LIMIT CYCLES OF PIECEWISE DIFFERENTIAL SYSTEMS WITH LINEAR HAMILTONIAN SADDLES AND LINEAR CENTERS

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Abstract. We study the continuous and discontinuous planar piecewise differential systems formed by linear centers together with linear Hamiltonian saddles separated by one or two parallel straight lines. When these piecewise differential systems are either continuous or discontinuous separated by one straight-line, they have no limit cycles. When these piecewise differential systems are continuous and are separated by two parallel straight lines they do not have limit cycles. On the other hand, when these piecewise differential systems are discontinuous and separated by two parallel straight lines (either two centers and one saddle, or two saddles and one center), we show that they can have at most one limit cycle, and that there exist such systems with one limit cycle. If the piecewise differential systems separated by two parallel straight lines have three linear centers, or three linear Hamiltonian saddles it is known that they have at most one limit cycle.

1. Introduction and statement of the main results

A limit cycle is a periodic orbit of a differential system isolated in the set of all periodic orbits of that system. The study of the limit cycles goes back essentially to Poincaré [24] at the end of the nineteenth century.

The existence of limit cycles became important in the applications to the real world, because many phenomena are related with their existence, see for instance the Van der Pol oscillator [27, 28], or the Belousov–Zhabotinskii reaction which is a classical reaction of non-equilibrium thermodynamics appearing in a non- linear chemical oscillator [3, 29]. The study of the continuous piecewise linear differential systems separated by one or two parallel straight lines appears in a natural way in the control theory, see for instance the books [2, 10, 12, 13, 18, 23]. The easiest continuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that such systems have at most one limit cycle, see [8, 15, 20, 21].

The study of the discontinuous piecewise linear differential systems separated by straight lines goes back to Andronov et al. [1] and until nowadays they have special attention from the mathematicians, mainly because these systems appear in mechanics, electrical circuits, economy, etc, see for instance the books [7, 25] and the surveys [22, 26].

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In the planar discontinuous piecewise differential systems here considered, the limit cycles can be of three kinds: those limit cycles placed at just one zone and the limit cycles placed in two or three zones, such limit cycles can be either sliding limit cycles or crossing limit cycles. We recall that the sliding limit cycles contain some segment of the lines of discontinuity, and the crossing limit cycles only contain isolated points of the lines of discontinuity. We will not treat the ones placed at just one zone because in our case since the systems in one zone are linear differential systems it is well known that such systems have no limit cycles. In this paper we only studied the crossing limit cycles, here also denoted simply limit cycles.

Again the easiest discontinuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that such systems can have three limit cycles, see [4, 5, 6, 9, 11, 14, 16]. It remains open to know if three is the maximum number of limit cycles that such systems can exhibit.

We now state the main results of the paper.

Theorem 1. *A continuous or discontinuous piecewise linear differential system separated by one straight line formed by one center and one Hamiltonian saddle has no limit cycles.*

The proof of Theorem 1 is given in section 3. The case where in the two regions there is a center, or in the two regions there is a saddle was studied in [17] and [19], respectively. In these papers the authors show that there are also no limit cycles in these cases.

Theorem 1 in the case of continuous or discontinuous piecewise differential systems can be extended to continuous or discontinuous piecewise linear differential systems separated by two parallel straight lines formed by either one center and two linear Hamiltonian saddles or one linear Hamiltonian saddle and two centers. The case in which in the three regions there is a center or in the three regions there is a saddle was studied in [17] and [19], respectively. In these papers the authors show that these systems have at most one limit cycle.

Theorem 2. *The following statements hold.*

- *(a) A continuous piecewise linear differential system separated by two parallel straight lines formed by two centers and one Hamiltonian saddle has no limit cycles.*
- *(b) A discontinuous piecewise linear differential system separated by two parallel straight lines formed by two centers and one Hamiltonian saddle can have at most one limit cycle. Moreover there are systems in this class having one limit cycle, see Figures 1 and 2.*

Theorem 2 is proved in section 4.

Theorem 3. *The following statements hold.*

- *(a) A continuous piecewise linear differential system separated by two parallel straight lines formed by one center and two Hamiltonian saddles has no limit cycles.*

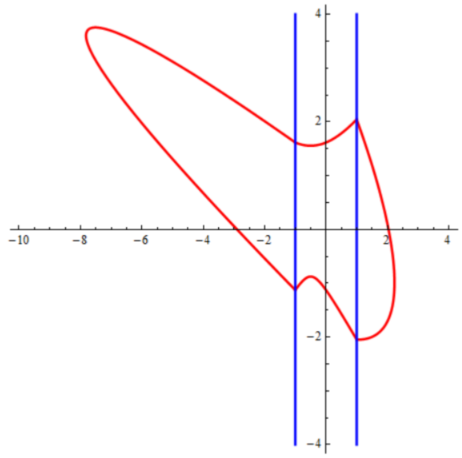

FIGURE 1. The limit cycle of the discontinuous piecewise differential system formed by the two linear centers (8) and (10), and the linear Hamiltonian saddle (9). This limit cycles is travelled in counterclockwise sense.

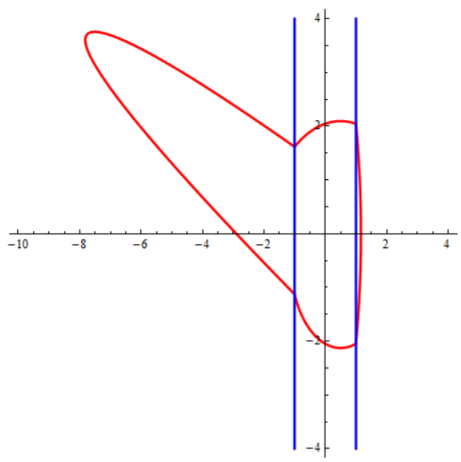

FIGURE 2. The limit cycle of the discontinuous piecewise differential system formed by the two linear centers (14) and (15), and the linear Hamiltonian saddle (16). This limit cycles is travelled in counterclockwise sense.

(b) *A discontinuous piecewise linear differential system separated by two parallel straight lines formed by one center and two Hamiltonian saddles can have at most one limit cycle. Moreover there are systems in this class having one limit cycles, see Figures 3 and 4.*

Theorem 3 is proved in section 5.

