# GLOBAL DYNAMICS FOR THE SZEKERES SYSTEM WITH NON-ZERO COSMOLOGICAL CONSTANT

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. The Szekeres system with cosmological constant term describes the evolution of the kinematic quantities for the Einstein field equations in dimension four. It is a Hamiltonian system (with Hamiltonian $H$). We restrict the dynamics on each one of the level surfaces $H = h$ with $h \in \mathbb{R}$ and using the Poincaré compactification on $\mathbb{R}^3$ we analyze the global dynamics of the Szekeres system.

It was known that the Szekeres system with cosmological constant term exhibits an attractor in the finite regime. Here we provide a new proof of the finite attractor and additionally we prove that also exhibits a repulsor in the finite regime, and that at infinity there is an atractor and a repulsor.

## 1. Introduction and statement of the main results

A Szekeres system represents the diagonal of the Einstein field equations $G - \Lambda g = T$ for a gravitational model where the energy-momentum tensor $T$ is that of a pressureless inhomogeneous fluid, $\Lambda > 0$ is the cosmological constant and $G = R - \frac{1}{2}Rg$ is the Einstein tensor for the background space, see for details [6]. The Szekeres system also can be obtained from the model of the silent Universe system, see for details [1]. Because of its importance in physical applications it has been widely investigated in the literature using Darboux functions, Jacobi's multiplier method, Painlevé method, Lie symmetries, ....

It was proved in [12] (see also [4, 9]) that with appropriate variables, the Szekeres system admits the Hamiltonian formalism with Hamiltonian

$$H = p_x p_y - \frac{\Lambda}{3}xy + \frac{x}{y^2},$$

2010 *Mathematics Subject Classification.* 34C29, 34C25,47H11.

*Key words and phrases.* Szekeres system, positive cosmological constant, global dynamics, Poincaré sphere.

and then its equations of motion are

$$\begin{aligned}\dot{x} &= p_y,\\ \dot{y} &= p_x,\\ \dot{p_x} &= \frac{\Lambda}{3}x + \frac{2x}{y^3},\\ \dot{p_y} &= \frac{\Lambda}{3}y - \frac{1}{y^2},\end{aligned}$$

where the dot denotes derivative with respect to the time $t$.

We introduce a rescaling of the time by $dt = y^3\, ds$ and with this new time the differential system becomes

$$\begin{aligned}\dot{x} &= p_y y^3,\\ \dot{y} &= p_x y^3,\\ \dot{p_x} &= \frac{\Lambda}{3}xy^3 + 2x,\\ \dot{p_y} &= \frac{\Lambda}{3}y^4 - y,\end{aligned}$$

now the dot denotes derivative with respect the new independent variable $s$.

We shall restrict the study of the dynamics of this system to each energy level $H = h$, $h \in \mathbb{R}$. In particular setting $H = h$ and solving this equation with respect to the variable $p_x$ we get

$$p_x = \frac{-3x + 3hy^2 + xy^3\Lambda}{3p_y y^2},$$

and so the restricted system on $H = h$ becomes, after a new rescaling of the time variable by $d\tau = 3p_y\, ds$ becomes

$$\begin{aligned}\dot{x} &= 3p_y^2 y^3,\\ \dot{y} &= y(3hy^2 + x(\Lambda y^3 - 3),\\ \dot{p_y} &= xp_y(\Lambda y^3 + 6),\end{aligned} \tag{1}$$

here the dot denotes derivative with respect to the variable $\tau$.

Note that the polynomial differential system (1) depends on two parameters: $\Lambda > 0$ and $h \in \mathbb{R}$. We will study its global dynamics in the compactification of $\mathbb{R}^3$ in function of the parameters $h$ and $\Lambda$. We note that the global description of the flow of a differential system in $\mathbb{R}^3$ is generally very difficult. In this paper using the Poincaré compactification we are able to do it. We recall that roughly speaking the Poincaré ball is obtained identifying $\mathbb{R}^3$ with the interior of the 3-dimensional ball of radius one centered at the origin, and extending analytically the flow of system (1) to the boundary $\mathbb{S}^2$ of that ball, and consequently to the infinity of $\mathbb{R}^3$. For more details see subsection 2.2.

