# COMPLETELY INTEGRABLE DIFFERENTIAL SYSTEMS ARE ESSENTIALLY LINEAR 

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#### Abstract

Let $\dot{x}=f(x)$ be a $C^{k}$ autonomous differential system with $k \in \mathbb{N} \cup\{\infty, \omega\}$ defined in an open subset $\Omega$ of $\mathbb{R}^{n}$. Assume that the system $\dot{x}=f(x)$ is $C^{r}$ completely integrable, i.e. there exist $n-1$ functionally independent first integrals of class $C^{r}$ with $2 \leq r \leq k$. If the divergence of system $\dot{x}=f(x)$ is non-identically zero, then any Jacobian multiplier is functionally independent of the $n-1$ first integrals. Moreover the system $\dot{x}=f(x)$ is $C^{r-1}$ orbitally equivalent to the linear differential system $\dot{y}=y$ in a full Lebesgue measure subset of $\Omega$. For Darboux and polynomial integrable polynomial differential systems we characterize their type of Jacobian multipliers.


## 1. Introduction and statement of results

Consider a $C^{k}$ autonomous differential system

$$
\begin{equation*}
\dot{x}=f(x), \quad x \in \Omega \subset \mathbb{R}^{n} \tag{1}
\end{equation*}
$$

where $k \in \mathbb{N} \cup\{\infty, \omega\}$, the dot denotes derivative with respect to the independent variable $t, \Omega$ is an open subset of $\mathbb{R}^{n}$, and $f(x)=$ $\left(f_{1}(x), \ldots, f_{n}(x)\right) \in C^{k}(\Omega)$. Recall that $\mathbb{N}$ is the set of positive integers, and $C^{\infty}$ and $C^{\omega}$ are respectively the sets of infinitely smooth functions and analytic functions.

A function $H(x)$ is a first integral of system (1) if it is continuous and defined in a full Lebesgue measure subset $\Omega_{1}$ of $\Omega$, and it is not locally constant on any positive Lebesgue measure subset of $\Omega_{1}$; moreover $H(x)$ is constant along each orbit of system (1) in $\Omega_{1}$. System (1) is $C^{r}$ completely integrable, if it has $n-1$ functionally independent $C^{r}$ first integrals in $\Omega$ with $1 \leq r \leq k$. Recall that $k$ functions $H_{1}(x), \ldots, H_{k}(x)$ are functionally independent in $\Omega$ if their gradients $\nabla H_{1}, \ldots, \nabla H_{k}$ have rank $k$ in a full Lebesgue measure subset of $\Omega$.

[^0]In this paper we denote by $\mathcal{X}$ the vector field associated to system (1). We will use $\partial_{i}$ to denote the partial derivative with respect to $x_{i}$ for $i=1, \ldots, n$. By convention $C^{r-1}=C^{r}$ if $r=\infty$ or $\omega$; and $\operatorname{div} \mathcal{X}=\operatorname{div} f=\partial_{1} f_{1}+\ldots+\partial_{n} f_{n}$.

A $C^{1}$ function $J$ is a Jacobian multiplier of system (1) if it is defined in a full Lebesgue measure subset $\Omega^{*} \subset \Omega$, and satisfies

$$
\operatorname{div}(J f) \equiv 0, \quad \text { i.e., } \quad \mathcal{X}(J)=-J \operatorname{div} \mathcal{X}, \quad x \in \Omega^{*} .
$$

If system (1) is two dimensional, a Jacobian multiplier is called an integrating factor.

We extend the usual ordering of $\mathbb{N}$ to the set $\mathbb{N} \cup\{\infty, \omega\}$ as follows: for all $k \in \mathbb{N}$ we have $k<\infty<\omega$. Two differential systems $\dot{x}=f(x)$ and $\dot{y}=g(y)$ are $C^{r}$ orbitally equivalent in a region $\mathcal{M}$ with $1 \leq r \leq \omega$ if there exists a $C^{r}$ invertible transformation $y=\Phi(x)$ defined on $\mathcal{M}$ such that $\Phi_{*} f \circ \Phi^{-1}(y)=q(y) g(y)$, where $\Phi_{*}$ is the tangent map of $\Phi$ and $q(y)$ is a non-vanishing scalar function defined on $\mathcal{M}$.