The paper is organized as follows: We first present a normal form of a linear differential system having a center (proved in [17]), and second we also present a normal

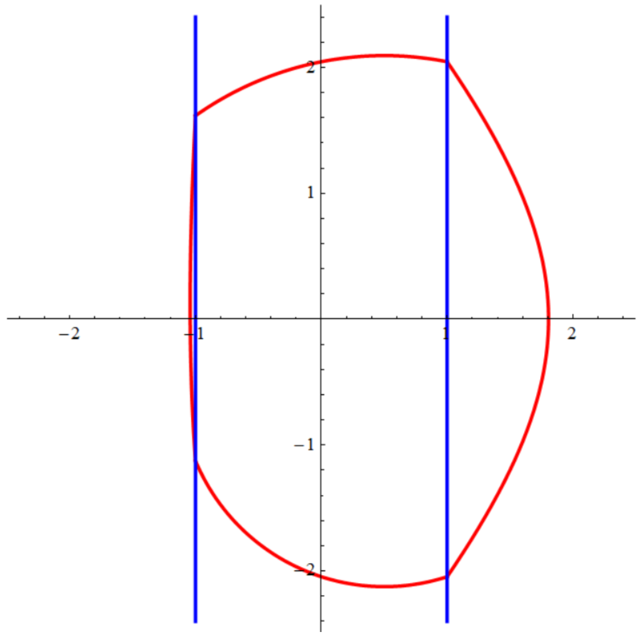

FIGURE 3. The limit cycle of the discontinuous piecewise differential system formed by two linear Hamiltonian saddles by the two linear Hamiltonian saddles (19) and (21), and the linear center (20). This limit cycles is travelled in counterclockwise sense.

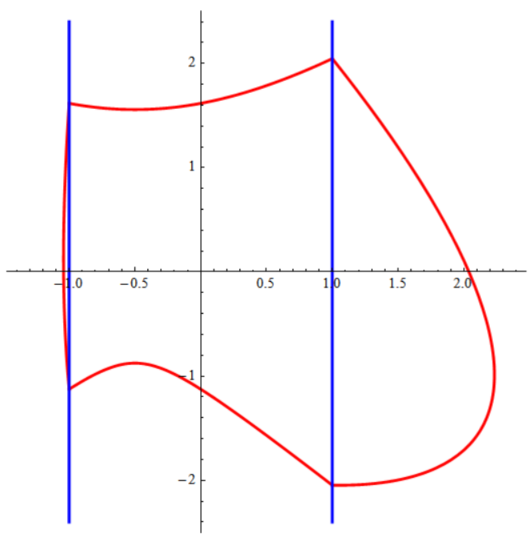

FIGURE 4. The limit cycle of the discontinuous piecewise differential system formed by two linear Hamiltonian saddles by the two linear Hamiltonian saddles (24) and (25), and the linear center (26). This limit cycles is travelled in counterclockwise sense.

form of a linear differential system having a linear Hamiltonian saddle (proved in [19]). These normal forms will be widely used in the proofs of the main results.

2. Preliminaries

The following propositions, proved in [17, Lemma 1] and [19, Proposition 1], respectively, prove the normal forms mentioned in the introduction.

Proposition 4. *Any linear differential system having a center can be written as*

$$\dot{x} = -bx - \Omega y + d, \quad \dot{y} = x + by + c, \tag{1}$$

where $\Omega = b^2 + \omega^2$ *with* $\omega \neq 0$.

The first integral of system (1) is

$$F(x, y) = -\frac{1}{2}x^2 - bxy - \frac{\Omega}{2}y^2 - cx + dy. \tag{2}$$

Proposition 5. *A differential system having a linear Hamiltonian saddle can be written as*

$$\dot{x} = -\beta x - \delta y + \mu, \quad \dot{y} = \alpha x + \beta y + \gamma, \tag{3}$$

with $\alpha \in \{0, 1\}$. *Moreover, when* $\alpha = 0$ *then* $\gamma = 0$, $b \neq 0$, *and when* $\alpha = 1$ *then* $\delta = \beta^2 - \omega^2$ *with* $\omega \neq 0$.

The first integral of system (3) is

$$H(x, y) = -\frac{\alpha}{2}x^2 - \beta xy - \frac{\delta}{2}y^2 - \gamma x + \mu y. \tag{4}$$

3. Proof of Theorem 1

Assume that we have a continuous piecewise linear differential system separated by one straight line and formed by one center and one Hamiltonian saddle. Without loss of generality we can assume that the straight line of separation is $x = 0$ and that we have system (1) in $x < 0$ with first integral in (2) and system (3) with first integral (4) in $x > 0$.

Note that if such piecewise linear differential systems (either continuous or discontinuous) have a periodic orbit candidate to be a limit cycle, such a periodic orbit must intersect the line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since F and H are two first integrals, we have that

$$F(0, y_1) = F(0, y_2) \quad \text{and} \quad H(0, y_1) = H(0, y_2)$$

that is

$$(y_1 - y_2)(2d - \Omega(y_1 + y_2)) = 0 \quad \text{and} \quad (y_1 - y_2)(2\mu - \delta(y_1 + y_2)) = 0. \tag{5}$$

If the piecewise differential system is continuous, then both systems must coincide in $x = 0$, and so we have that $\beta = b$, $\gamma = c$, $\mu = d$ and $\delta = \Omega$. Then the solutions (y_1, y_2) of this last system satisfying the necessary condition $y_1 < y_2$ are

$$y_1 = \frac{2d}{\Omega} - y_2.$$

If the piecewise differential system is discontinuous, then the solutions (y_1, y_2) of system (5) satisfying the necessary condition $y_1 < y_2$ are

$$\mu_1 = \frac{d\delta}{\Omega}, \quad y_1 = \frac{2d}{\Omega} - y_2.$$

So the periodic orbits of either the discontinuous or the continuous piecewise differential systems are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of the theorem.

4. Proof of Theorem 2

Assume that we have a piecewise linear differential system separated by two parallel straight lines and formed by two centers and one Hamiltonian saddle. Without loss of generality we can assume that the straight lines of separation are $x = -1$ and $x = 1$. We have to consider only two different cases (the other combination of systems in the different zones that are not contained in the cases given below are equivalent to one of them doing the symmetry with respect to the y-axis).