In [7, 8] the authors made a detailed analysis on the dynamics in the Poincaré disc of the Szekeres system when $\Lambda = 0$ and it was found that the orbits come from the infinity of $\mathbb{R}^4$ and go to infinity. In this paper we extend this analysis by considering that $\Lambda > 0$. Other studies related with this model can be also obtained in [10].

The main result of this paper about the finite dynamics in the Poincaré ball is the following one.

**Theorem 1.** *The following statement hold for system* (1)*.*

(a) *It has the rational first integral*

$$I = \frac{9x^2 - 18hxy^2 - 18v^2y^3 + 9h^2y^4 - 6x^2y^3\Lambda + 6hxy^5\Lambda - 3v^2y^6\Lambda + x^2y^6\Lambda^2}{9p_y^2y^4}.$$

(b) *The planes* $y = 0$, $p_y = 0$ *and the surface* $9x^2 - 18hxy^2 - 18p_y^2y^3 + 9h^2y^4 - 6x^2y^3\Lambda + 6hxy^5\Lambda - 3p_y^2y^6\Lambda + x^2y^6\Lambda^2 = 0$ *are invariant by the flow of system* (1)*.*

(c) *The* $x$*-axis, the* $p_y$*-axis and the curve*

$$C_h = \left\{(x, y, p_y) \in \mathbb{R}^3 : p_y = 0, x = -\frac{3hy^2}{y^3\lambda - 3}\right\} \text{ with } y^3\lambda - 3 \neq 0,$$

*are filled up with equilibrium points.*

(d) *The* $x$*-axis for* $x \neq 0$ *has a* 2*-dimensional stable and a* 2*-dimensional unstable manifolds. While the curve* $C_h$ *when* $hy(y^3\Lambda - 3) \neq 0$ *has a* 3*-dimensional stable manifold in the arc of the curve whose points satisfy* $\dfrac{3hy^2(6 + y^3\Lambda)}{y,^3\Lambda - 3} > 0$ *(i.e. this arc is an attractor), and a* 3*-dimensional unstable manifold in the arc of the curve whose points satisfy* $\dfrac{3hy^2(6 + y^3\Lambda)}{y^3\Lambda - 3} < 0$ *(i.e. this arc is a repulsor).*

Theorem 1 is proved in section 3.

It follows from Theorem 1 that there is an attractor and a repulsor of the flow in the finite dynamics.

The main result of the paper on the infinite dynamics in the Poincaré ball (that is in the Poincaré sphere) is the following one.

**Theorem 2.** *The phase portraits of system* (1) *in the local chart* $U_1$ *of the infinity* $\mathbb{S}^2$ *is topologically equivalent to the one described in Figure 1. More precisely, on the sphere* $\mathbb{S}^2$ *of the infinity there are three maximal circles filled up with equilibria. The equations of these circles in the local chart* $U_1$ *are* $z_1 = z_3 = 0$*, and* $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$*. Moreover the following statements hold.*

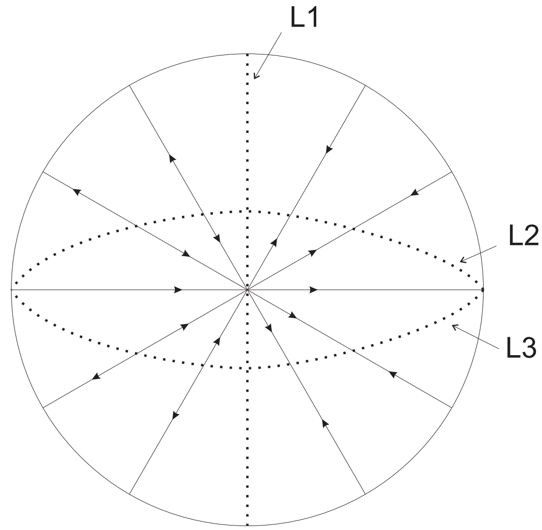


FIGURE 1. The phase portrait in the local chart $U_1$ of the infinity. The straight line $L1$ is $z_1 = z_3 = 0$. The straight line $L2$ is $z_2 = \sqrt{\Lambda/3}, z_3 = 0$. The straight line $L3$ is $z_2 = -\sqrt{\Lambda/3}, z_3 = 0$.

