677y - 0.22530x^2 - 0.45898xy \\
&\quad - 0.29892y^2 + 0.15219x^3 - 0.42001x^2y + xy^2,
\end{aligned}$$

has the singular points

$$(1,2),\quad (0.58..,-0.80..),\quad (-0.59..,0.80..),\quad (0.42..,-0.90..),\quad (4,2),$$
$$(2,2),\quad (0.01..,-0.45..),\quad (-0.70..,-0.70..),\quad (-0.70..,-0.70..)$$

in the configuration $(0,-,2+,-,+;+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 2 - y - 2x^2 - 2y^2 + x^2y + y^3, \\ Q(x,y) &= -1.45224 - 1.29953x - 0.3233y + 1.58257x^2 - 1.35024xy \\ &\quad + 0.52471y^2 - 0.79129x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(4,2), \quad (2,2), \quad (1,2), \quad (1,2), \quad (0.03..,2), \quad (-0.70..,-0.70..), \\ &(0.42..,-0.90..), \quad (-0.59..,0.80..), \quad (0.86..,-0.5) \end{aligned}$$

in the configuration $(0,+,-,+,-,+,-;+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.63728 - 1.01236x - 1.77583y + 0.35343x^2 - 1.55544xy \\ &\quad - 0.36373y^2 - 0.29452x^3 + 0.8541x^2y + y^3, \\ Q(x,y) &= 0.0517 - 0.95142x - 0.03677y - 0.2253x^2 - 0.45898xy \\ &\quad - 0.29892y^2 + 0.15219x^3 - 0.42001x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(4,2), \quad (2,2), \quad (1,2), \quad (-0.70..,-0.70..), \quad (-0.70..,-0.70..), \\ &(0.42..,-0.90..), \quad (0.01..,-0.45..), \quad (0.58..,-0.8..), \quad (-0.59..,0.80..) \end{aligned}$$

in the configuration $(0,2+,-,+,-;+,-)$,

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -1.05986 - 1.17457x - 1.90014y + 0.73053x^2 - 1.80467xy \\ &\quad - 0.10155y^2 - 0.34171x^3 + 0.83072x^2y + y^3, \\ Q(x,y) &= 0.39154 - 0.82097x + 0.06321y - 0.52855x^2 - 0.25854xy \\ &\quad - 0.50976y^2 + 0.19014x^3 - 0.40121x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(4,2), \quad (2,2), \quad (1,2), \quad (-0.70..,-0.70..), \quad (0.42..,-0.90..), \\ &(0.41..,-0.88..), \quad (0.58..,-0.80..), \quad (0.58..,-0.80..), \quad (-0.59..,0.80..) \end{aligned}$$

in the configuration $(0,2+,-,2+;2-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.89569 + 1.6814x - 3.69593y + 0.46307x^2 + 1.50863xy \\ &\quad - 1.85672y^2 - 1.2361x^3 + 1.89441x^2y + y^3, \\ Q(x,y) &= -0.15127 + 1.1644x - 1.54493y - 0.13917x^2 + 1.94772xy \\ &\quad - 1.4716y^2 - 0.58739x^3 + 0.39711x^2y + xy^2, \end{aligned}$$

has the singular points

$(4,2)$, $(1,2)$, $(-1.18..,-0.93..)$, $(-0.70..,-0.70..)$, $(-0.70..,-0.70..)$, $(-0.86..,-0.5)$, $(0.42..,-0.90..)$, $(0.58..,-0.80..)$, $(0.86..,-0.5)$

in the configuration $(2+,-,2+,-;0,-)$.

The cubic system (1) with

$$\begin{aligned}
P(x,y) &= 2 - y - 2x^2 - 2y^2 + x^2y + y^3, \\
Q(x,y) &= -1.31108 - 1.47454x - 0.43766y + 0.99081x^2 - 1.26273xy \\
&\quad + 0.5466y^2 - 0.49541x^2y + xy^2,
\end{aligned}$$

has the singular points

$(4,2)$, $(2,2)$, $(2,2)$, $(1.14..,2)$, $(1,2)$, $(-0.70..,-0.70..)$,
$(0.42..,-0.95..)$, $(0.58..,-0.80..)$, $(-0.59..,0.80..)$

in the configuration $(3+,-,+,-;0,-)$.

The cubic system (1) with

$$\begin{aligned}
P(x,y) &= 0.22343x - 3.70599y + 1.91805x^2 - 0.91189xy \\
&\quad + 2.33258y^2 + 0.20499x^3 - 1.65986x^2y + y^3, \\
Q(x,y) &= 0.33464x - 4.37889y + 2.26518x^2 - 1.91377xy \\
&\quad + 4.02829y^2 + 0.29371x^3 - 2.21375x^2y + xy^2
\end{aligned}$$

has the singular points

$(2.68..,1.46..)$, $(1.7,1.3)$, $(1.7,1.3)$, $(1.2,0.64)$, $(0.1,1.1)$,
$(0.6,0.2)$, $(-0.5,0.1)$, $(-1.1,0.65)$, $(0,0)$

in the configuration $(+,-,+,-,+,-;0,+)$.

Configuration $(5;3)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,-,2+,-;2+,-)$, $(0,+,-,+,-;2+,-)$, $(0,2+,-,+;2-,+)$, $(0,3+,-;2-,+)$, $(2+,-,+,-;0,+,-)$ and $(4+,-;0,2-)$.

The cubic system (1) with

$$\begin{aligned}
P(x,y) &= -169.055 + 488.263x - 94.7783y - 431.31x^2 + 174.286xy \\
&\quad - 5.32869y^2 + 116.227x^3 + -63.2897x^2y + y^3, \\
Q(x,y) &= -24.6823 + 73.049x - 14.8523y - 66.5269x^2 + 29.1387xy \\
&\quad - 1.43955y^2 + 18.7953x^3 - 11.8453x^2y + xy^2,
\end{aligned}$$

has the singular points

$(2,2)$, $(2,2)$, $(1.93..,3.44..)$, $(0.62..,-1.07..)$, $(1.91..,2.85..)$,
$(0.8,2)$, $(0.57,-0.9)$, $(0.58..,-0.80..)$, $(0.86..,-0.5)$

in the configuration $(0,-,2+,-;2+,-)$.

The cubic system (1) with

$$P(x,y) = -0.26624 - 0.127x + 1.42367y + 0.21549x^2 + 4.60997xy + 3.27449y^2 - 2.03241x^3 - 2.64176x^2y + y^3,$$
$$Q(x,y) = 0.26673 + 0.74274x - 1.54738y + 0.42201x^2 - 4.81204xy - 2.55609y^2 + 1.32549x^3 + 1.2861x^2y + xy^2,$$

has the singular points

$(1.7, 1.3)$, $(1.2, 0.64)$, $(0.2, -1.2)$, $(-0.15.., -0.41..)$, $(-1.63.., 1.29..)$, $(-1.63.., 1.29..)$, $(0, 0.14)$, $(-0.83.., 0.45..)$, $(-0.9, 0.65)$

in the configuration $(0, +, -, +, -; 2+, -)$.

The cubic system (1) with

$$P(x,y) = 2 - y - 2x^2 + xy - 2y^2 + x^2y + y^3,$$
$$Q(x,y) = -1.45224 - 1.29953x - 0.3233y + 1.58257x^2 - 1.35024xy + 0.52471y^2 - 0.79129x^2y + xy^2,$$

has the singular points

$(4, 2)$, $(4, 2)$, $(2, 2)$, $(1, 2)$, $(0.18.., 2)$,
$(-0.70.., -0.70..)$, $(0.42.., -0.90..)$, $(-0.59.., 0.80)$, $(0.86.., -0.5)$

in the configuration $(0, 2+, -, +; 2-, +)$.