(i) We have a linear center in the regions $x < -1$ and $x > 1$; and we have a linear Hamiltonian saddle in the region $x \in (-1, 1)$.
(ii) We have a linear center in the regions $x < -1$ and $x \in (-1, 1)$; and we have a linear Hamiltonian saddle in the region $x > 1$.

We will study each of the cases separately.

Case (i). Note that if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle, such a periodic orbit must intersect the lines $x = \pm 1$ in exactly four points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since F_1, F_2 and H_1 are three first integrals and we denote the parameters of the first integrals F_1 and H_1 with subindexes one, and the parameters of the first integral F_2 with subindexes 2, we have that

$$F_1(-1, y_1) - F_1(-1, y_2) = 0, \quad H_1(-1, y_2) - H_1(1, y_3) = 0,$$
$$F_2(1, y_3) - F_2(1, y_4) = 0, \quad H_1(1, y_4) - H_1(-1, y_1) = 0,$$

that is

$$\begin{aligned}
&(y_1 - y_2)(2(b_1 + d_1) - \Omega_1(y_1 + y_2)) = 0,\\
&4\gamma_1 + 2(\beta_1 + \mu_1)y_2 + 2(\beta_1 - \mu_1)y_3 - \delta_1(y_2^2 - y_3^2) = 0,\\
&(2(b_2 - d_2) + \Omega_2(y_3 + y_4))(y_3 - y_4) = 0,\\
&-4\gamma_1 - 2(\beta_1 + \mu_1)y_1 - 2(\beta_1 - \mu_1)y_4 + \delta_1(y_1^2 - y_4^2) = 0.
\end{aligned} \tag{6}$$

Assume first that it is a continuous piecewise differential system. Then both systems (1) (with $i = 1$) and (3) must coincide in $x = -1$ and systems (3) and (1) (with $i = 2$) must coincide in $x = 1$. Doing so we get

$$\Omega_i = \delta_i,\ b_i = \beta_i,\ i = 1, 2,\ d_1 = \mu_1,\ d_2 = \mu_1 + \beta_2 - \beta_1,\ c_1 = 1 + \gamma_1 - \alpha_1,\ c_2 = -1 + \gamma_2 + \alpha_2.$$

Then the solutions (y_1, y_2, y_3, y_4) of this last system satisfying the necessary condition $y_1 < y_2$ are

$$y_2 = \frac{2(\beta_1 + \mu_1)}{\delta_1} - y_1,\ y_3 = \frac{\mu_1 - \beta_1}{\delta_1} \pm \frac{\sqrt{\Delta}}{\delta_1}, \quad y_4 = \frac{\mu_1 - \beta_1}{\delta_1} \pm \frac{\sqrt{\Delta}}{\delta_1},$$

where $\Delta = \beta_1^2 - 4\gamma_1\delta_1 + (\mu_1 - y_1\delta_1)^2 - 2\beta_1(\mu_1 + y_1\delta_1)$. Hence all the periodic orbits of the continuous piecewise differential system are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of the theorem for the continuous piecewise differential systems in Case (i).

Assume now that it is a discontinuous piecewise differential system. Since $\Omega_1, \Omega_2 > 0$ the solution of the first and third equations of (6) is

$$y_1 = \frac{2(b_1 + d_1)}{\Omega_1} - y_2, \quad y_3 = \frac{2(d_2 - b_2)}{\Omega_2} - y_4.$$

Introducing these solutions into the second and fourth equations of (6) we get

$$\begin{aligned} e_1 = {} & 4((d_2 - b_2)\beta_1\Omega_2 + \gamma_1\Omega_2^2 + (b_2 - d_2)\Omega_2\nu_1 + (b_2 - d_2)^2\delta_1) + 2\Omega_2^2(\beta_1 + \mu_1)y_2 \\ & - 2\Omega_2(\beta_1\Omega_2 - \Omega_2\mu_1 - 2(b_2 - d_2)\delta_1)y_4 - \Omega_2^2\delta_1(y_2^2 - y_4^2) = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} e_2 = {} & 4(-(b_1 + d_1)\beta_1\Omega_1 - \gamma_1\Omega_1^2 - (b_1 + d_1)\Omega_1\nu_1 + (b_1 + d_1)^2\delta_1) - 2\Omega_1^2(\beta_1 - \nu_1)y_4 \\ & + 2\Omega_1((\beta_1 + \Omega_1)\nu_1 - 2(b_1 + d_1)\delta_1)y_2 + \Omega_1^2\delta_1(y_2^2 - y_4^2) = 0. \end{aligned}$$

Taking $e_3 = \Omega_1^2 e_1 - \Omega_2^2 e_2$, and solving $e_3 = 0$ in y_4 we get

$$y_4 = \frac{A_0}{A_1} + \frac{A_2}{A_1} y_2,$$

where

$$\begin{aligned} A_0 = {} & -\Omega_1\Omega_2((\mu_1 - \beta_1)d_2\Omega_1 + \beta_1(b_1 + d_1)\Omega_2 + b_2\Omega_1(\beta_1 - \mu_1) + (b_1 + d_1)\Omega_2\mu_1) \\ & + ((b_2 - d_2)^2\Omega_1^2 + (b_1 + d_1)^2\Omega_2^2)\delta_1, \\ A_1 = {} & \Omega_1^2\Omega_2((\beta_1 - \mu_1)\Omega_2 + (d_2 - b_2)\delta_1), \quad A_2 = \Omega_1\Omega_2^2((\beta_1 + \mu_1)\Omega_1 - (d_1 + b_1)\delta_1), \end{aligned}$$

whenever $A_1 \neq 0$. The case with $A_1 = 0$ yields $\beta_1 = \mu_1 + (b_2 - d_2)\delta_1/\Omega_2$. Introducing it into $e_3 = 0$ and solving in y_2 we obtain $y_2 = y_1 = (b_1 + d_1)/\Omega_1$ which is not possible. So we can assume that $A_1 \neq 0$. Now introducing y_4 into equation (7) and solving in y_2 we get

$$y_{2\pm} = \frac{b_1 + d_1}{\Omega_1} \pm \frac{\sqrt{\Delta}}{2A_3},$$

where

$$A_3 = \Omega_1^2\Omega_2^2\delta_1(2\Omega_1\Omega_2\mu_1 + (b_2 - d_2)\Omega_1\delta_1 - (b_1 + d_1)\Omega_2\delta_1)(2\beta_1\Omega_1\Omega_2 - ((b_2 + d_2)\Omega_1 + (b_1 + d_1)\Omega_2)\delta_1),$$