The next is our first main result.
Theorem 1. Let $\dot{x}=f(x)$ be the $C^{k}$ autonomous differential system (1) with $k \in(\mathbb{N} \backslash\{1\}) \cup\{\infty, \omega\}$ defined in $\Omega$. Assume that system (1) is $C^{r}$ completely integrable in $\Omega$ with $2 \leq r \leq k$, $\operatorname{div} \mathcal{X} \not \equiv 0$, and that the Lebesgue measure of the set of its singularities is zero. Let $H_{1}(x), \ldots, H_{n-1}(x)$ be $n-1$ functionally independent $C^{r}$ first integrals. Then the following statements hold.
(a) System (1) has always a $C^{r-1}$ Jacobian multiplier defined in a full Lebesgue measure subset of $\Omega$.
(b) If $J(x)$ is a $C^{r-1}$ Jacobian multiplier of system (1), then $J(x)$ is functionally independent of $H_{1}(x), \ldots, H_{n-1}(x)$.
(c) There exists a full Lebesgue measure subset $\Omega_{0} \subset \Omega$ in which system (1) is $C^{r-1}$ orbitally equivalent to the linear differential system

$$
\begin{equation*}
\dot{y}=y . \tag{2}
\end{equation*}
$$

We remark that statement $(a)$ is not new, it can be obtained from [20, Theorem 1.1]. We include its proof here for completeness. Statement (b) is new. Statement (c) generalizes and improves Theorem 1 of [11] extending it from two dimensional differential systems to any finite dimensional differential systems.

The flow box theorem states the existence of $n-1$ functionally independent first integrals in a neighborhood of a regular point of the
differential system $\dot{x}=f(x)$ by making it diffeomorphic to the differential system $\left(\dot{y}_{1}, \dot{y}_{2}, \ldots, \dot{y}_{n}\right)=(1,0, \ldots, 0)$. While Theorem 1 under the assumptions of the existence of $n-1$ functionally independent first integrals and the divergence non-identically zero for the $C^{k}$ differential system $\dot{x}=f(x)$ defined in an open subset $\Omega$ of $\mathbb{R}^{n}$, shows that the system is diffeomorphic to the linear differential system $\left(\dot{y}_{1}, \ldots, \dot{y}_{n}\right)=\left(y_{1}, \ldots, y_{n}\right)$ in an open and dense subset of $\Omega$.

We want to note that completely integrable differential systems defined in an open subset $\Omega$ of $\mathbb{R}^{n}$ with divergence identically zero in general cannot be diffeomorphic in an open and dense subset of $\Omega$ to the linear differential system $\dot{y}=y$. For instance this is the case of superintegrable Hamiltonian systems when $n$ is even, due to the Liouville-Arnold Theorem, see for more details $[1,2,4,12]$ and the references therein.

In the next proposition we characterize the zero Lebesgue measure subset mentioned in statement (c) of Theorem 1.

Proposition 2. Under the assumptions of Theorem 1, we can assume without loss of generality that

$$
D(x)=\operatorname{det}\left(\begin{array}{ccc}
\partial_{1} H_{1}(x) & \ldots & \partial_{n-1} H_{1}(x) \\
\vdots & \vdots & \vdots \\
\partial_{1} H_{n-1}(x) & \ldots & \partial_{n-1} H_{n-1}(x) .
\end{array}\right) \not \equiv 0
$$

Then the zero Lebesgue measure subset $\Omega \backslash \Omega_{0}$ of statement (c) of Theorem 1 is

$$
\left\{x \in \Omega \mid D(x) f_{n}(x) \operatorname{div} f(x)=0\right\} .
$$

Theorem 1 shows the existence of Jacobian multipliers for completely integrable differential systems. We now characterize the class of the Jacobian multipliers of these integrable differential systems.

Let $\mathbb{C}[x]$ be the ring of all polynomials in the variables $\left(x_{1}, \ldots, x_{n}\right)=$ $x$ with coefficients in $\mathbb{C}$. A function $H(x)$ is of Darboux type if it is of the form

$$
f_{1}^{k_{1}}(x) \ldots f_{r}^{k_{r}}(x) \exp \left(\frac{g(x)}{h(x)}\right),
$$

where $f_{i}, g, h \in \mathbb{C}[x]$ with $g$ and $h$ coprime, and $k_{i} \in \mathbb{C}$ for $i=1, \ldots, r$. Recall that the notion of Darboux function was considered by Darboux [8, 9] in 1878 for studying the existence of first integrals through invariant algebraic curves (or surfaces or hypersurfaces) of polynomial differential systems. A Darboux first integral of the system $\dot{x}=f(x)$ is a first integral of Darboux type. A polynomial differential system
$\dot{x}=f(x)$ in $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$ is Darboux integrable if it has $n-1$ functionally independent Darboux first integrals.