The cubic system (1) with

$$P(x,y) = -0.32744 - 0.74422x - 1.5832y + 0.38015x^2 - 1.37547xy - 0.62723y^2 - 0.24965x^3 + 0.68371x^2y + y^3,$$
$$Q(x,y) = 0.29506 - 0.74081x + 0.11453y - 0.2043x^2 - 0.31762xy - 0.50588y^2 + 0.18743x^3 - 0.55384x^2y + xy^2,$$

has the singular points

$(4, 2)$, $(2, 2)$, $(1, 2)$, $(-0.70.., -0.70..)$, $(-0.70.., -0.70)$,
$(0.42.., -0.90..)$, $(-0.59.., 0.80..)$, $(0.40.., -0.34..)$, $(0.86.., -0.5)$

in the configuration $(0, 3+, -; 2-, +)$.

The cubic system (1) with

$$P(x,y) = -2.65273 - 1.49182x - 3.75008y + 1.40703x^2 - 4.00078xy + 1.89442y^2 - 0.6781x^3 + 1.66983x^2y + y^3,$$
$$Q(x,y) = -1.53136 - 1.32802x - 1.58746y + 0.60227x^2 - 2.37969xy + 1.47477y^2 - 0.1491x^3 + 0.22071x^2y + xy^2,$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (-1.077..,-0.89..),\quad (-0.70..,-0.70..),$$
$$(-0.70..,-0.70..),\quad (-0.86..,-0.5),\quad (-0.59..,0.80..),\quad (0.86..,-0.5)$$

in the configuration $(2+,-,+,-;0,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) =& -0.66741-0.85292x-1.66839y+0.72781x^2-1.57638xy \\ & -0.42671y^2-0.28612x^3+0.63752x^2y+y^3, \\ Q(x,y) =& 0.4278-0.69837x+0.14779y-0.34005x^2-0.23917xy \\ & -0.58417y^2+0.20166x^3-0.5358x^2y+xy^2, \end{aligned}$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (-0.70..,-0.70..),\quad (0.42..,-0.90..),$$
$$(0.21..,-0.75..),\quad (-0.59..,0.80..),\quad (0.86..,-0.5),\quad (0.86..,-0.5)$$

in the configuration $(4+,-;0,2-)$.

Configuration $(4;4)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,-,+,-;3+,-)$, $(0,+,-,+;+,-,+,-)$, $(0,+,-,+;2+,2-)$, $(0,2+,-;+,-,+,-)$, $(0,2+,-;2+,2-)$, $(0,3+;3-,+)$, $(+,-,+,-;0,+,-,+,-)$, $(+,-,+,-;0,2+,-)$, $(3+,-;0,-,+,-)$, $(3+,-;0,2-,+)$ and $(4+;0,3-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.81818+0.06798x-1.1351y+2.57036x^2+0.06798xy \\ &\quad -1.95328y^2-0.71989x^3-0.75432x^2y+y^3, \\ Q(x,y) &= 1.40791-0.95813x-0.04956y+1.71975x^2+0.04187xy \\ &\quad -1.45747y^2-1.31898x^3+0.09218x^2y+xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.33..,1.66..),\quad (1.31..,1.67..),\quad (0.-1),\quad (0,-1),$$
$$(0.79..,1.3),\quad (0.15..,-1.01),\quad (0.53..,-1.13..),\quad (-0.53..,1.13..)$$

in the configuration $(0,-,+,-;3+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.22404+1.65337x-1.64431y+1.96013x^2-1.71128xy \\ &\quad -0.58091y^2+1.00465x^3-1.94643x^2y+y^3, \\ Q(x,y) &= 0.68907+0.70571x-0.70214y+2.06091x^2-1.41422xy \\ &\quad -0.32726y^2-1.35748x^3-0.18828x^2y+xy^2, \end{aligned}$$

has the singular points

$$(1.5, 2), \quad (1.33.., 1.66..), \quad (-2.68.., 3.93..), \quad (0.79.., 1.5), \quad (0.53.., 1.13..),$$
$$(-0.53.., -1.13..), \quad (-0.53.., -1.13..), \quad (-0.53.., 1.13..), (-0.22.., 0.78..)$$

in the configuration $(0, +, -, +; +, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 7.55853 - 10.0716x - 1.24914y + 5.0247x^2 + 10.2185xy \\ &\quad - 7.80767y^2 - 6.16078x^3 + 2.69095x^2y + y^3, \\ Q(x, y) &= 2.06465 - 2.88002x - 0.05144y + 2.29759x^2 + 1.7866xy \\ &\quad - 2.11609y^2 - 1.76752x^3 + 0.05748x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.47.., 2.02..), \quad (1.5, 2), \quad (1.5, 2), \quad (1.33.., 1.66..), \quad (1.31.., 1.67..),$$
$$(0.84.., 1.3), \quad (0.53.., 1.10..), \quad (0, -1), \quad (-0.53.., 1.13..)$$

in the configuration $(0, +, -, +; 2+, 2-)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 0.05521 + 0.04668x - 2.22167y + 1.23859x^2 - 1.12651xy \\ &\quad - 0.9332y^2 + 0.49019x^3 - 0.36319x^2y + y^3, \\ Q(x, y) &= 0.25156 + 0.58542x - 4.02491y + 1.58971x^2 - 8.43119xy \\ &\quad - 5.19507y^2 + 3.01039x^3 + 2.43304x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.6, 0.2), \quad (-1.63.., 1.29..),$$
$$(-1.63.., 1.29..), \quad (-0.75.., 0.28..), \quad (-1.3, 0.65), \quad (-0.35.., 0.09..)$$

in the configuration $(0, 2+, -; +, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= -10.3655 - 8.98918x - 27.6779y + 9.36214x^2 + 6.33352xy \\ &\quad + 6.75136y^2 + 0.8147x^3 + 7.51867x^2y + y^3, \\ Q(x, y) &= 8.32828 + 3.6089x + 13.3552y - 4.42819x^2 + 3.32785xy \\ &\quad - 4.90853y^2 - 1.30944x^3 - 7.55403x^2y + xy^2, \end{aligned}$$

has the singular points

$$(-2.72.., -6.63..), \quad (-0.06.., 3.14..), \quad (1.47.., 0.49..),$$
$$(2.94.., -1.33..), \quad (2.94.., -1.33..), \quad (2.17.., -1.78..),$$
$$(-1.09.., 0.42..), \quad (1.09.., -0.52..), \quad (2.35.., -1.46..)$$

in the configuration $(0, 2+, -; 2+, 2-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.59672 + 0.29792x - 1.21191y + 2.50785x^2 + 0.29792xy \\ &\quad - 1.80862y^2 - 0.96331x^3 - 0.81568x^2y + y^3, \\ Q(x,y) &= 0.9278 - 0.5773x - 0.30067y + 2.22057x^2 + 0.42271xy \\ &\quad - 1.22846y^2 - 2.3668x^3 + 0.26153x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.5,2), \quad (0,-1), \quad (0,-1), \quad (1.33..,1.66..), \quad (0.79..,1.5), \\ (0.53..,1.13..), \quad (0.53..,-1.13..), \quad (-0.77..,1.26..), \quad (-0.22..,0.78..) \end{gathered}$$

in the configuration $(0,3+;3-,+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.66741 - 0.85292x - 1.66839y + 0.72781x^2 - 1.57638xy \\ &\quad - 0.42671y^2 - 0.28612x^3 + 0.63752x^2y + y^3, \\ Q(x,y) &= 0.4278 - 0.69837x + 0.14779y - 0.34005x^2 - 0.23917xy \\ &\quad - 0.58417y^2 + 0.20167x^3 - 0.5358x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (4,2), \quad (2,2), \quad (1,2), \quad (-0.70..,-0.70..), \quad (0.42..,-0.90..), \\ (0.21..,-0.75..), \quad (-0.59..,0.80..), \quad (0.86..,-0.5), \quad (0.86..,-0.5) \end{gathered}$$