$$\Delta = 4\Omega_1^4\Omega_2^2\delta_1((\beta_1 - \mu_1)\Omega_2 + (-b_2 + d_2)\delta_1)^2(2\Omega_1\Omega_2\mu_1 + (b_2 - d_2)\Omega_1\delta_1 - (b_1 + d_1)\Omega_2\delta_1) (-2\beta_1\Omega_1\Omega_2 + (b_2\Omega_1 - d_2\Omega_1 + (b_1 + d_1)\Omega_2)\delta_1)(2\Omega_1\Omega_2((\beta_1 - \mu_1)d_2\Omega_1 + \beta_1(b_1 + d_1)\Omega_2 + 2\gamma_1\Omega_1\Omega_2 + (b_1 + d_1)\Omega_2\mu_1 + b_2\Omega_1(-\beta_1 + \mu_1)) + (b_2 - d_2)\Omega_1 - (b_1 + d_1)\Omega_2) ((b_2 - d_2)\Omega_1 + (b_1 + d_1)\Omega_2)\delta_1),$$

whenever $A_3 \neq 0$, and if $A_3 = 0$ then there is at most one solution y_2.

When $A_3 \neq 0$,since

$$y_{1\pm} = \frac{2(d_1 + b_1)}{\Omega_1} - y_{2\pm} = \frac{d_1 + b_1}{\Omega_1} \mp \frac{\sqrt{\Delta}}{2A_3} = y_{2\mp},$$

there is at most one solution with $y_1 > y_2$ and $y_3 < y_4$. In summary, we have proved that at most we can have one limit cycle.

Now we shall prove that the discontinuous piecewise linear differential system having a center, a saddle and a center has one limit cycle. This will complete the proof of Theorem 2 in Case (i) when it is discontinuous.

The Hamiltonians of the three linear systems in Case (i) are

$$F_1(x, y) = -8y - \frac{y^2}{4} - 4(x + 2y)^2,$$
$$H_1(x, y) = x - y + x^2 - y^2,$$
$$F_2(x, y) = -8x8y - 4x^2 - 8xy - 5y^2,$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\dot{x} = 8 + 16x + \frac{65}{2}y, \qquad \dot{y} = -8(x + 2y); \tag{8}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\dot{x} = -1 - 2y, \qquad \dot{y} = -1 - 2x; \tag{9}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\dot{x} = 8 - 8x - 10y, \qquad \dot{y} = 8 + 8x + 8y. \tag{10}$$

These three linear differential systems are a center, a saddle and a center because the determinant of their linear part are 4, -4 and 16, respectively.

The discontinuous piecewise differential system formed by the three linear differential systems (8), (9) and (10) in order to have one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, these points must satisfy system (6), and this system has a unique solution satisfying $y_1 > y_2$

and $y_3 < y_4$, namely

$$(y_1, y_2, y_3, y_4) = \left(\frac{16}{65} + \frac{\sqrt{4873}}{36\sqrt{2}}, \frac{16}{65} - \frac{\sqrt{4873}}{36\sqrt{2}}, -\frac{97\sqrt{4873}}{2340\sqrt{2}}, \frac{97\sqrt{4873}}{2340\sqrt{2}}\right). \tag{11}$$

Drawing the corresponding limit cycle associated to this solution we obtain the limit cycle of Figure 1.

Case (ii). Note that if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle, such a periodic orbit must intersect the lines $x = \pm 1$ in exactly four points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since F_1, F_2 and H_1 are three first integrals, we have that

$$\begin{aligned} F_1(-1, y_1) - F_1(-1, y_2) = 0, \quad F_2(-1, y_2) - F_2(1, y_3) = 0,\\ H_1(1, y_3) - H_1(1, y_4) = 0, \quad F_2(1, y_4) - F_2(-1, y_1) = 0, \end{aligned}$$

that is

$$\begin{aligned} &(y_1 - y_2)(2(b_1 + d_1) - \Omega_1(y_1 + y_2)) = 0,\\ &4c_2 + 2(b_2 + d_2)y_2 + 2(b_2 - d_2)y_3 - \Omega_2(y_2^2 - y_3^2) = 0,\\ &(2(\beta_1 - \mu_1) + \delta_1(y_3 + y_4))(y_3 - y_4) = 0,\\ &4c_2 + 2(b_2 + d_2)y_1 + 2(b_2 - d_2)y_4 - \Omega_2(y_1^2 - y_4^2) = 0. \end{aligned} \tag{12}$$

Assume first that it is a continuous piecewise differential system. Then both systems (1) (with $i = 1$) and (1) (with $i = 2$) must coincide in $x = -1$, and systems (3) and (1) (with $i = 2$) must coincide in $x = 1$. Doing so we get

$$b_2 = b_1 = \beta_1,\ \Omega_2 = \Omega_1 = \delta_1,\ d_2 = d_1 = \mu_1,\ c_2 = c_1 = \gamma_1, \alpha_2 = 1.$$

Then the solutions (y_1, y_2, y_3, y_4) of this last system satisfying the necessary condition $y_1 < y_2$ are

$$y_2 = \frac{2(\beta_1 + \mu_1)}{\delta_1} - y_1,\ y_3 = \frac{\mu_1\beta_1}{\delta_1} \mp \frac{\sqrt{\Delta}}{\delta_1}, \quad y_4 = \frac{\mu_1 - \beta_1}{\delta_1} \pm \frac{\sqrt{\Delta}}{\delta_1},$$

where $\Delta = \beta_1^2 - 4\gamma_1\delta_1 - 2\beta_1\mu_1 + \mu_1^2 - 2\delta_1(\beta_1 + \mu_1)y_1 + \delta_1 y_1^2$. Hence all the periodic orbits of the continuous piecewise differential system are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of Theorem 2 for the continuous piecewise differential systems in Case (ii).