(a) *In the sphere $\mathbb{S}^2$ of the infinity at each equilibrium of the circles $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$ arrive two orbits one in each side of the circle if $z_1 > 0$, or exit two orbits one in each side of the circle if $z_1 < 0$.*

(b) *In the Poincaré ball $B$ at each infinite equilibrium of the circles $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$ arrive a 2-dimensional stable manifold if $z_1 > 0$, or exit a 2-dimensional unstable manifold if $z_1 < 0$.*

(c) *In the Poincaré ball $B$ the circles $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$ have a 3-dimensional stable manifold if $z_1 < 0$ (i.e. these circles are attractors), or a 3-dimensional unstable manifold $z_1 < 0$ (i.e. these circles are repulsors).*

Theorem 2 is proved in section 4.

We note that from statement (b) of Theorem 2 at infinity there is an attractor and a repulsor.

## 2. Preliminaries

2.1. **Normally hyperbolic theory.** Let $\phi$ be a smooth flow on a manifold $M$ and assume that $C$ is a submanifold of $M$ consisting entirely of equilibrium points for the flow $\phi_t$. $C$ is said to be *normally hyperbolic* if the tangent bundle of $M$ over $C$ splits into three subbundles $TC$ (the tangent bundle of $C$), $E^s$ and $E^u$ that are invariant under $d\phi_t$, and such that $d\phi_t$ contracts $E^s$ exponentially and $d\phi_t$ expands $E^u$ exponentially.

From the normally hyperbolic theory one has the usual existence of smooth stable and unstable manifolds to normally hyperbolic submanifolds of equilibrium points.

**Theorem 3.** *Let $C$ be a normally hyperbolic submanifold of equilibrium points for $\phi_t$. Then there exist smooth stable and unstable manifolds tangent along $C$ to $E^s \oplus TC$ and $E^u \oplus TC$, respectively.*

The proof of this theorem can be found in [5].

2.2. **Poincaré compactification in $\mathbb{R}^3$.** Poincaré in his Ph.D. introduced what we call now the Poincaré compactification of the polynomial vector fields in the plane $\mathbb{R}^2$, see [11] and Chapter 5 of [3]. This compactification was extended to polynomial vector fields in $\mathbb{R}^n$, see [2].

Consider in $\mathbb{R}^3$ a polynomial vector field $X = (P_1, P_2, P_3)$ of degree $n$, i.e. $P_i = P_i(x, y, z)$ are polynomials and $n = \max\{deg(P_i)\} : i = 1, 2, 3\}$. Let

$$\begin{aligned} &\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : ||y|| = 1\},\\ &\mathbb{S}_+ = \{y \in \mathbb{S}^3 : y_4 > 0\} \;\; \text{and,}\\ &\mathbb{S}_- = \{y \in \mathbb{S}^3 : y_4 < 0\}. \end{aligned}$$

be the unit sphere in $\mathbb{R}^4$, the northern hemisphere of $\mathbb{S}^3$ and the southern hemisphere of $\mathbb{S}^3$, respectively. The tangent space of $\mathbb{S}^3$ at the point $y$ is denoted by $T_y\mathbb{S}^3$, and the tangent hyperplane $T_{(0,0,0,1)}\mathbb{S}^3 = \{(y_1, y_2, y_3, 1) \in \mathbb{R}^4\}$ is identified with $\mathbb{R}^3$.