in the configuration $(+,-,+,-;0,+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.23671 - 1.23041x - 0.91629y + 2.60321x^2 + 0.82175xy \\ &\quad - 1.153y^2 + 1.03428x^3 - 2.83549x^2y + y^3, \\ Q(x,y) &= 0.70928 - 2.21598x + 0.07739y + 2.09491x^2 + 0.81736xy \\ &\quad - 0.63189y^2 - 0.00953x^3 - 1.81021x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.5,2), \quad (1.31..,1.67..), \quad (0.79..,1.3), \quad (0.79..,1.28..), \quad (0,-1), \\ (0.53..,1.10..), \quad (0.53..,1.10..), \quad (0.53..,-1.13..), \quad (-0.53..,1.13..) \end{gathered}$$

in the configuration $(+,-,+,-;0,2+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.32167 + 1.43788x - 1.57293y + 2.16797x^2 - 1.37002xy \\ &\quad - 0.86391y^2 0.40451x^3 - 1.5578x^2y + y^3, \\ Q(x,y) &= 0.37924 + 1.38958x - 0.92865y + 1.40129x^2 - 2.49725xy \\ &\quad + 0.57086y^2 + 0.54712x^3 - 1.42164x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned}&(1.5, 2), \quad (1.33.., 1.66..), \quad (0.79.., 1.5),\\ &(-0.53.., -1.13..), \quad (0.53.., 1.13..), \quad (-0.53.., 1.13..),\\ &(-0.22364.., 0.78..), \quad (-0.22393.., 0.78..), \quad (-0.22393.., -0.78..)\end{aligned}$$

in the configuration $(3+, -; 0, -, +, -)$.

The cubic system (1) with

$$\begin{aligned}P(x, y) &= -0.03342 + 0.13443x + 0.1563y + 0.34256x^2 - 1.88158xy\\ &\quad + 0.97155y^2 + 0.62347x^3 - 1.09981x^2y + y^3,\\ Q(x, y) &= -0.02792 + 0.01194x + 0.24347y + 0.1814x^2 - 0.64693xy\\ &\quad - 0.12559y^2 + 0.3054x^3 - 0.94879x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.7, 1.3), \quad (0.3, 0.2), \quad (0.3, 0.2), \quad (1.2, 0.64), \quad (0.2, -1.2),\\ &(0.62.., 0.50..), \quad (0.40.., 0.31..), \quad (-0.5, 0.1), \quad (-0.1, 0.1)\end{aligned}$$

in the configuration $(3+, -; 0, 2-, +)$.

The cubic system (1) with

$$\begin{aligned}P(x, y) &= 3.1825 + 0.66392x + 1.92747y + 1.30631x^2 - 4.16906xy\\ &\quad - 2.89871y^2 - 1.25514x^3 + 2.09021x^2y + y^3,\\ Q(x, y) &= -0.0862 + 0.15006x - 0.09326y + 0.0254x^2 + 0.76885xy\\ &\quad - 0.60459y^2 + 0.82558x^3 - 1.92383x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(2, 2), \quad (1.55.., 1.90..), \quad (1.5, 1.8), \quad (1.33.., 1.81..), \quad (1.33.., 1.81..),\\ &(1, 2), \quad (-0.86.., -0.5), \quad (-0.70.., -0.5), \quad (0.42.., -0.90..)\end{aligned}$$

in the configuration $(4+; 0, 3-)$.

Configuration $(4; 3; 1)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -, +; 2-, +; +)$, $(0, +, -, +; 2+, -; -)$, $(0, 2+, -; 2+, -; -)$, $(0, 2+, -; 2-, +; +)$, $(0, 3+; 3-; +)$, $(0, 3+; 2-, +; -)$, $(0, -, +, -; 2+, -; +)$, $(0, -, +, -; 3+; -)$, $(+, -, +, -; 0, +, -; +)$, $(+, -, +, -; 0, 2+; -)$, $(3+, -; 0, +, -; -)$, $(3+, -; 0, 2-; +)$, $(+, -, +, -; 2+, -; 0)$, $(3+, -; 2-, +; 0)$, $(4+; 3-; 0)$, and $(4+; 0, 2-; -)$.

No we show that the configurations $(0, 2+, -; 2+, -; -)$ and $(0, 3+; 2-, +; -)$ are not possible.

For the configuration $(0, 2+, -; 2+, -; -)$, we denote by p_0, p_1, p_2, p_3 the points in the 0 level (being p_3 the point with negative index), by p_4, p_5, p_6 the points in the 1st level (being p_6 the negative index) and by p_7 the point in the

2nd level. Consider the straight line $L_{6,7}$. The points p_1, p_2, p_4 and p_5 have positive index. The line $L_{6,7}$ can separate the set of points $\{p_1, p_2, p_4, p_5\}$ in two ways. First one point p_ℓ is on one side of $L_{6,7}$ and the other three points are on the other side of $L_{6,7}$. Second two points are on one side of $L_{6,7}$ and the other two points are on the other side of $L_{6,7}$. In the first case applying (2) to $L_{0,3}L_{0,\ell}L_{6,7}$ we have a contradiction. In the second case denote by p_k the closest point of the set $\{p_1, p_2, p_4, p_5\}$ to $L_{6,7}$. Applying (2) to $L_{0,3}L_{0,k}L_{6,7}$ we reach a contradiction.

For the configuration $(0, 3+; 2-, +; -)$ denote by p_0, p_1, p_2, p_3 the points in the 0 level, by p_4, p_5, p_6 the points in the 1st level (being p_4 with positive index) and by p_7 the point in the 2nd level. Consider the straight line $L_{0,4}$. If the three singular points with negative index are in the same side of $L_{0,4}$ then applying (2) to $L_{0,1}L_{0,4}L_{2,3}$ we have a contradiction. If the three singular points with negative index are not all in the same side of $L_{0,4}$ then applying (2) to $L_{0,1}L_{0,3}L_7$ being L_7 the straight line passing to p_7 being parallel to $L_{5,6}$ we reach a contradiction.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 2.66278 - 2.03402x - 3.32725y + 0.13241x^2 + 5.23787xy \\
&\quad - 2.25453y^2 - 1.23728x^3 - 4.87209x^2y + y^3, \\
Q(x,y) &= 1.07346 - 1.81894x - 3.55618y + 0.48056x^2 + 4.04944xy \\
&\quad + 1.16146y^2 + 0.42202x^3 + 0.58928x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{array}{ccc}
(-2.72.., -6.63..), & (2.94.., -1.33..), & (2.17.., -1.78..), \\
(-0.06.., 3.14..), & (1.09.., -0.52..), & (-1.09.., 0.42..), \\
(-1.09.., 0.42..), & (1.026.., -0.25..), & (0.45.., 0.56..)
\end{array}
$$

in the configuration $(0, +, -, +; 2-, +; +)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 9.65409 - 1.51164x - 5.00183y + 6.8444x^2 + 1.47388xy \\
&\quad - 7.15989y^2 - 11.7997x^3 + 9.61314x^2y + y^3 \\
Q(x,y) &= 1.21914 - 0.151691x + 0.432748y + 2.77906x^2 - 1.75697xy \\
&\quad - 1.74289y^2 - 2.74733x^3 + 1.64549x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{array}{ccccc}
(-0.79.., 2), & (0.53.., -1.13..), & (1.45.., 2.14..), & (1.5, 2), & (1.5, 2), \\
(1.31.., 1.67..), & (0.81.., 1.3), & (0.53.., 1.10..), & (1.33.., 1.66..) &
\end{array}
$$

in the configuration $(0, +, -, +; 2+, -; -)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 1.58089 - 3.87837x - 8.08004y + 0.71079x^2 + 5.41967xy \\
&\quad - 0.64709y^2 - 0.58716x^3 - 2.83003x^2y + y^3 \\
Q(x,y) &= -1.29881 - 0.50969x - 2.43789y + 2.54356x^2 + 4.06429xy \\
&\quad + 1.05356y^2 - 0.17975x^3 + 0.78554x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(-2.72.., -6.63..), \quad (2.31.., -1.98..), \quad (2.94.., -1.33..), \\
(2.94.., -1.33..), \quad (2.17.., -1.78..), \quad (-0.06.., 3.14..), \\
(2.03.., -1.46..), \quad (1.09.., -0.52..), \quad (-1.09.., 0.42..)
\end{gathered}
$$