Assume now that it is a discontinuous piecewise differential system. If $\delta_1 = 0$ then the third equation in (12) yields $\beta_1 = \mu_1$, and any solution of the other three equations in (12) yield a continuous of solutions. If $\delta_1 \neq 0$, since $\Omega_1 > 0$ the solution of the first and third equations is

$$y_1 = \frac{2(d_1 + b_1)}{\Omega_1} - y_2, \quad y_3 = \frac{2(\mu_1 - \beta_1)}{\delta_1} - y_4.$$

Introducing these solutions into the second and fourth equations in (6) we get

$$\begin{aligned} e_1 = {}& 4(\beta_1 - \mu_1)^2\Omega_2 - 4(b_2 - d_2)(\beta_1 - \mu_1)\delta_1 + 4c_2\delta_1^2 + 4c_2\delta_1^2 + 2(b_2 + d_2)\delta_1^2 y_2\\ & 2\delta_1(2(\beta_1 - \mu_1)\Omega_2 + (d_2 - b_2)\delta_1)y_4 - \Omega_2\delta_1^2(y_2^2 - y_4^2) = 0 \end{aligned} \tag{13}$$

and

$$e_2 = 4(b_1+d_1)(b_2+d_2)\Omega_1 + 4c_2\Omega_1^2 - 4(b_1+d_1)^2\Omega_2 + 2(b_2-d_2)\Omega_1^2 y_4 \\ - 2\Omega_1((b_2+d_2)\Omega_1 - 2(b_1+d_1)\Omega_2)y_2 - \Omega_1^2\Omega_2(y_2^2 - y_4^2) = 0.$$

Taking $e_3 = \Omega_1^2 e_1 - \delta_1^2 e_2$ and solving $e_3 = 0$ in y_4 we get

$$y_4 = \frac{A_0 + A_2 y_2}{\delta_1\Omega_1^2((b_2-d_2)\delta_1 + (\mu_1-\beta_1)\Omega_2)},$$

where

$$A_0 = (\beta_1-\mu_1)^2\Omega_1^2\Omega_2 - (b_2-d_2)(\beta_1-\mu_1)\Omega_1^2\delta_1 + (b_1+d_1)((b_1+d_1)\Omega_2 - (b_2+d_2)\Omega_1)\delta_1^2,$$
$$A_2 = \Omega_1((b_2+d_2)\Omega_1 - (b_1+d_1)\Omega_2)\delta_1^2,$$

whenever $\mu_1 \neq \beta_1 + (d_2-b_2)\delta_1/\Omega_2$. When $\mu_1 = \beta_1 + (d_2-b_2)\delta_1/\Omega_2 = 0$ solving $e_3 = 0$ in y_2 we obtain $y_2 = y_1 = 2(\mu_1+\beta_1)/\delta_1$ which is not possible. So we can assume that $\mu_1 \neq \beta_1 + (d_2-b_2)\delta_1/\Omega_2$. Now introducing y_4 into equation (13) and solving in y_2 we get

$$y_{2\pm} = \frac{b_1+d_1}{\Omega_1} \pm \frac{\sqrt{\Delta}}{2A_3},$$

where

$$A_3 = -\Omega_1^2\Omega_2\delta_1^2((\beta_1-\mu_1)\Omega_1\Omega_2 + 2d_2\Omega_1\delta_1 - (b_1+d_1)\Omega_2\delta_1) \\ ((\beta_1-\mu_1)\Omega_1\Omega_2 - 2b_2\Omega_1\delta_1 + (b_1+d_1)\Omega_2\delta_1),$$
$$\Delta = 4\Omega_1^4\Omega_2\delta_1^2(\beta_1\Omega_2 - \mu_1\Omega_2 + (d_2-b_2)\delta_1)^2((\beta_1-\mu_1)\Omega_1\Omega_2 + 2d_2\Omega_1\delta_1 - (b_1+d_1)\Omega_2\delta_1) \\ (\beta_1\Omega_1\Omega_2 - \mu_1\Omega_1\Omega_2 - 2b_2\Omega_1\delta_1 + (b_1+d_1)\Omega_2\delta_1)((\beta_1-\mu_1)^2\Omega_1^2\Omega_2 \\ - 2(b_2-d_2)(\beta_1-\mu_1)\Omega_1^2\delta_1 + (2\Omega_1((b_1+d_1)(b_2+d_2) + 2c_2\Omega_1) - (b_1+d_1)^2\Omega_2)\delta_1^2),$$

whenever $A_3 \neq 0$, and if $A_3 = 0$ then there is at most one solution y_2.

When $A_3 \neq 0$, since

$$y_{1\pm} = \frac{b_1+d_1}{\Omega_1} - y_{2\pm} = (b_1+d_1)/\Omega_1 \mp \frac{\sqrt{\Delta}}{2A_3} = y_{2\mp},$$

there is at most one solution with $y_1 > y_2$ and $y_3 < y_4$. In summary, we have proved that at most we can have one limit cycle.

Now we shall prove that this discontinuous piecewise linear differential system has one limit cycle. This will complete the proof of Theorem 2 in Case(*ii*) when it is discontinuous and so the proof of Theorem 2.

The Hamiltonians of the three linear systems in Case (ii) are

$$F_1(x,y) = -8y - \frac{1}{4}y^2 - 4(x+2y)^2,$$
$$F_2(x,y) = x - y - x^2 - y^2,$$
$$H_1(x,y) = -4x + \frac{1}{2}x^2 - \frac{1}{2}y^2,$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\dot{x} = -8 - 16x - \frac{65}{2}y, \qquad \dot{y} = 8(x + 2y); \tag{14}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\dot{x} = -1 - 2y, \qquad \dot{y} = -1 + 2x; \tag{15}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\dot{x} = -y, \qquad \dot{y} = 4 - x. \tag{16}$$

These three linear differential systems are two centers and a saddle because the determinant of their linear part are 4, 4 and -1, respectively.

The discontinuous piecewise differential system formed by the three linear differential systems (14), (15) and (16) in order to have one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, these points must satisfy system (12), and this system has the unique solution (11) satisfying $y_1 > y_2$ and $y_3 < y_4$. Drawing the corresponding limit cycle associated to this solution we obtain the limit cycle of Figure 2.

5. Proof of Theorem 3

Assume that we have a piecewise linear differential system separated by two parallel straight lines and formed by two Hamiltonian saddles and one center. Without loss of generality we can assume that the straight lines of separation are $x = -1$ and $x = 1$. We have to consider only two different cases (the other combination of systems in the different zones that are not contained in the cases given below are equivalent to one of them doing the symmetry with respect to the y-axis).