Let $\mathbb{C}(x)$ be the field of all rational functions in the variables $x$ with coefficients in $\mathbb{C}$. A function is Liouvillian if it belongs to the Liouvillian field extension of $\mathbb{C}(x)$, for more details on the Liouvillian field extension see for instance [16]. A polynomial differential system $\dot{x}=f(x)$ in $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$ is Liouvillian integrable if it has $n-1$ functionally independent Liouvillian first integrals.

Our next result provides the class of functions where belong the Jacobian multipliers of an integrable polynomial differential system.

Theorem 3. Assume that system (1) with $f=\left(f_{1}, \ldots, f_{n}\right)$ is an $n$-dimensional polynomial differential system with $f_{1}, \ldots, f_{n}$ relatively prime. Then the following statements hold.
(a) If system (1) is Liouvillian integrable, then it has a Darboux Jacobian multiplier.
(b) If system (1) is Darboux integrable, then it has a rational Jacobian multiplier.
(c) If system (1) is polynomial integrable, then it has a polynomial Jacobian multiplier.

Statement (a) with $n=2$ is due to Singer [16], Christopher [6] provided a different proof, see also [7, 14]. Statement (a) with $n>2$ was proved recently by Zhang in [19]. We include this statement here for completeness. Statement $(b)$ was proved in [5] for $n=2$. The proof of statement $(c)$ with $n=2$ follows from [5, 10].

This paper is organized as follows. In the next section we will prove our results. In section 3 we present an application of Theorem 1.

## 2. Proof of the main Results

For proving Theorem 1 we need the following results. The first one is due to Olver see [13, Theorem 2.16], it reveals the essential property of functional dependence.

Theorem 4. Assume that $M \subset \mathbb{R}^{n}$ is a $C^{\infty}$ manifold, and $g_{1}, \ldots, g_{k}$ are real $C^{1}$ functions on $M$. Then $g_{1}, \ldots, g_{k}$ are functionally dependent on $M$ if and only if for all $x \in M$, there exists a neighborhood $U$ of $x$ and a $C^{1}$ real function $F\left(z_{1}, \ldots, z_{k}\right)$ in $k$ variables such that

$$
F\left(g_{1}(x), \ldots, g_{k}(x)\right) \equiv 0, \quad x \in U
$$

The second result characterizes a first integral of system (1) in function of $n-1$ functionally independent first integrals.

Lemma 5. Assume that an n-dimensional $C^{1}$ autonomous differential system $\dot{x}=f(x)$ has $n-1$ functionally independent $C^{1}$ first integrals in $\Omega$, namely $H_{1}(x), \ldots, H_{n-1}(x)$. If $H$ is a $C^{1}$ first integral of system (1) defined in an open subset $\Omega_{0} \subset \Omega$, then for any $x_{0} \in \Omega_{0}$ there exists a neighborhood $U_{0} \subset \Omega_{0}$ of $x_{0}$, and a $C^{1}$ function $\Phi$ in $n-1$ variables, such that

$$
H(x)=\Phi\left(H_{1}(x), \ldots, H_{n-1}(x)\right), \quad x \in \Omega_{0} .
$$

Proof. The proof can be obtained from Theorem 4. See also for instance, Theorem 2.17 of [13] or Proposition 3 of [21, Chapter 1].

The third result provides a method for constructing first integrals using Jacobian multipliers.

Lemma 6. If $J_{1}(x)$ and $J_{2}(x)$ are Jacobian multipliers of system $\dot{x}=$ $f(x)$ in $\Omega$, then $J_{1}(x) / J_{2}(x)$ is a first integral of the system in $\Omega \backslash$ $\left\{J_{2}(x)=0\right\}$.

Proof. The proof follows from direct calculations and it is easy, see for instance [3].

We must mention that the idea of the proof of statement $(a)$ of Theorem 1 partially comes from [19, 20], and the proof of statement (c) partially comes from [11].