in the configuration $(0, 2+, -; 2-, +; +)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 1.83083 + 0.06503x - 1.12582y + 2.53496x^2 + 0.06503xy \\
&\quad - 1.95665y^2 - 0.6756x^3 - 0.77083x^2y + y^3, \\
Q(x,y) &= 1.29796 - 0.93243x - 0.13022y + 2.02728x^2 + 0.06757xy \\
&\quad - 1.42818y^2 - 1.70372x^3 + 0.23551x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(1.5, 2), \quad (0, -1), \quad (0, -1), \quad (1.38.., 1.89..), \quad (1.31.., 1.67..), \\
(0.79.., 1.3), \quad (0.53.., 1.10..), (0.53.., -1.13..), \quad (-0.53.., 1.13..)
\end{gathered}
$$

in the configuration $(0, 3+; 3-; +)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= -19.764x - 20y - y^2 + 4.38561x^2 + 3.55687xy - 0.238337x^3 \\
&\quad - 0.136263x^2y + 0.118392xy^2, \\
Q(x,y) &= -395.28x - 400y + 109.356x^2 + 113.222xy - 6.63869x^3 \\
&\quad - 5.42657x^2y + 2.58145xy^2 + y^3,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(0, 0), \quad (0, 0), \quad (0, -20), \quad (10, -9), \quad (9.1, -6), \\
(5, -7.87..), \quad (9, -7.87..), \quad (9, -7), \quad (8.9, -7.86..)
\end{gathered}
$$

in the configuration $(0, -, +, -; 2+, -; +)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 1.4391 - 0.77711x - 0.99307y + 2.5933x^2 + 0.5588xy \\
&\quad - 1.43217y^2 + 0.4201x^3 - 2.10839x^2y + y^3, \\
Q(x,y) &= 0.96301 - 1.94997x + 0.11714y + 1.74667x^2 + 0.61791xy \\
&\quad - 0.84587y^2 + 0.01895x^3 - 1.49701x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$(1.80..,2.37..),\quad (1.5,2),\quad (1.33..,1.66..),\quad (1.31..,1.67..),\quad (0,-1),$$
$$(0.79..,1.3),\quad (-0.53..,1.13..),\quad (-0.53..,1.13..),\quad (0.53..,-1.13..)$$

in the configuration $(0,-,+,-;3+;-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 2.18148+0.82957x-1.24948y+2.49468x^2-0.38158xy \\ &\quad -2.43096y^2-1.68474x^3+0.4477x^2y+y^3, \\ Q(x,y) &= 4.93445+6.99657x-1.41267y+1.60951x^2-4.56412xy \\ &\quad -6.34712y^2-12.1694x^3+12.8727x^2y+xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.31..,1.67..),\quad (1.31..,1.67..),\quad (0.79..,1.3),\quad (0,-1),$$
$$(0.53..,1.10..),\quad (-0.68..,1.53..),\quad (-0.53..,1.13..),\quad (0.53..,-1.13..)$$

in the configuration $(+,-,+,-;0,+,-;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) =&1.2053-1.29832x-0.90547y+2.60744x^2+0.86151xy \\ &-1.11077y^2+1.12318x^3-2.94351x^2y+y^3, \\ Q(x,y) =&0.84068-2.22268x+0.16297y+1.75407x^2+0.7763xy \\ &-0.67771y^2+0.38682x^3-1.93396x^2y+xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.31..,1.70..),\quad (1.33..,1.66..),\quad (1.31..,1.67..),\quad (0,-1),$$
$$(0.79..,1.3),\quad (0.79..,1.3),\quad (0.53..,-1.13..),\quad (-0.53..,1.13..)$$

in the configuration $(+,-,+,-;0,2+;-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.21847-3.23165x-1.86475y+8.41861x^2+4.65148xy \\ &\quad +2.12508y^2-5.14896x^3-3.22687x^2y+y^3, \\ Q(x,y) &= -1.88545-9.51989x-4.2883y+29.3482x^2+5.9392xy \\ &\quad +5.94989y^2-18.5052x^3-1.99464x^2y+xy^2, \end{aligned}$$

has the singular points

$$(1.74..,1.89..),\quad (-0.2,0.5),\quad (1.74..,-1.89..),$$
$$(0.85..,-0.10..),\quad (0.85..,-0.10..),\quad (-0.19..,0.53..),$$
$$(0.85..,0.20..),\quad (0.69..,-0.21..),\quad (0.55..,-0.33..)$$

in the configuration $(3+,-;0,+,-;-)$.

The cubic system (1) with

$$P(x,y) = -3.98259 + 3.15619x7.96671y - 2.13359x^2 + 8.36509xy- 0.53375y^2 - 0.59508x^3 - -1.55334x^2y + y^3,$$
$$Q(x,y) = 1.96752 - 0.56429x + 2.74268y + 1.01136x^2 - 2.82378xy - 1.28245y^2 + 0.63638x^3 - 0.87949x^2y + xy^2,$$

has the singular points

$$(2,2),\quad (1.5,1.8),\quad (1.33..,1.81..),\quad (1,2),\quad (1.23..,1.84..),$$
$$(-0.86..,-0.5),\quad (-0.70..,-0.5),\quad (-0.70..,-0.5),\quad (0.42..,-0.90..)$$

in the configuration $(3+,-;0,2-;+)$.

The cubic system (1) with

$$P(x,y) = 1.20565 - 1.29812x - 0.90534y + 2.60678x^2 + 0.8613xy - 1.11099y^2 + 1.12365x^3 - 2.9434x^2y + y^3,$$
$$Q(x,y) = 0.65655 - 2.33095x + 0.09598y + 2.10096x^2 + 0.88451xy - 0.56057y^2 + 0.14222x^3 - 1.99345x^2y + xy^2,$$

has the singular points

$$(1.5,2),\quad (1.31..,1.67..),\quad (0.79..,1.3),\quad (0.79..,1.3),\quad (0,-1),$$
$$(0.52..,1.11..),\quad (0.53..,1.10..),\quad (0.53..,-1.13..),\quad (-0.53..,1.13..)$$

in the configuration $(+,-,+,-;2+,-;0)$.

The cubic system (1) with

$$P(x,y) = -0.70509 - 0.69397x + 4.27287y - 0.13448x^2 + 4.13789xy - 3.43606y^2 - 1.67007x^3 - 0.16262x^2y + y^3,$$
$$Q(x,y) = -2.11104 - 2.03548x + 12.7928y + 1 - 0.81903x^2 + 2.2063xy - 9.51663y^2 - 5.57219x^3 + 1.30903x^2y + xy^2,$$

has the singular points

$$(1.7,1.3),\quad (-0.2,0.2),\quad (-0.2,0.2),\quad (-1.29..,-1.05..),\quad (1.2,0.64),$$
$$(0.3,0.2),\quad (-0.5,0.1),\quad (-1.63..,1.29..),\quad (-0.8,0.65),$$

in the configuration $(3+,-;2-,+;0)$.

The cubic system (1) with

$$P(x,y) = 1.1096 + 0.1224x - 1.02991y + 1.24418x^2 - 1.13951y^2 - 0.42839x^3 - 0.89532x^2y + y^3,$$
$$Q(x,y) = 0.51363 - 0.8668x - 0.03189y + 0.65715x^2 - 0.54553y^2 - 1.46621x^3 + 0.11163x^2y + xy^2,$$

has the singular points

$$(1.5, 2), \quad (0.62.., 1.20..), \quad (0.53, 1.133),$$
$$(0.534.., 1.1338..), \quad (0.534.., 1.1338..), \quad (-0.534.., -1.1338..),$$
$$(0, -1), \quad (0.534.., -1.1338..), \quad (-0.534.., 1.1338..)$$

in the configuration $(4+; 3-; 0)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 3.25912 + 0.55182x + 1.8978y + 1.18903x^2 - 4.42068xy \\ &\quad - 2.91515y^2 - 1.59916x^3 + 2.65276x^2y + y^3, \\ Q(x, y) &= -0.10693 + 0.18039x - 0.08524y + 0.05712x^2 + 0.83693xy \\ &\quad - 0.60014y^2 + 0.91865x^3 - 2.07602x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1, 2), \quad (0.42.., -0.90..), \quad (1.331.., 1.8113..),$$
$$(1.33, 1.81), \quad (1.336.., 1.812..), \quad (1.336.., 1.812..),$$
$$(2, 2), \quad (-0.86.., -0.5), \quad (-0.70.., -0.5)$$

in the configuration $(4+; 0, 2-; -)$.