(i) We have a linear Hamiltonian saddle in the regions $x < -1$ and $x > 1$; and we have a linear center in the region $x \in (-1, 1)$.
(ii) We have a linear Hamiltonian saddle in the regions $x < -1$ and $x \in (-1, 1)$; and we have a linear center in the region $x > 1$.

We will study each of these two cases separately.

Case (i). Note that if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle such a periodic orbit must intersect the lines $x = \pm 1$ in exactly four points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since F_1, H_1 and H_2 are three first integrals, we have that

$$\begin{aligned} H_1(-1, y_1) - H_1(-1, y_2) = 0, \quad F_1(-1, y_2) - F_1(1, y_3) = 0, \\ H_2(1, y_3) - H_2(1, y_4) = 0, \quad F_1(1, y_4) - F_1(-1, y_1) = 0, \end{aligned}$$

that is

$$(17)\qquad \begin{aligned} &(y_1-y_2)(2(\beta_1+\mu_1)-\delta_1(y_1+y_2))=0,\\ &4c_1+2(b_1+d_1)y_2+2(b_1-d_1)y_3-\Omega_1(y_2^2-y_3^2)=0,\\ &(2(\beta_2-\mu_2)+\delta_2(y_3+y_4))(y_3-y_4)=0,\\ &4c_1+2(b_1+d_1)y_1+2(b_1-d_1)y_4-\Omega_1(y_1^2-y_4^2)=0. \end{aligned}$$

Assume first that it is a continuous piecewise differential system. Then system (3) (with $i=1$) and system (1) (with $i=1$) must coincide in $x=-1$, and system (3) (with $i=2$) and system (1) (also with $i=1$) must coincide in $x=1$. Doing so we get

$$\beta_i=b_1,\ \mu_i=d_1,\ \delta_i=\Omega_1,\ i=1,2,\ \alpha_1=1-c_1+\gamma_1,\ \alpha_2=1+c_1-\gamma_2.$$

Then the solutions (y_1,y_2,y_3,y_4) of system (17) satisfying the necessary condition $y_1<y_2$ are

$$y_2=\frac{2(b_1+d_1)}{\Omega_1}-y_1,\ y_3=\frac{d_1-b_1}{\Omega_1}\pm\frac{\sqrt{\Delta}}{\Omega_1},\quad y_4=\frac{d_1-b_1}{\Omega_1}\pm\frac{\sqrt{\Delta}}{\Omega_1},$$

where $\Delta=b_1^2-4c_1\Omega_1+(d_1-\Omega_1y_1)^2-2b_1(d_1+\Omega_1y_1)$. Hence all the periodic orbits of the continuous piecewise differential system are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of Theorem 3 for the continuous piecewise differential systems in Case (i).

Assume now that it is a discontinuous piecewise differential system. If $\delta_1=0$ the first equation in (17) yields $\beta_1=-\mu_1$ and any solution of the other three equations in (17) yield a continuous of solutions. If $\delta_2=0$ then the third equation in (17) yields $\beta_2=\mu_2$ and any solution of the other three equations in (17) yield a continuous of solutions.

If $\delta_1\delta_2\neq 0$, then the solution of the first and third equations is

$$y_1=\frac{2(\beta_1+\mu_1)}{\delta_1}-y_2,\quad y_3=\frac{2(\mu_2-\beta_2)}{\delta_2}-y_4.$$

Introducing these solutions into the second and fourth equations in (6) we get

$$(18)\qquad \begin{aligned} e_1={}&4(\beta_2-\mu_2)^2\Omega_1-4(b_1-d_1)(\beta_2-\mu_2)\delta_2+4c_1\delta_2^2+2\delta_2^2(b_1+d_1)y_2\\ &+2\delta_2(2\beta_2\Omega_1-2\mu_2\Omega_1-b_1\delta_2+d_1\delta_2)y_4-\Omega_1\delta_2^2(y_2^2-y_4^2)=0 \end{aligned}$$

and

$$\begin{aligned} e_2={}&4(\beta_1+\mu_1)^2\Omega_1-4(b_1+d_1)(\beta_1+\mu_1)\delta_1-4c_1\delta_1^2-2\delta_1^2(b_1-d_1)y_4\\ &-2\delta_1(2\beta_1\Omega_1+2\mu_1\Omega_1-b_1\delta_1-d_1\delta_1)y_2+\Omega_1\delta_1^2(y_2^2-y_4^2)=0. \end{aligned}$$

Taking $e_3=\delta_1^2e_1+\delta_2^2e_2$ and solving $e_3=0$ in y_4 we get

$$y_4=\frac{A_0}{A_1}+\frac{A_2}{A_1}y_2$$

where

$$A_0=-(\beta_2-\mu_2)^2\Omega_1\delta_1^2+(b_1-d_1)(\beta_2-\mu_2)\delta_1^2\delta_2-(\beta_1+\mu_1)((\beta_1+\mu_1)\Omega_1-(b_1+d_1)\delta_1)\delta_2^2,$$

$$A_1=\delta_1^2\delta_2(-\beta_2\Omega_1+\mu_2\Omega_1+b_1\delta_2-d_1\delta_2),\quad A_2=(-\beta_1\Omega_1-\mu_1\Omega_1+b_1\delta_1+d_1\delta_1)\delta_2^2\delta_1,$$

whenever $A_1 \neq 0$. The case with $A_1 = 0$ yields $\beta_2 = \mu_2 + \delta_2(b_1 - d_1)/\Omega_1$. Introducing it into $e_3 = 0$, and solving in y_2 we obtain $y_2 = y_1 = (\beta_1 + \mu_1)/\delta_1$ which is not possible. So we can assume that $A_1 \neq 0$. Now introducing y_4 into equation (18) and solving in y_2 we get

