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NONINTEGRABILTY OF A HALPHEN SYSTEM

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We study the Halphen system with real variables and real constants. We show that in the case where at least one constant is nonzero, this system does not admit any first integral that can be described by formal power series. It hence follows that analytic first integrals do not exist. Furthermore, we prove that first integrals of the Darboux type also do not exist.

Keywords: Halphen system, analytic first integral

1. Introduction to the problem

We consider the system

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1,x_2,x_3) = x_2x_3 - x_1(x_2+x_3) - \alpha_1^2(x_1-x_2)(x_3-x_1),\\
\dot{x}_2 &= F_2(x_1,x_2,x_3) = x_3x_1 - x_2(x_3+x_1) - \alpha_2^2(x_2-x_3)(x_1-x_2),\\
\dot{x}_3 &= F_3(x_1,x_2,x_3) = x_1x_2 - x_3(x_1+x_2) - \alpha_3^2(x_3-x_1)(x_2-x_3),
\end{aligned} \tag{1}$$

where x_1, x_2, and x_3 are real variables and α_1, α_2, and α_3 are real constants. We call system (1) the second Halphen system (Halphen himself called it the second system [1]) because system (1) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ becomes the so-called classical Halphen system. The classical Halphen system is a famous model (see, e.g., [1]–[3]), which first appeared in Darboux's work [1] and was later solved by Halphen [3]. One of the circumstances making this system famous is that this system, as was shown, is equivalent to the Einstein field equations for a diagonal self-dual Bianchi-IX metric with a Euclidean signature (see [2], [4]). The classical Halphen system also arises in similarity reductions of associativity equations on a three-dimensional Frobenius manifold [5].

From the standpoint of integrability, the classical Halphen system has been intensively studied using different theories. One of the main results in this direction is that system (1) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ can be explicitly integrated because we can express its general solution in terms of elliptic integrals (see [3], [6], [7]), but the first integrals are not global and are multivalued nonalgebraic functions (see [8]). Other results that we mention are in [9], where the so-called Darboux polynomials were used to prove that system (1) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ does not admit a nonconstant algebraic first integral, and finally in [10], where a complete characterization of the formal and analytic first integrals was provided.

Our first aim in this paper is to show the nonexistence of first integrals of system (1) that can be described by formal series. We restrict ourself to the case where $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. The case

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