Configuration $(3; 5)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -; 2+, -, +, -)$, $(0, +, -; 3+, 2-)$, $(0, 2+; 2-, +, -, +)$, $(0, 2+; 3-, 2+)$, $(2+, -; 0, -, 2+, -)$, $(2+, -; 0, +, -, +, -)$, $(2+, -; 0, 2+, 2-)$, $(3+; 0, 2-, +, -)$ and $(3+; 0, 3-, +)$.

We will show that the configurations $(0, +, -; 3+, 2-)$, $(0, 2+; 3-, 2+)$, $(2+, -; 0, 2+, 2-)$ and $(3+; 0, 3-, +)$ are not possible.

For the configuration $(0, +, -; 3+, 2-)$ denote by p_0, p_1, p_2 the points in the 0 level and by p_3, p_4, p_5, p_6, p_7 the points in the 1st level being p_6, p_7 the points with negative index. Applying (2) to $R = L_{0,1}L_{0,2}L_{6,7}$ we reach a contradiction.

For the configuration $(0, 2+; 3-, 2+)$, denote by p_0, p_1, p_2 the points in the 0 level and by p_3, p_4, p_5, p_6, p_7 the points in the 1st level being p_6, p_7 the points with positive index. Applying (2) to $R = L_{0,1}L_{0,2}L_{6,7}$ we have a contradiction.

For the configuration $(2+, -; 0, 2+, 2-)$ denote by p_1, p_2, p_3 the points in the 0 level in counterclockwise sense being p_3 the one with negative index, and by p_0, p_4, p_5, p_6, p_7 the points in the 1st level in counterclockwise sense (being p_6, p_7 with negative index). Note that there exists $k_1, k_2 \in \{1, 2, 3\}$ so the conic $C = L_{0,6}L_{0,7}$ satisfies that $C(p_{k_1})C(p_{k_2}) > 0$. Note that at least p_{k_1} or p_{k_2} has positive index. Without loss of generality we assume it is p_{k_1}. Applying formula (2) to $R = L_{0,6}L_{0,7}L_{k_2k_3}$ with $k_3 \in \{1,2,3\}$ so that $k_3 \neq k_1$ and $k_3 \neq k_2$ we reach a contradiction.

For the configuration $(3+;0,3-,+)$ denote by p_1, p_2, p_3 the points in the 0 level and by p_0, p_4, p_5, p_6, p_7 the points in the 1st level in counterclockwise sense. Consider $C = L_{0,4}L_{0,7}$. Note that there exists $\ell \in \{1,2,3\}$ so that the signs of $C(p_\ell)$ and $C(p_5)$ are different. Applying (2) to $R = CL_{k_0,k_1}$ being $k_0, k_1 \in \{1,2,3\}$ with $k_0 \neq \ell$ and $k_1 \neq \ell$, we get a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.07010 - 0.01445x - 0.73135y + 0.32004x^2 + 0.70005xy \\ &\quad + 0.60163y^2 + 0.15478x^3 - 1.47483x^2y + y^3, \\ Q(x,y) &= 0.06439 - 0.12082x - 0.54808y + 0.16133x^2 + 1.65512xy \\ &\quad - 0.45545y^2 - 0.11253x^3 - 1.2832x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7,1.3), \quad (1.2,0.64), \quad (0.2,-1.2), \quad (0.2,-1.2), \quad (-0.1,0.1), \\ (0.400..,0.31..), \quad (0.3,0.2), \quad (-0.5,0.1), \quad (0.09..,0.12..) \end{gathered}$$

in the configuration $(0,+,-;2+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -6.8337 - 23.5185x - 3.02691y + 25.4219x^2 + 25.1177xy \\ &\quad + 4.80679y^2 - 15.2956x^3 - 13.3315x^2y + y^3, \\ Q(x,y) &= 2.77818 + 10.0337x + 0.59214y - 6.70244x^2 - 11.4714xy \\ &\quad - 2.18605y^2 + 5.10052x^3 + 3.55814x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.5,2), \quad (1.33..,1.66..), \quad (-0.79..,2), \quad (-0.79..,2), \quad (0,-1), \\ (-0.78,1.99), \quad (0.53..,1.10..), \quad (-0.67..,1.61..), \quad (-0.53..,1.13..) \end{gathered}$$

in the configuration $(0,2+;2-,+,-,+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.16718x - 0.84559y + 0.67325x^2 - 0.13985xy \\ &\quad + 0.40119y^2 - 0.07882x^3 - 0.88085x^2y + y^3, \\ Q(x,y) &= 0.2161x + 0.17696y + 0.25951x^2 - 0.94753xy \\ &\quad - 0.26355y^2 + 0.01953x^3 - 0.47296x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7,1.3), \quad (1.2,0.64), \quad (1.2,0.64), \quad (0.2,-1.2), \quad (0,0), \\ (0.41..,0.71..), \quad (0.40..,0.71..), \quad (-0.5,0.1), \quad (-1,0.55) \end{gathered}$$

in the configuration $(2+,-;0,-,2+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -1.95227 + 1.56068x - 9.84555y + 0.02429x^2 + 6.40808xy\\ &\quad - 0.55368y^2 - 0.11501x^3 - 1.07423x^2y + y^3,\\ Q(x,y) &= 1.82196 - 1.19243x - 4.12989y - 0.03222x^2 + 4.35035xy\\ &\quad - 2.42007y^2 + 0.08933x^3 - 1.0964x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(4.82..,2.80..),\quad (3,1),\quad (-4.10..,0),\quad (2.60..,0.66..),\quad (2.51..,0.69..),\\ &(2.53..,0.65..),\quad (2.55,0.63..),\quad (2.55..,0.63..),\quad (2.4,-0.8)\end{aligned}$$

in the configuration $(2+,-;0,+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 1.73639 - 0.38881x - 0.96698y + 2.16264x^2 + 0.17322xy\\ &\quad - 1.70337y^2 + 0.53463x^3 - 1.68794x^2y + y^3,\\ Q(x,y) &= 0.30263 + 3.3525x - 0.00289y + 0.01622x^2 - 4.26596xy\\ &\quad - 0.30552y^2 + 0.52779x^3 + 0.1948x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.5,2),\quad (1.33..,1.66..),\quad (-0.79..,2),\quad (-0.78,1.99),\quad (0,-1),\\ &(0.53..,1.10..),\quad (0.29..,1.04..),\quad (-0.53..,1.13..),\quad (-0.53..,1.13..)\end{aligned}$$

in the configuration $(3+;0,2-,+,-)$.

Configuration $(3;4;1)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,+,-;+,-,+,-;+)$, $(0,+,-;2+,2-;+)$, $(0,+,-;3+,-;-)$, $(0,2+;3-,+;+)$, $(0,2+;2+,2-;-)$, $(0,2+;+,-,+,-;-)$, $(2+,-;0,2-,+;+)$, $(2+,-;0,2+,-;-)$, $(2+,-;0,-,+,-;+)$, $(2+,-;0,+,-,+;-)$, $(3+;0,2-,+;-)$, $(3+;0,3-;+)$, $(3+;0,-,+,-;-)$, $(2+,-;2+,2-;0)$, $(2+,-;+,-,+,-;0)$, and $(3+;3-,+;0)$.

We will show that the configurations $(0,+,-;2+,2-;+)$, $(0,2+;2+,2-;-)$, $(0,2+;+,-,+,-;-)$, $(2+,-;0,-,+,-;+)$, $(2+,-;0,+,-,+;-)$, $(3+;0,-,+,-;-)$ and $(2+,-;+,-,+,-;0)$ are not possible.