$$y_{2\pm} = \frac{\beta_1 + \mu_1}{\delta_1} \pm \frac{\sqrt{\Delta}}{2A_3},$$

where

$$\begin{aligned} A_3 &= \Omega_1\delta_1^2\delta_2^2(\beta_2\Omega_1\delta_1 - \mu_2\Omega_1\delta_1 + \beta_1\Omega_1\delta_2 + \mu_1\Omega_1\delta_2 - 2b_1\delta_1\delta_2) \\ &\quad (\beta_2\Omega_1\delta_1 - \mu_2\Omega_1\delta_1 - \beta_1\Omega_1\delta_2 - \mu_1\Omega_1\delta_2 + 2d_1\delta_1\delta_2), \\ \Delta &= 4\Omega_1\delta_1^4\delta_2^2(\beta_2\Omega_1 - \mu_2\Omega_1 + (-b_1 + d_1)\delta_2)^2((\beta_2 - \mu_2)\Omega_1\delta_1 + (\beta_1 + \mu_1)\Omega_1\delta_2 - 2b_1\delta_1\delta_2) \\ &\quad ((\beta_2 - \mu_2)\Omega_1\delta_1 - (\beta_1 + \mu_1)\Omega_1\delta_2 + 2d_1\delta_1\delta_2)((\beta_2 - \mu_2)^2\Omega_1\delta_1^2 - 2(b_1 - d_1)(\beta_2 - \mu_2)\delta_1^2\delta_2 \\ &\quad + (4c_1\delta_1^2 - (\beta_1 + \mu_1)^2\Omega_1 + 2(b_1 + d_1)(\beta_1 + \mu_1)\delta_1)\delta_2^2), \end{aligned}$$

whenever $A_3 \neq 0$, and if $A_3 = 0$ then there is at most one solution y_2.

When $A_3 \neq 0$, since

$$y_{1\pm} = \frac{2(\beta_1 + \mu_1)}{\delta_1} - y_{2\pm} = \frac{\beta_1 + \mu_1}{\delta_1} \mp \frac{\sqrt{\Delta}}{2A_3} = y_{2\mp},$$

there is at most one solution with $y_1 > y_2$ and $y_3 < y_4$. In summary, we have proved that at most we can have one limit cycle.

Now we shall prove that the discontinuous piecewise linear differential system having a saddle, a center and a saddle has one limit cycle. This will complete the proof of Theorem 3 in Case (i) when it is discontinuous.

The Hamiltonians of the three linear systems in Case (i) are

$$\begin{aligned} H_1(x, y) &= 16x + 2y + x^2 - \frac{1}{48}(309 - \sqrt{157881})xy - \frac{65}{1536}(405 - \sqrt{157881}))y^2, \\ F_1(x, y) &= x - y - x^2 - y^2, \\ H_2(x, y) &= -8x + x^2 - y^2, \end{aligned}$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\begin{aligned} \dot{x} &= 2 - \frac{1}{48}(309 - \sqrt{157881})x - \frac{65}{768}(405 - \sqrt{157881})y, \\ \dot{y} &= -16 - 2x + \frac{1}{48}(309 - \sqrt{157881})y; \end{aligned} \tag{19}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\dot{x} = -1 - 2y, \qquad \dot{y} = -1 + 2x; \tag{20}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\dot{x} = -2y, \qquad \dot{y} = 8 - 2x. \tag{21}$$

These three linear differential systems are a saddle, a center and a saddle, because the determinant of their linear part are $-8569/48 + 7\sqrt{157881}/16 < 0$, 4 and -4, respectively.

The discontinuous piecewise differential system formed by the three linear differential systems (19), (20) and (21) in order to have one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, these points must satisfy system (12), and this system has the unique solution (11) satisfying $y_1 > y_2$ and $y_3 < y_4$. Drawing the corresponding limit cycle associated to this solution we obtain the limit cycle of Figure 3.

Case (*ii*). Note that if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle, such a periodic orbit must intersect the lines $x = \pm 1$ in exactly four points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since H_1, H_2 and F_1 are three first integrals, we have that

$$\begin{aligned} H_1(-1, y_1) - H_1(-1, y_2) = 0, \quad H_2(-1, y_2) - H_1(1, y_3) = 0, \\ F_1(1, y_3) - F_1(1, y_4) = 0, \quad H_2(1, y_4) - H_2(-1, y_1) = 0, \end{aligned}$$

that is

$$\begin{aligned} &(y_1 - y_2)(2(\beta_1 + \mu_1) - \delta_1(y_1 + y_2)) = 0, \\ &4\gamma_2 + 2(\beta_2 + \mu_2)y_2 + 2(\beta_2 - \mu_2)y_3 - \delta_2(y_2^2 - y_3^2) = 0, \\ &(2(b_1 - d_1) + \Omega_1(y_3 + y_4))(y_3 - y_4) = 0, \\ &-4\gamma_2 - 2(\beta_2 + \mu_2)y_1 - 2(\beta_2 - \mu_2)y_4 + \delta_2(y_1^2 - y_4^2) = 0. \end{aligned} \tag{22}$$

Assume first that it is a continuous piecewise differential system. Then both systems (3) (with $i = 1$ and with $i = 2$) must coincide in $x = -1$, and system (3) (with $i = 2$) and system (1) (with $i = 1$) must coincide in $x = 1$. Doing so we get

$$\beta_i = b_1,\ \mu_i = d_1,\ \delta_i = \Omega_1,\ i = 1, 2,\ \alpha_1 = 1 + c_1 + \gamma_1 - 2\gamma_2,\ \alpha_2 = 1 + c_1 - \gamma_2.$$

Then the solutions (y_1, y_2, y_3, y_4) of system (22) satisfying the necessary condition $y_1 < y_2$ are

$$y_2 = \frac{2(b_1 + d_1)}{\Omega_1} - y_1,\ y_3 = \frac{d_1 - b_1}{\Omega_1} \pm \frac{\sqrt{\Delta}}{\Omega_1}, \quad y_4 = \frac{d_1 - b_1}{\Omega_1} \pm \frac{\sqrt{\Delta}}{\Omega_1},$$

where $\Delta = b_1^2 - 4\gamma_2\Omega_1 + (d_1 - \Omega_1 y_1)^2 - 2b_1(d_1 + \Omega_1 y_1)$. Hence all the periodic orbits of the continuous piecewise differential system are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of the theorem for the continuous piecewise differential systems in Case (*ii*).

Assume now that it is a discontinuous piecewise differential system. If $\delta_1 = 0$ then the first equation in (22) yields $\beta_1 = -\mu_1$, and any solution of the other three equations in (22) yield a continuous of solutions.