Proof of Theorem 1. Since $H_{1}(x), \ldots, H_{n-1}(x)$ are $C^{r}$ first integrals of system (1) in $\Omega$, by definition we have

$$
\begin{array}{r}
\partial_{1} H_{1} f_{1}+\ldots+\partial_{n-1} H_{1} f_{n-1}+\partial_{n} H_{1} f_{n}=0, \\
\vdots  \tag{3}\\
\partial_{1} H_{n-1} f_{1}+\ldots+\partial_{n-1} H_{n-1} f_{n-1}+\partial_{n} H_{n-1} f_{n}=0 .
\end{array}
$$

Since $H_{1}, \ldots, H_{n-1}$ are functionally independent in $\Omega$ we can assume without loss of generality that

$$
D(\mathcal{H}):=\operatorname{det}\left(\partial_{1} \mathcal{H}, \cdots, \partial_{n-1} \mathcal{H}\right) \neq 0, \quad x \in \Omega_{0} \subset \Omega,
$$

where $\Omega_{0}$ is a full Lebesgue measure subset of $\Omega$, and

$$
\begin{aligned}
\mathcal{H} & :=\left(H_{1}, \ldots, H_{n-1}\right)^{T}, \\
\partial_{i} \mathcal{H} & :=\left(\partial_{i} H_{1}, \ldots, \partial_{i} H_{n-1}\right)^{T}, \quad i=1, \ldots, n .
\end{aligned}
$$

where $T$ denotes the transpose of a matrix
For $i=1, \ldots, n-1$, set

$$
D_{i}(\mathcal{H}):=\operatorname{det}\left(\partial_{1} \mathcal{H}, \cdots, \partial_{i-1} \mathcal{H}, \partial_{n} \mathcal{H}, \partial_{i+1} \mathcal{H}, \cdots, \partial_{n-1} \mathcal{H}\right) .
$$

Using Cramer's rule to solve (3) with respect to $f_{1}, \ldots, f_{n-1}$, we get

$$
\begin{equation*}
f_{i}(x)=-\frac{D_{i}(\mathcal{H})}{D(\mathcal{H})} f_{n}(x), \quad i=1, \ldots, n-1 . \tag{4}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
D(\mathcal{H})\left(f_{1}(x), \ldots, f_{n-1}(x)\right)=-\left(D_{1}(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})\right) f_{n}(x) . \tag{5}
\end{equation*}
$$

Set

$$
\begin{equation*}
Q(x)=\frac{D(\mathcal{H})}{f_{n}(x)} \tag{6}
\end{equation*}
$$

It is clear that $Q$ is defined in a full Lebesgue measure subset $\Omega_{Q} \subset$ $\Omega_{0} \subset \Omega$, and is a $C^{r-1}$ function in $\Omega_{Q}$. Moreover we get from (5) that

$$
\begin{equation*}
D_{i}(\mathcal{H})(x)=-Q(x) f_{i}(x), \quad i=1, \ldots, n-1 . \tag{7}
\end{equation*}
$$

We claim that $Q(x)$ is a Jacobian multiplier of system (1) in $\Omega_{0}$. We now prove this claim. It follows from (6) and (7) that

$$
\begin{equation*}
\sum_{i=1}^{n-1} \partial_{i}\left(Q f_{i}\right)+\partial_{n}\left(Q f_{n}\right)=-\sum_{i=1}^{n-1} \partial_{i} D_{i}(\mathcal{H})+\partial_{n} D(\mathcal{H}) \tag{8}
\end{equation*}
$$

Next we only need to prove the right hand side of (8) is identically zero. Using the derivative of a determinant we get easily that

$$
\begin{aligned}
& \partial_{n} D(\mathcal{H})= \sum_{i=1}^{n-1} \operatorname{det}\left(\partial_{1} \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H}, \partial_{n} \partial_{i} \mathcal{H}, \partial_{i+1} \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}\right), \\
& \partial_{i} D_{i}(\mathcal{H})= \operatorname{det}\left(\partial_{1} \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H}, \partial_{i} \partial_{n} \mathcal{H}, \partial_{i+1} \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}\right) \\
&+\sum_{j=1, j \neq i}^{n-1} \operatorname{det}\left(\partial_{1} \mathcal{H}, \ldots, \partial_{j-1} \mathcal{H}, \partial_{i} \partial_{j} \mathcal{H}, \partial_{j+1} \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H}\right. \\
&\left.\partial_{n} \mathcal{H}, \partial_{i+1} \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}\right) .
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\partial_{n} D(\mathcal{H}) & -\sum_{i=1}^{n-1} \partial_{i} D_{i}(\mathcal{H}) \\
& =-\sum_{i=1}^{n-1} \sum_{j=1, j \neq i}^{n-1} \operatorname{det}\left(\partial_{1} \mathcal{H}, \ldots, \partial_{j-1} \mathcal{H}, \partial_{i} \partial_{j} \mathcal{H}, \partial_{j+1} \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H},\right. \\
& =0,
\end{aligned}
$$