For the configuration $(0,+,-;2+,2-;+)$, denote by p_0,p_1,p_2 the points in the 0 level and by p_3,p_4,p_5,p_6 the points in the first level being p_3 and p_4 the two consecutive points with positive index and p_5,p_6 the two consecutive points with negative index. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{5,6}$ we reach a contradiction.

For the configuration $(0,2+;2+,2-;-)$, denote by p_0,p_1,p_2 the points in the 0 level and by p_3,p_4,p_5,p_6 the points in the first level being p_3 and p_4 the two consecutive points with positive index and p_5,p_6 the two consecutive points with negative index. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{3,4}$ we reach a contradiction.

For the configuration $(0, 2+; +, -, +, -; -)$ denote by p_0, p_1, p_2 the points in the 0 level, by p_3, p_4, p_5, p_6 the points in the 1st level in counterclockwise sense and by p_7 the point in the 2nd level. Denote by p_{k_1} the closest point to p_0 in the set $\{p_3, p_5\}$. Denote by p_ℓ the closest point to L_{0,k_1} in the set $\{p_4, p_6, p_7\}$. Denote by p_{k_2} the point in $\{p_3, p_5\}$ different from p_{k_1}. There exists $j_1, j_2 \in \{1, 2\}$ so that L_{j_1,k_2} leaves p_{j_2} on one side of L_{j_1,k_2} and the remaining points on the other side of L_{j_1,k_2}. Applying formula (2) to $L_{0,k_1}L_{0,\ell}L_{j_1,k_2}$ we reach a contradiction.

For the configuration $(2+, -; 0, -, +, -; +)$, we denote by p_1, p_2, p_3 the points in the 0 level in counterclockwise sense and by p_0, p_4, p_5, p_6 the points in the first level also in counterclockwise sense. If $L_{0,4}L_{0,6}$ leave a point with positive index p_ℓ of the 0-level (that without loss of generality we can assume that it is p_1) then applying formula (2) to $L_{0,4}L_{0,6}L_{2,3}$ we reach a contradiction. Otherwise, p_ℓ has negative index (that we assume that it is p_1). Note that either $L_{1,4}$ leaves the points p_5, p_7 on the same side of $L_{1,4}$, or $L_{1,6}$ leaves the points p_5, p_7 on the same side of $L_{1,4}$. Without loss of generality we assume that it is $L_{1,4}$. Then there exists $k_1 \in \{2, 3\}$ so that $L_{0,6}L_{0,k_1}L_{1,4}$ leaves the other point in the set $\{p_1, p_2\}$ on the same side of p_5, p_7. Applying formula (2) to $L_{0,6}L_{0,k_1}L_{1,4}$ we reach a contradiction.

For the configuration $(2+, -; 0, +, -, +; -)$, denote by p_1, p_2, p_3 the points in the 0 level (being p_1, p_2 with positive index and p_3 with negative index) and denote by p_0, p_4, p_5, p_6 the points in the 1st level in counterclockwise sense and by p_7 the point in the 2nd level. There exists $k_0 \in \{1, 2\}$ so that the conic $C = L_{0,5}L_{0,7}$ evaluated on p_{k_0} and on p_4 have the same sing, or the conic C evaluated on p_{k_0} and on p_6 have the same sign (note that C evaluated on p_4 and on p_6 have different sign). Applying (2) to $R = CL_{k_1,k_2}$ with $k_1, k_2 \in \{1, 2, 3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

For the configuration $(3+; 0, -, +, -; -)$, denote by p_1, p_2, p_3 the points in the 0 level, by p_0, p_4, p_5, p_6 the points in the 1st level in counterclockwise sense and by p_7 the point in the 2nd level. There exits $k_1, k_2 \in \{1, 2, 3\}$ so that $L_{k_1,5}$ leaves k_2 on one side of $L_{k_1,5}$ and the rest of the points in the other side of $L_{k_1,5}$. Denote by k_3 the remaining point in the 0 level (different from k_0 and k_1). Denote by p_ℓ the closest point to L_{0,k_3} in the set $\{4, 6, 7\}$. Applying formula (2) to $L_{0,k_3}L_{0,\ell}L_{k_1,5}$ we reach a contradiction.

For the configuration and $(2+, -; +, -, +, -; 0)$ denote by p_1, p_2, p_3 the points in the 0 level (being p_1, p_2 with positive index and p_3 with negative index) by p_4, p_5, p_6, p_7 the points in the 1st level in counterclockwise sense (being p_4 and p_6 with positive index) and by p_0 the point in the 2nd level. There exists $k_0 \in \{1, 2\}$ so that the conic $C = L_{0,5}L_{0,7}$ evaluated on p_{k_0} and on p_4 have the same sign, or the conic C evaluated on p_{k_0} and on p_6

have the same sign (note that the conic C evaluated on p_4 and on p_6 have different signs). Applying (2) to $R = CL_{k_1,k_2}$ with $k_1, k_2 \in \{1,2,3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.0629x - 0.62613y + 0.71196x^2 + 0.66221xy \\ &\quad + 0.73704y^2 - 0.2665x^3 - 1.2599x^2y + y^3, \\ Q(x,y) &= 0.50188x - 1.27453y + 0.92819x^2 + 1.64485xy \\ &\quad - 1.06428y^2 - 1.57961x^3 + 0.31205x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7, 1.3), \quad (1.41.., 0.95..), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.2, -1.2), \\ (0.40.., 0.31..), \quad (-1.63.., 1.29..), \quad (-1.1, 0.65), \quad (0, 0) \end{gathered}$$

in the configuration $(0, +, -; +, -, +, -; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 3851.43 - 7426.34x - 7061.69y + 2914.87x^2 + 12386.9xy \\ &\quad - 813.074y^2 + 589.519x^3 - 4778.06x^2y + y^3, \\ Q(x,y) &= -1388.63 + 2677.53x + 2546.87y - 1050.86x^2 - 4466.74xy \\ &\quad + 291.434y^2 - 212.616x^3 + 1722.41x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.74.., 1.89..), \quad (1.74.., 1.89..), \quad (1.74.., -1.89..), \quad (1.10.., 0), \quad (0.1, 0.5), \\ (0.85.., 0.10..), \quad (0.85.., -0.08..), \quad (-0.19.., 0.53..), \quad (-0.39.., 0.54..) \end{gathered}$$

in the configuration $(0, +, -; 3+, -; -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.14321 - 0.04623x - 0.28656y + 0.69539x^2 + 0.70614xy \\ &\quad + 1.11684y^2 - 0.2855x^3 - 1.50591x^2y + y^3, \\ Q(x,y) &= 0.44569 + 0.84699x - 0.88299y - 0.11115x^2 - 3.36763xy \\ &\quad - 1.88701y^2 + 0.63082x^3 + 1.12932x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.40.., 0.31..), \quad (-0.5, 0.1), \\ (-1.63.., 1.29..), \quad (-1.1, 0.65), \quad (-1.1, 0.65), \quad (-1.08.., 0.65..) \end{gathered}$$

in the configuration $(0, 2+; 3-, +; +)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.06804x - 0.06224y + 0.27383x^2 - 1.1449xy + 0.89605y^2\\&\quad + 0.53725x^3 - 1.23682x^2y + y^3,\\Q(x,y) &= 0.17249x - 0.23800y - 0.2008x^2 + 0.52425xy - 0.31002y^2\\&\quad - 0.91859x^3 + 0.42637x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7, 1.3),\quad (0.2, -1.2),\quad (-0.30.., -0.39..),\quad (0.3, 0.2),\quad (0.3, 0.2),\\(0.40.., 0.31..),\quad (-1.63.., 1.29..),\quad (-1.1, 0.65),\quad (0, 0)\end{gathered}$$