If $\delta_1 \neq 0$ since $\Omega_1 > 0$, the solution of the first and third equations is

$$y_1 = \frac{2(\mu_1 + \beta_1)}{\delta_1} - y_2, \quad y_3 = \frac{2(d_1 - b_1)}{\Omega_1} - y_4.$$

Introducing these solutions into the second and fourth equations in (22) we get

$$\begin{aligned} e_1 = &4\Omega_1(-(b_1-d_1)(\beta_2-\mu_2)+\gamma_2\Omega_1)+4\Omega_1^2(b_1-d_1)^2\delta_2+2\Omega_1^2(\beta_2+\mu_2)y_2 \\ &-2\Omega_1(\beta_2\Omega_1-\mu_2\Omega_1-2b_1\delta_2+2d_1\delta_2)y_4-\Omega_1^2\delta_2(y_2^2-y_4^2)=0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} e_2 = &-4(\beta_1+\mu_1)(\beta_2+\mu_2)\delta_1-4\gamma_2\delta_1^2+4(-\beta_1^2+2\beta_1\mu_1+\mu_1^2)\delta_2-2\delta_1^2(\beta_2-\mu_2)y_4 \\ &+2\delta_1(\beta_2\delta_1+\mu_2\delta_1-2\beta_1\delta_2-2\mu_1\delta_2)y_2+\delta_1^2\delta_2(y_2^2-y_4^2)=0. \end{aligned}$$

Taking $e_3=\delta_1^2e_1+\Omega_1^2e_2$ and solving $e_3=0$ in y_4 we get

$$y_4=\frac{A_0}{A_1}+\frac{A_2}{A_1}y_2,$$

where

$$\begin{aligned} A_0 = &-\Omega_1\delta_1((\beta_1+\mu_1)(\beta_2+\mu_2)\Omega_1-(b_1-d_1)(\beta_2-\mu_2)\delta_1)-\delta_2((\beta_1+\mu_1)^2\Omega_1^2 \\ &+(b_1-d_1)^2\delta_1^2), \\ A_1 = &\Omega_1\delta_1^2(\Omega_1(\beta_2-\mu_2)+(d_1-b_1)\delta_2), \quad A_2=\Omega_1^2\delta_1((\beta_2+\mu_2)\delta_1-(\beta_1+\mu_1)\delta_2), \end{aligned}$$

whenever $\beta_2\neq\mu_2+(b_1-d_1)\delta_2/\Omega_1$. When $\beta_2=\mu_2+(b_1-d_1)\delta_2/\Omega_1$ solving $e_3=0$ in y_2 we obtain $y_2=y_1=(\mu_1+\beta_1)/\delta_1$ which is not possible. So we can assume that $\beta_2\neq\mu_2+(b_1-d_1)\delta_2/\Omega_1$. Now introducing y_4 into equation (23) and solving in y_2 we get

$$y_{2\pm}=\frac{\beta_1+\mu_1}{\delta_1}\pm\frac{\sqrt{\Delta}}{2A_3}$$

where

$$\begin{aligned} A_3 = &\Omega_1^2\delta_1^2\delta_2(2\mu_2\Omega_1\delta_1-\beta_1\Omega_1\delta_2-\mu_1\Omega_1\delta_2+b_1\delta_1\delta_2-d_1\delta_1\delta_2)(2\beta_2\Omega_1\delta_1-\beta_1\Omega_1\delta_2 \\ &-\mu_1\Omega_1\delta_2-b_1\delta_1\delta_2+d_1\delta_1\delta_2), \\ \Delta = &-4\Omega_1^2\delta_1^4\delta_2(\beta_2\Omega_1-\mu_2\Omega_1+(-b_1+d_1)\delta_2)^2(2\beta_2\Omega_1\delta_1-((\beta_1+\mu_1)\Omega_1+(b_1-d_1)\delta_1)\delta_2) \\ &(2\mu_2\Omega_1\delta_1-((\beta_1+\mu_1)\Omega_1+(d_1-b_1)\delta_1)\delta_2)(2\Omega_1\delta_1((\beta_1+\mu_1)(\beta_2+\mu_2)\Omega_1-(b_1-d_1) \\ &(\beta_2-\mu_2)\delta_1+2\gamma_2\Omega_1\delta_1)-((\beta_1+\mu_1)\Omega_1+(b_1-d_1)\delta_1)((\beta_1+\mu_1)\Omega_1+(d_1-b_1)\delta_1)\delta_2), \end{aligned}$$

whenever $A_3\neq 0$, and if $A_3=0$ then there is at most one solution y_2.

When $A_3\neq 0$, since

$$y_{1\pm}=\frac{2(\beta_1+\mu_1)}{\delta_1}-y_{2\pm}=\frac{\beta_1+\mu_1}{\delta_1}\mp\frac{\sqrt{\delta}}{2A_3}=y_{2\mp},$$

there is at most one solution with $y_1>y_2$ and $y_3<y_4$. In summary, we have proved that at most we can have one limit cycle.

Now we shall prove that the discontinuous piecewise linear differential system having two saddles and a center has one limit cycle. This will complete the proof of Theorem 3 in Case (*ii*) when it is discontinuous and so the proof of Theorem 3.

The Hamiltonians of the three linear systems in Case (ii) are

$$H_1(x,y) = 16x + 2y + x^2 - \frac{1}{48}(309-\sqrt{157881})xy - \frac{65}{1536}(405-\sqrt{157881}))y^2,$$
$$H_2(x,y) = x - y + x^2 - y^2,$$
$$F_1(x,y) = -8x + 8y - 4x^2 - 8xy - 5y^2,$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\begin{aligned} \dot{x} &= 2 - \frac{1}{48}(309-\sqrt{157881})x - \frac{65}{768}(405-\sqrt{157881})y, \\ \dot{y} &= -16 - 2x + \frac{1}{48}(309-\sqrt{157881})y; \end{aligned} \tag{24}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\dot{x} = -1 - 2y, \qquad \dot{y} = -1 - 2x; \tag{25}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\dot{x} = 8 - 8x - 10y, \qquad \dot{y} = 8 + 8x + 8y. \tag{26}$$

These three linear differential systems are two saddles and a center because the determinant of their linear part are $-8569/48 + 7\sqrt{157881}/16 < 0$, -4 and 16, respectively.

The discontinuous piecewise differential system formed by the three linear differential systems (24), (25) and (26) in order to have one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, these points must satisfy system (22), and this system has the unique solution (11) satisfying $y_1 > y_2$ and $y_3 < y_4$. Drawing the corresponding limit cycle associated to this solution we obtain the limit cycle of Figure 4.

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