because in the last equality we have used the fact that

$$
\begin{aligned}
& \operatorname{det}\left(\partial_{1} \mathcal{H}, \ldots, \partial_{j-1} \mathcal{H}, \partial_{i} \partial_{j} \mathcal{H}, \partial_{j+1} \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H}, \partial_{n} \mathcal{H}, \partial_{i+1} \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}\right) \\
& \quad+\operatorname{det}\left(\partial_{1} \mathcal{H}, \ldots, \partial_{j-1} \mathcal{H}, \partial_{n} \mathcal{H}, \partial_{j+1} \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H}, \partial_{j} \partial_{i} \mathcal{H}, \partial_{i+1} \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}\right) \\
& \quad=0
\end{aligned}
$$

Now it follows from (8) that

$$
\sum_{i=1}^{n-1} \partial_{i}\left(Q f_{i}\right)+\partial_{n}\left(Q f_{n}\right) \equiv 0
$$

That is $Q(x)$ is a Jacobian multiplier, and consequently statement ( $a$ ) follows.

For proving (b) we first prove that $Q(x)$ is functionally independent of $H_{1}(x), \ldots, H_{n-1}(x)$. Since

$$
A:=\operatorname{det}\left(\begin{array}{cccc}
\partial_{1} Q & \cdots & \partial_{n-1} Q & \partial_{n} Q \\
\partial_{1} H_{1} & \cdots & \partial_{n-1} H_{1} & \partial_{n} H_{1} \\
\vdots & \vdots & \vdots & \vdots \\
\partial_{1} H_{n-1} & \cdots & \partial_{n-1} H_{n-1} & \partial_{n} H_{n-1}
\end{array}\right)=\sum_{i=1}^{n} \partial_{i} Q Q_{i}^{*} \text {, }
$$

where $Q_{i}^{*}=(-1)^{1+i} M_{1 i}$ with $M_{1 i}$ the minor defined to be the determinant of the $(n-1) \times(n-1)$-matrix that results from the matrix by removing the first row and the $i$ th column. Clearly

$$
D(\mathcal{H})=M_{1 n}, \quad D_{i}(\mathcal{H})=(-1)^{n-1-i} M_{1 i}, \quad i=1, \ldots, n-1 .
$$

So we have

$$
\begin{aligned}
A & =\sum_{i=1}^{n-1} \partial_{i} Q(-1)^{n} D_{i}(\mathcal{H})+\partial_{n} Q(-1)^{n+1} D(\mathcal{H}) \\
& =\sum_{i=1}^{n-1} \partial_{i} Q(-1)^{n}\left(-Q f_{i}\right)+\partial_{n} Q(-1)^{n+1} Q f_{n}=(-1)^{n+1} Q \mathcal{X}(Q),
\end{aligned}
$$

where in the second equality we have used (6) and (7). Recall that $\mathcal{X}$ is the vector field associated to system (1).

Since $Q$ is not a first integral of system (1), it follows that $\mathcal{X}(Q)=$ $Q \operatorname{div} \mathcal{X}$ is a nonzero function on $\Omega_{Q}$. This implies that $A$ can only vanish in a zero Lebesgue measure subset of $\Omega$. This shows that $Q, H_{1}, \ldots, H_{n-1}$ are functionally independent on $\Omega$.

By assumption $J(x)$ is a $C^{r-1}$ Jacobian multiplier, then by Lemma 6 we obtain that $Q / J$ is a first integral of system (1). Then we get from Lemma 5 that $Q / J$ can be locally expressed as

$$
\begin{equation*}
Q / J=\Phi\left(H_{1}, \ldots, H_{n-1}\right), \tag{9}
\end{equation*}
$$

where $\Phi$ is a $C^{1}$ function. If $J$ is functionally dependent of $H_{1}, \ldots, H_{n-1}$, we get from Theorem 4 that $J$ can be locally expressed as a $C^{1}$ function of $H_{1}, \ldots, H_{n-1}$. This together with (9) shows that $Q$ is functionally dependent of $H_{1}, \ldots, H_{n-1}$, a contradiction. This proves that $J$ is functionally independent of $H_{1}, \ldots, H_{n-1}$, so statement (b) follows.