in the configuration $(2+, -; 0, 2-, +; +)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.22436y - 0.02526x^2 - 1.54369xy + 1.08593y^2\\&\quad + 0.75301x^3 - 1.20812x^2y + y^3,\\Q(x,y) &= 0.48854y - 0.959x^2 - 0.4867xy + 0.17134y^2\\&\quad - 0.37163x^3 + 0.49914x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7, 1.3),\quad (1.18.., 0.99..),\quad (0.2, -1.2),\quad (0, 0),\quad (0, 0),\\(0.40.., 0.31..),\quad (0.3, 0.2),\quad (-1.63.., 1.29..),\quad (-1.1, 0.65)\end{gathered}$$

in the configuration $(2+, -; 0, 2+, -; -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -0.39665y + 0.53911x^2 - 0.11032xy + 0.79396y^2 + 0.07436x^3\\&\quad - 1.25172x^2y + y^3,\\Q(x,y) &= -3.10565y + 2.30737x^2 + 7.80916xy - 1.5185y^2 - 4.29942x^3\\&\quad + 0.24677x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7, 1.3),\quad (1.2, 0.64),\quad (0.2, -1.2),\quad (0, 0),\quad (0, 0)\\(0.40.., 0.31..),\quad (0.32.., 0.13..),\quad (-1.63.., 1.29..),\quad (-1.1, 0.65)\end{gathered}$$

in the configuration $(3+; 0, 2-, +; -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.09811x - 0.282y + 0.40366x^2 - 0.85096xy + 0.7576y^2\\&\quad + 0.36605x^3 - 1.16101x^2y + y^3,\\Q(x,y) &= 0.22473x + 0.2107y + 0.25304x^2 - 1.27757xy - 0.29072y^2\\&\quad + 0.13727x^3 - 0.43116x^2y + xy^2,\end{aligned}$$

has the singular points

$$(1.7, 1.3), \quad (1.2, 0.64), \quad (0, 0), \quad (-0.5, 0.1), \quad (-0.5, 0.1),$$
$$(0.2, -1.2), \quad (0.40.., 0.43..), \quad (-1.1, 0.65), \quad (1.07.., 0.58..)$$

in the configuration $(3+; 0, 3-; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.1436x - 0.01373y + 0.21951x^2 - 1.67597xy \\ &\quad + 0.83957y^2 + 0.74789x^3 - 1.18366x^2y + y^3, \\ Q(x,y) &= -0.13225x + 0.19145y + 0.47895x^2 - 0.57423xy \\ &\quad - 0.15865y^2 + 0.06518x^3 - 0.81562x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.3, 0.2), \quad (0.3, 0.2),$$
$$(0.56.., 0.59..), \quad (0.40.., 0.43..), \quad (-1.1, 0.65), \quad (0, 0)$$

in the configuration $(2+, -; 2+, 2-; 0)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -7.77285 - 26.0531x - 3.25264y + 27.9708x^2 + 27.8513xy \\ &\quad + 5.52021y^2 - 17.0304x^3 - 14.6074x^2y + y^3, \\ Q(x,y) &= 3.1688 + 11.088x + 0.68603y - 7.76257x^2 - 12.6084xy \\ &\quad - 2.48277y^2 + 5.82205x^3 + 4.08885x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.5, 2), \quad (1.33.., 1.66..), \quad (-0.94.., 2.55..), \quad (-0.79.., 2), \quad (0, -1),$$
$$(-0.78, 1.99), \quad (-0.78, 1.99), \quad (0.53.., 1.10..), \quad (-0.53.., 1.13..)$$

in the configuration $(3+; 3-, +; 0)$.

Configuration $(3; 3; 2)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -; 2+, -; +, -)$, $(0, +, -; 2-, +; 2+)$ $(0, +, -; 3+; 2-)$, $(0, 2+; 2-, +; +, -)$, $(0, 2+; 3-; 2+)$, $(0, 2+; 2+, -; 2-)$, $(2+, -; 0, +, -; +, -)$, $(2+, -; 0, 2-; 2+)$, $(2+, -; 0, 2+; 2-)$, $(3+; 0, +, -; 2-)$, $(3+; 0, 2-; +, -)$, $(2+, -; 2-, +; 0, +)$, $(3+; 3-; 0, +)$, $(2+, -; 2+, -; 0, -)$, and $(3+; 2-, +; 0, -)$.

We will see that the configurations $(0, +, -; 2-, +; 2+)$, $(0, +, -; 3+; 2-)$, $(0, 2+; 2+, -; 2-)$, $(2+, -; 0, 2+; 2-)$, $(3+; 0, +, -; 2-)$ and $(3+; 2-, +; 0, -)$ are not possible.

For the configuration $(0, +, -; 2-, +; 2+)$ denote by p_1, p_2 the points in the 0-level and by p_3, p_4 the points in the 1st level with negative index. Applying (2) with $L_{0,1}L_{0,2}L_{3,4}$ we have a contradiction.

For the configuration $(0, 2*; 3*; 2-)$ (which in particular contains the configurations $(0, +, -; 3+; 2-)$ and $(0, 2+; 2+, -; 2-)$) we note that there exists at least one point p_ℓ in the 0 level that has positive index. Without loss of generality we call it p_1. We denote by p_0, p_1, p_2 the points in the 0 level, by p_3, p_4, p_5 the points in the first level and by p_6, p_7 the points in the 2nd level. There exists $k_1, k_2 \in \{3, 4, 5\}$ so that $C = L_{0,k_1} L_{0,k_2}$ evaluated on p_6, p_7, p_1 has the same sign. Denote by k_3 the remaining index in $\{3, 4, 5\}$ different from k_1, k_2. Applying (2) to $L_{0,k_1} L_{0,k_2} L_{2,k_3}$ we reach a contradiction.

For the configuration $(2+, -; 0, 2+; 2-)$ denote by p_1, p_2, p_3 the points in the 0-level being p_3 the negative, by p_0, p_4, p_5 the points in the 1st-level and by p_6, p_7 the points in the 2nd level. The two straight lines $C = L_{0,4} L_{0,5}$ intersect the sides of the triangle with vertices p_1, p_2, p_3 in four points. So one side of this triangle has two of these four points. If p_3 is in the same component of $\mathbb{R}^2 \setminus C$ then applying (2) with $CL_{1,2}$ we get a contradiction. If p_3 is not in the same component than $\mathbb{R}^2 \setminus C$ there exists $\ell \in \{1, 2\}$ such that p_ℓ is in the same component of $\mathbb{R}^2 \setminus C$ than p_6 and p_7. Then applying (2) to $CL_{3,\ell}$ we have a contradiction.

For the configuration $(3+; 0, +, -; 2-)$ we denote by p_1, p_2, p_3 the points in the 0 level, by p_4, p_5 the points in the 1st level (being p_4 the point with positive index) and by p_6, p_7 the points in the 2nd level. We can rename p_1, p_2, p_3 so that $R = L_{0,1} L_{0,2} L_{3,4}$ evaluated on p_5, p_6 and p_7 have the same sign. Applying formula (2) to R we reach a contradiction.