Consider the central projections

$$f_+ : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \to \mathbb{S}_+, \quad f_- : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \to \mathbb{S}_-$$

where

$$f_\pm(x) = \pm\frac{(x_1, x_2, x_3, 1)}{\Delta(x)} \quad \text{with} \quad \Delta(x) = \left(1 + \sum_{i=1}^{3} x_i^2\right)^{1/2}.$$

Using these central projections $\mathbb{R}^3$ is identified with $\mathbb{S}_+$ and $\mathbb{S}_-$. Note that the equator of $\mathbb{S}^3$ is the 2-dimensional sphere $\mathbb{S}^2 = \{y \in \mathbb{S}^3 : y_4 = 0\}$.

The maps $f_\pm$ define two copies of the vector field $X$ on $\mathbb{S}^3$, one $Df_+ \circ X$ in $\mathbb{S}_+$, and the other $Df_- \circ X$ in $\mathbb{S}_-$. Denote by $\overline{X}$ the vector field on $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$, that restricted to $\mathbb{S}_+$ coincides with $Df_+ \circ X$, and restricted to $\mathbb{S}_-$ coincides with $Df_- \circ X$. We can extend analytically the vector field $\overline{X}(y)$ to the whole sphere $\mathbb{S}^3$ setting $p(X) = y_4^{n-1}\overline{X}(y)$.

Using that $\mathbb{S}^3$ is a differentiable manifold, to compute the expression of the vector field $p(X)$, we consider the eight local charts $(U_i, F_i)$, $(V_i, G_i)$, where

$$U_i = \{y \in \mathbb{S}^3 : y_i > 0\} \quad \text{and} \quad V_i = \{y \in \mathbb{S}^3 : y_i < 0\}, \quad \text{for } i = 1, 2, 3, 4.$$

Note that the diffeomorphisms $F_i\colon U_i \to \mathbb{R}^3$ and $G_i\colon V_i \to \mathbb{R}^3$ for $i = 1,2,3,4$ are the inverse of the central projections from the origin to the tangent hyperplane at the points $(\pm 1,0,0,0)$, $(0,\pm 1,0,0)$, $(0,0,\pm 1,0)$ and $(0,0,0,\pm 1)$, respectively.

The analytical vector field $p(X)$ in the local charts $U_1$, $U_2$ and $U_3$ become, after a rescaling of the time variable,

$$z_3^n\big(-z_1P_1+P_2,-z_2P_1+P_3,-z_3P_1\big), \quad \text{where } P_i = P_i(1/z_3, z_1/z_3, z_2/z_3),$$
$$z_3^n\big(-z_1P_2+P_1,-z_2P_2+P_3,-z_3P_2\big), \quad \text{where } P_i = P_i(z_1/z_3, 1/z_3, z_2/z_3),$$
$$z_3^n\big(-z_1P_3+P_1,-z_2P_3+P_2,-z_3P_3\big), \quad \text{where } P_i = P_i(z_1/z_3, z_2/z_3, 1/z_3),$$

respectively. The expression for $p(X)$ in $U_4$ is $z_3^{n+1}(P_1,P_2,P_3)$ and the expression for $p(X)$ in $V_i$ is the same as in $U_i$ multiplied by $(-1)^{n-1}$, for all $i = 1,2,3,4$.

In what follows we will consider only the orthogonal projection of $p(X)$ from $\mathbb{S}_+$ to $y_4 = 0$ and we will denote it again by $p(X)$. Observe that the projection of the closed $\mathbb{S}_+$ is a 3-dimensional closed ball of radius one, denoted by $B$, whose interior is diffeomorphic to $\mathbb{R}^3$. Its boundary, $\mathbb{S}^2$, corresponds to the infinity of $\mathbb{R}^3$. Moreover $p(X)$ is defined in the whole closed ball $B$ in such way that the flow on the boundary, given by $z_3 = 0$ in any local chart, is invariant. The vector field induced by $p(X)$ on $B$ is called the *Poincaré compactification* of $X$, $B$ is called the *Poincaré ball*, and $\mathbb{S}^2$ is called the *Poincaré sphere*.