By statement (a) system (1) has a $C^{r-1}$ Jacobian multiplier $J(x)$. From (b) it follows that the functions $J, H_{1}, \ldots, H_{n-1}$ are functionally independent. So there exists a full Lebesgue measure subset $\Omega_{0} \subset \Omega$ such that $\nabla J, \nabla H_{1}, \ldots, \nabla H_{n-1}$ have rank $n$ at all points of $\Omega_{0}$. Taking the invertible change of variables

$$
y_{n}=J(x), \quad y_{i}=J(x) H_{i}, \quad i=1, \ldots, n-1, \quad x \in \Omega_{0}
$$

we have

$$
\begin{aligned}
\dot{y}_{n} & =\dot{J}=J \operatorname{div} f=y_{n} \operatorname{div} f \\
\dot{y}_{i} & =\dot{J} H_{i}+J \dot{H}=\dot{J} H_{i}=J \operatorname{div} f H_{i}=y_{i} \operatorname{div} f .
\end{aligned}
$$

This proves that system (1) is $C^{r-1}$ orbitally equivalent to the linear system (2). Hence statement (c) follows. This completes the proof of the theorem.

Proof of Proposition 2. We note that $D(x)=D(\mathcal{H})(x)$. The latter was defined in the proof of Theorem 1. By assumption it follows that $D(x)$ is a $C^{1}$ function and does not vanish on a full Lebesgue measure subset of $\Omega$. From (4) of the proof of Theorem 1 we have that $f_{n}(x)$ does not vanish on a full Lebesgue measure subset of $\Omega$. Otherwise all $f_{i}$ 's are almost zero, and so system (1) has a positive Lebesgue measure subset of singularities, a contradiction.

Choose the Jacobian multiplier $Q=D(x) / f_{n}(x)$, then from the proof of Theorem 1 we get that

$$
\begin{aligned}
\operatorname{det}\left(\begin{array}{ccc}
\partial_{1} Q & \cdots & \partial_{n} Q \\
\partial_{1}\left(Q H_{1}\right) & \cdots & \partial_{n}\left(Q H_{1}\right) \\
\vdots & \vdots & \vdots \\
\partial_{1}\left(Q H_{n-1}\right) & \cdots & \partial_{n}\left(Q H_{n-1}\right)
\end{array}\right) & =(-1)^{n+1} Q^{n} \mathcal{X}(Q) \\
& =(-1)^{n+1}\left(\frac{D(x)}{f_{n}(x)}\right)^{n+1} \operatorname{div} \mathcal{X}
\end{aligned}
$$

This shows that the transformation from system $\dot{x}=f(x)$ to $\dot{y}=y$ defined by

$$
y_{1}=Q(x) H_{1}(x), \ldots, y_{n-1}=Q(x) H_{n-1}(x), y_{n}=Q(x)
$$

is a $C^{r-1}$ diffeomorphism on $\Omega_{0}:=\left\{x \in \Omega: D(x) f_{n}(x) \operatorname{div} \mathcal{X}(x) \neq 0\right\}$. Obviously $\Omega \backslash \Omega_{0}=\left\{x \in \Omega: D(x) f_{n}(x) \operatorname{div} \mathcal{X}(x)=0\right\}$ is a zero Lebesgue measure subset of $\Omega$. This proves the proposition.

Proof of Theorem 3. Recall that statement (a) was proved in [16, 19]. We now prove statements (b) and (c).

Let $H_{1}(x), \ldots, H_{n-1}(x)$ be $n-1$ functionally independent Darboux first integrals of the polynomial differential system (1). Taking $G_{i}(x)=$ $\log H_{i}(x), i=1, \ldots, n-1$. Then $G_{1}(x), \ldots, G_{n-1}(x)$ are also functionally independent first integrals of system (1).