Finally, for the configuration $(3+; 2-, +; 0, -)$ denote by p_1, p_2, p_3 the points in the 0-level, by p_4, p_5, p_6 the points in the first level (being p_6 the point with positive index) and by p_0, p_7 the points in the 2nd level. There exists $\ell_0, \ell_1, \ell_2 \in \{4, 5, 7\}$ so that $C = L_{0,\ell_0} L_{0,\ell_1}$ evaluated on p_6 and on p_{ℓ_2} have the same sign. Moreover, there exists $k_0 \in \{1, 2, 3\}$ so that C evaluated on p_6 and on p_{k_0} has different sign. Applying the Euler-Jacobi formula (2) to $R = CL_{k_1,k_2}$ with $k_1 \neq k_2$, $k_1, k_2 \in \{1, 2, 3\}$ being $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

The cubic system (1) with

$$
\begin{aligned}
P(x, y) = {}& -0.63917 - 0.03915x + 1.54354y + 0.0948x^2 + -0.09761xy \\
& - 0.9535y^2 + 0.01618x^3 - 0.20338x^2 y + y^3, \\
Q(x, y) = {}& -1.41632 - 0.11903x + 10.7244y + 0.19431x^2 - 5.71385xy \\
& - 2.63511y^2 + 0.03392x^3 + 0.57344x^2 y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{gathered}
(269.39.., -130.87..), \quad (-4.10.., 0), \quad (-4.10.., 0), \quad (2.49.., -0.27..), \\
(2.48.., 0.55..), \quad (2.55.., 0.63..), \quad (2.60.., 0.66..), \quad (3, 1), \quad (2.34.., 0.65..)
\end{gathered}
$$

in the configuration $(0, +, -; 2+, -; +, -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 1.74803 - 0.42482x - 1.08034y + 2.55396x^2 + 0.46753xy \\ &\quad - 1.82837y^2 - 0.29212x^3 - 1.28883x^2y + y^3, \\ Q(x,y) &= 0.13374 + 0.26732x + 0.08368y + 2.04637x^2 - 2.03967xy \\ &\quad - 0.05005y^2 - 0.10762x^3 - 1.02755x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.63.., 2.14..), \quad (1.5, 2), \quad (1.33.., 1.66..), \quad (1.31.., 1.67..), \quad (0, -1), \\ &(0, 79.., 1.3), \quad (-0.79.., 2), \quad (-0.79.., 2), \quad (0.53.., 1.10..)\end{aligned}$$

in the configuration $(0, 2+; 2-, +; +, -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -\,50.1135x - y + 11.1057x^2 - 25.6451xy + 0.0683157x^3 \\ &\quad + 2.20461x^2y - 3.23766xy^2 + 0.101397y^3, \\ Q(x,y) &= 17.5763x^2 + 0.082767xy + y^2 - 0.029669x^3 + 3.41881x^2y \\ &\quad + 0.0242532xy^2 + 0.318429y^3,\end{aligned}$$

has the singular points

$$\begin{aligned}&(0, 0), \quad (0, 0), \quad (-5, -5), \quad (7, -5), \quad (-2, -4.5), \\ &(2, -4.5), \quad (0, -3.14..), \quad (-1/2, -3.5), \quad (1/2, -3.5)\end{aligned}$$

in the configuration $(0, 2+; 3-; 2+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 4.51755 + 4.06796x - 1.97571y + 2.2832x^2 - 1.57532xy \\ &\quad - 5.49326y^2 - 7.53714x^3 + 7.1848x^2y + y^3, \\ Q(x,y) &= 1.15404 + 1.92249x - 0.24618y + 1.94662x^2 - 2.79227xy \\ &\quad - 1.40022y^2 - 2.77672x^3 + 2.09418x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.58.., 2.03..), \quad (1.5, 2), \quad (1.5, 2), \quad (1.33.., 1.66..), \quad (0, -1) \\ &(1.31.., 1.67..), \quad (0.79.., 1.3), \quad (-0.79.., 2), \quad (0.53.., 1.10..)\end{aligned}$$

in the configuration $(2+, -; 0, +, -; +, -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 1.67312x - y - 0.826899x^2 + 1.90157xy - 0.987919x^3 \\ &\quad - 1.24626x^2y + 1.42219xy^2 + y^3, \\ Q(x,y) &= -1.6012x^2 - 0.222771xy + y^2 + 0.820301x^3 + 0.865023x^2y \\ &\quad - 0.850119xy^2 - y^3,\end{aligned}$$

has the singular points

$$(0,0),\quad (0,0),\quad (-5,-5),\quad (7,-5),\quad (0,1),$$
$$(-2,-2,1),\quad (2,-2.1),\quad (1.89..,-2),\quad (-1.89..,-2)$$

in the configuration $(2+,-;0,2-;2+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -4.34579 + 10.04x + 39.1074y - 22.3386x^2 - 61.6263xy \\ &\quad + 25.3103y^2 + 25.8492x^3 - 9.25676x^2y + y^3, \\ Q(x,y) &= -5.04257 + 11.6444x + 45.5646y - 25.7126x^2 - 72.0976xy \\ &\quad + 27.8699y^2 + 29.631x^3 - 9.55345x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.7,1.3),\quad (1.2,0.64),\quad (0.2,-1.2),\quad (0.1,0.1),\quad (0.1,0.1),$$
$$(0.79..,0.23..),\quad (0.6,0.2),\quad (0.7,0.1),\quad (-1.63..,1.29..)$$

in the configuration $(3+;0,2+;+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.04785 - 3.26417x - 2.41163y + 3.37543x^2 + 6.19441xy \\ &\quad + 0.14689y^2 - 1.158x^3 - 3.93093x^2y + y^3, \\ Q(x,y) &= -0.47186 + 1.4699x + 1.8721y - 1.52x^2 - 3.50447xy \\ &\quad - 1.81296y^2 + 0.52146x^3 + 1.39268x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.74..,1.89..),\quad (1.74..,-1.89..),\quad (1.10..,0),\quad (0.90..,0),\quad (0.90..,0),$$
$$(0.1,0.5),\quad (0.850..,-0.1008..),\quad (0.856..,-0.109..),\quad (-0.19..,0.53..)$$

in the configuration $(2+,-;2-,+;0,+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.06826x - 0.04133y + 0.28897x^2 - 1.25411xy \\ &\quad + 0.89334y^2 + 0.59877x^3 - 1.2708x^2y + y^3, \\ Q(x,y) &= 0.17828x + 0.30521y + 0.19261x^2 - 2.31316xy \\ &\quad - 0.38029y^2 + 0.67984x^3 - 0.45642x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.7,1.3),\quad (1.2,0.64),\quad (0.2,-1.2),\quad (0.3,0.2),\quad (0.3,0.2),$$
$$(-0.5,0.1),\quad (-1.1,0.65),\quad (-0.13..,0.03..),\quad (0,0)$$

in the configuration $(3+;3-;0,+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) =& 0.05945x - 0.05036y + 0.28675x^2 - 1.16699xy \\ & + 0.90087y^2 + 0.58306x^3 - 1.29606x^2y + y^3, \\ Q(x,y) =& 0.05071x + 0.07078y + 0.13478x^2 - 0.04988xy \\ & - 0.18464y^2 + 0.27164x^3 - 1.11276x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.3, 0.2), \quad (0.3, 0.2), \\ &(0.62.., 0.50..), \quad (0.40.., 0.31..), \quad (-0.5, 0.1), \quad (0, 0) \end{aligned}$$

in the configuration $(2+, -; 2+, -; 0, -)$.

Acknowledgements

The first author is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

References

[1] V. Arnold, A. Varchenko and S. Goussein-Zude, *Singularités des applications différentialbes*, Mir, Moscow, 1982.
[2] A. N. Berlinskii, *On the behavior of the integral curves of a differential equation*, Izv. Vyssh. Uchebn. Zaved. Mat. **2** (1960), 3–18.
[3] R.F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
[4] C. Chicone and T. Jinghuang, *On general properties of quadratic systems*, Amer. Math. Monthly **81** (1982), 167–178.
[5] A. Cima, A. Gasull and F. Manõsas, *Some applications of the Euler-Jacobi formula to differential equations*, Proc. Amer. Math. **118** (1993), 151–163.
[6] A. Cima and J. Llibre, *Configurations of fans and nests of limit cycles for polynomial vector fields in the plane*, J. Differential Equations **82** (1989), 71–97.
[7] A. Gasull and J. Torregrosa, *Euler-Jacobi formula for double points and applications to quadratic and cubic systems*, Bull. Belg. Math. Soc. **6** (1999), 337–346.
[8] A. G. Khovanskii, *Index of a polynomial vector field*, Funktsional Anal. i Prilozhen **13** (1979), 49–58.
[9] J. Llibre and C. Valls, *On the singularities of the planar cubic polynomial differential systems and the Euler Jacobi formula*, to appear in Qualitative Theory of Dynamical Systems, (2020).
[10] N.G. Lloyd, *Degree Theory*, Cambridge Tracks in Mathematics, **73**, Cambridge University Pres, New York, 1978.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.ist.utl.pt