## 3. Proof of Theorem 1

Since

$$\frac{dI}{d\tau} = \frac{\partial I}{\partial x}x' + \frac{\partial I}{\partial y}y' + \frac{\partial I}{\partial y_p}y_p' = 0,$$

we obtain that $I$ is a first integral of system (1), i.e. $I$ is constant on the orbits of system (1). So statement (a) is proved.

Since $I$ is a rational first integral it is clear that the planes $y = 0$, $p_y = 0$ and the surface $9x^2-18hxy^2-18p_y^2y^3+9h^2y^4-6x^2y^3\Lambda+6hxy^5\Lambda-3p_y^2y^6\Lambda+x^2y^6\Lambda^2 = 0$ are invariant by the flow of system (1). Hence statement (b) is proved.

Computing the finite singular points of system (1) we obtain that the two straight lines $x = y = 0$ (the $p_y$-axis) and $y = p_y = 0$ (the $x$-axis), together with the curve $C_h$ (which only exists if $y^3\lambda - 3 \neq 0$) are filled with singular points. Therefore statement (c) is proved.

The eigenvalues of the Jacobian matrix at the equilibrium points $(x,0,0)$ of the $x$-axis are $6x, -3x, 0$. Therefore the $x$-axis for $x \neq 0$ is normally hyperbolic. So from Theorem 3 the $x$-axis for $x \neq 0$ has a 2-dimensional stable and a 2-dimensional unstable manifolds. The stable manifold is formed by orbits which tend asymptotically to the line and the unstable manifold if formed by orbits that tend asymptotically toward the line in the negative time direction.

The eigenvalues of the Jacobian matrix at the equilibrium points of the curve $C_h$ are

$$-\frac{3hy^2(6+y^3\Lambda)}{y^3\Lambda-3}, \quad -\frac{3hy^2(6+y^3\Lambda)}{y^3\Lambda-3}, \quad 0.$$

Therefore this curve of equilibria is normally hyperbolic when $hy(y^3\Lambda-3) \neq 0$. Applying Theorem 3 we get that it has a 3-dimensional stable manifold in the line of equilibrium points $(x, y, p_y) = (-(3hy^2)/(y^3\lambda - 3), y, 0)$ satisfying $\dfrac{3hy_0^2(6+y_0^3\Lambda)}{y_0^3\Lambda-3} > 0$, and a 3-dimensional unstable manifold in the line of equilibrium points $(x, y, p_y) = (-(3hy^2)/(y^3\lambda - 3), y, 0)$ satisfying $\dfrac{3hy_0^2(6+y_0^3\Lambda)}{y_0^3\Lambda-3} < 0$. This completes the proof of statement (d).

## 4. Proof of Theorem 2

In this section we study the dynamics of system (1) at infinity and near infinity using the Poincaré compactification of the system in $\mathbb{R}^3$ described in the subsection 2.2.

The expression of the Poincaré compactification $p(X)$ of system (1) in the local chart $U_1$ is

$$\begin{aligned}\dot{z}_1 &= -z_1(3z_1^3z_2^2 - 3hz_1^2z_3^2 + 3z_3^3 - z_1^3\Lambda),\\ \dot{z}_2 &= z_2(-3z_1^3z_2^2 + 6z_3^3 + z_1^3\Lambda),\\ \dot{z}_3 &= -3z_1^3z_2^2z_3.\end{aligned} \tag{2}$$

When $z_3 = 0$ (which correspond to the points on the sphere $\mathbb{S}^2$ of the infinity) system (2) becomes

$$\begin{aligned}\dot{z}_1 &= -z_1^4(3z_2^2 - \Lambda),\\ \dot{z}_2 &= z_1^3z_2(-3z_2^2 + \Lambda).\end{aligned} \tag{3}$$

We have three circles on the sphere of the infinity filled up with equilibria which are $z_1 = 0$, $z_2 = \sqrt{\Lambda/3}$ and $z_2 = -\sqrt{\Lambda/3}$.