We assume without loss of generality that

$$
\mathcal{G}(x):=\operatorname{det}\left(\begin{array}{ccc}
\partial_{1} G_{1}(x) & \cdots & \partial_{n-1} G_{1}(x) \\
\vdots & \vdots & \vdots \\
\partial_{1} G_{n-1}(x) & \cdots & \partial_{n-1} G_{n-1}(x)
\end{array}\right)
$$

is not zero in $\mathbb{R}^{n}$ except perhaps a zero Lebesgue measure subset. Then we get from the proof of Proposition 2 that $f_{n} \neq 0$ in a full Lebesgue measure subset of $\Omega$. Since $H_{i}(x)$ is a Darboux function, we assume that it is of the form

$$
H_{i}(x)=g_{i 1}^{k_{i 1}} \ldots g_{i r_{i}}^{k_{i r_{i}}} \exp \left(\frac{q_{i}(x)}{h_{i}(x)}\right),
$$

where $g_{i j}, q_{i}, h_{i}$ are polynomials, and $k_{i j} \in \mathbb{C}, j=1, \ldots, r_{i}$. Computing the partial derivative of $G_{i}(x)=\log H_{i}(x)$ with respect to $x_{s}$ for $s=$ $1, \ldots, n$, we get

$$
\partial_{s} G_{i}(x)=\sum_{j=1}^{r_{i}} k_{i j} \frac{\partial_{s} g_{i j}}{g_{i j}}+\frac{h_{i} \partial_{s} q_{i}-q_{i} \partial_{s} h_{i}}{h_{i}^{2}} .
$$

This is a rational function. So $\mathcal{G}(x)$ is also a rational function.
By Theorem 1 and its proof it follows that system (1) has the rational Jacobian multiplier $Q(x)=\mathcal{G}(x) / f_{n}(x)$, because $f_{n}$ is a polynomial. Hence statement (b) follows.

Finally we prove statement (c). Let $H_{1}, \ldots, H_{n-1}$ be $n-1$ functionally independent polynomial first integrals of the polynomial differential system (1). Here we will use the notations defined in the proof of Theorem 1. We assume that $D(\mathcal{H}):=\operatorname{det}\left(\partial_{1} \mathcal{H}, \cdots, \partial_{n-1} \mathcal{H}\right) \neq 0$ in $\mathbb{R}^{n}$ except perhaps a zero Lebesgue measure subset. Recall that $\mathcal{H}=\left(H_{1}, \ldots, H_{n-1}\right)^{T}$ and $\partial_{i} \mathcal{H}=\left(\partial_{i} H_{1}, \ldots, \partial_{i} H_{n-1}\right)^{T}, i=1, \ldots, n$. The proof of Theorem 1 shows that

$$
D(\mathcal{H})\left(f_{1}(x), \ldots, f_{n-1}(x)\right)=-\left(D_{1}(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})\right) f_{n}(x) .
$$

Since $D(\mathcal{H}), D_{1}(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})$ are polynomials, and $f_{1}(x), \ldots, f_{n}(x)$ are relatively prime polynomials, it verifies that $f_{n}(x)$ divides $D(\mathcal{H})(x)$.

Hence system (1) has the polynomial Jacobian multiplier

$$
Q(x)=D(\mathcal{H})(x) / f_{n}(x) .
$$

This proves statement $(c)$, and consequently the theorem.

## 3. An application of Theorem 1

Consider the differential system

$$
\begin{equation*}
\dot{x}=-y-z, \quad \dot{y}=x, \quad \dot{z}=x z, \tag{10}
\end{equation*}
$$

which is the only completely integrable case of the Rössler differential system constructed by Rössler [15] in 1976. This unique integrable Rössler differential system was first proved in [18]. Recently system (10) was studied from the Poisson dynamics point of view, see [17].

We can check that system (10) has the two functionally independent first integrals

$$
H_{1}(x, y, z)=\frac{1}{2}\left(x^{2}+y^{2}\right)+z, \quad H_{2}(x, y, z)=e^{-y} z,
$$

and the Jacobian multiplier $J=e^{-y}$. We can check that the transformation of variables

$$
\begin{equation*}
u=J(x, y, z), \quad v=J(x, y, z) H_{1}(x, y, z), \quad w=J(x, y, z) H_{2}(x, y, z), \tag{11}
\end{equation*}
$$

is invertible in the region $\Omega_{0}:=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x \neq 0\right\}$, because the Jacobian determinant of this transformation is $x e^{-4 y}$. By Theorem 1, system (10) is transformed to

$$
\dot{u}=u, \quad \dot{v}=v, \quad \dot{w}=w,
$$

via the change of variables (11) in $\Omega_{0}$. We note that the divergence of system (10) is $x$.

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