Doing a rescaling of the independent variable system (3) reduces to system

$$\dot{z}_1 = -z_1, \qquad \dot{z}_2 = -z_2.$$

So the origin of this differential system is a star node, and all orbits different from the origin end at the origin following the straight lines through the origin. Going back to the differential system (3) we obtain that the phase portrait on the local chart $U_1$ of the sphere $\mathbb{S}^2$ is the one of Figure 1.

The eigenvalues of the Jacobian matrix at the points of the circle $z_1 = z_3 = 0$ for system (2) and (3) are all zero and so they do not provide any information about the dynamics of the system near them.

The eigenvalues of the Jacobian matrix of system (2) at the points of the circles $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$ are $0, -2\Lambda z_1^3$ and $-\Lambda z_1^3$. By the normally hyperbolicity theorem on the sphere $\mathbb{S}^2$ of the infinity at each equilibria of the circles $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$ arrive two orbits one in each side of the circle if $z_1 > 0$, or exit two orbits one in each side of the circle if $z_1 < 0$. Hence statement (a) is proved.

Again by the normally hyperbolicity theorem at each equilibria of the circles $z_2 = \pm\sqrt{\Lambda/3}, z_3 = 0$ arrive a 2-dimensional stable manifold inside the Poincaré ball if $z_1 > 0$, or exit a 2-dimensional unstable manifold if $z_1 < 0$. This proves statement (b) and (c).

## 5. Conclusions

It was known that the Szekeres system with cosmological constant term exhibits an attractor in the finite regime, see [10]. Here we provide a new proof of the finite attractor (see statement (d) of Theorem 1) and additionally we prove that also exhibits a repulsor in the finite regime, and that at infinity there is an atractor and a repulsor (see statement (c) of Theorem 2).

Moreover we describe completely the flow on the infinite sphere and consequently the dynamics near the infinity, see Theorem 2.

## Acknowledgements

The first author is supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2019.

## References

[1] M. Bruni, S. Matarrese and O. Pantano, *Dynamics of silent universe*, Astrophys. J. **445** (1995), 958–977.

[2] A. Cima and J. Llibre, *Bounded polynomial vector fields*, Trans. Amer. Math. Soc. **318** (1990), no. 2, 557–579.

[3] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer Verlag, New York, 2006.

[4] A. Gierzkiewicz and Z.A. Golda, *On integrability of the Szekeres system. I*, J. Nonlinear Math. Phys. **23** (2016), no. 4, 494–506.

[5] W.M. Hirsch, C.C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics **583**, Springer, 1977.

[6] A. Krasisnski, *Inhomogeneous Cosmological Models*, Cambridge U.P., Cambridge, (1997).

[7] J. Llibre and C. Valls, *On the dynamics of the Szekeres system*, Phys. Lett. A **383** (2019), no. 4, 301–305.

[8] J. Llibre and C. Valls, Dynamics of the Szekeres system, J. Math. Phys **62** (2021), no. 8, Paper No. 082502, 7 pp.

[9] A. Paliathanasis and P.G.L. Leach, *Symmetries and singularities of the Szekeres system*, Phys. Lett. A **381**, (2017), no. 15, 1277–1280.

[10] A. Paliathanasis and G. Leon, *Global dynamics and evolution for the Szekeres system with nonzero cosmological constant term*, preprint. arXiv:2109.069041.

[11] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, Journal de Mathématiques **37** (1881), 375–422; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris, 1951, pp 3–84.

[12] A. Zampeli and A. Paliathanasis, *Quantization of inhomogeneous spacetimes with cosmological constant term*, Class. Quantum Grav. **38** (2021), no. 16, Paper No. 165012, 16 pp.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

*Email address*: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

*Email address*: cvalls@math.ist.utl.pt