THE CO–CIRCULAR CENTRAL CONFIGURATIONS OF THE 5–BODY PROBLEM

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Abstract. Chenciner in 2001 asked: *Is the regular n–gon with equal masses the unique central configuration such that all the bodies lie on a circle, and the center of mass coincides with the center of the circle*? This question has a positive answer for $n = 3$. Hampton in 2003 proved that also this question has a positive answer for $n = 4$. Here we provide a positive answer for $n = 5$.

1. Introduction

The main problem of the classical Celestial Mechanics is the *n-body problem*; i.e. the description of the motion of n particles of positive masses under their mutual Newtonian gravitational forces. This problem is completely solved only when $n = 2$, and for $n > 2$ there are only few partial results.

Consider the Newtonian n–body problem in the plane $\mathbb{R}^2$, i.e.

$$\ddot{\mathbf{r}}_i = \sum_{j=1,\ j\neq i}^{n} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad \text{for} \quad i = 1, \ldots, n.$$

Here m_i are the masses of the bodies, $\mathbf{r}_i \in \mathbb{R}^2$ are their positions, and $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ are their mutual distances. The vector $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \mathbb{R}^{2n}$ is called the *configuration* of the system. The differential equations are well–defined if the configuration is of non–collision type, i.e. $r_{ij} \neq 0$ when $i \neq j$.

The *total mass* and the *center of mass* of the n bodies are

$$M = m_1 + \ldots + m_n, \qquad c = \frac{1}{M}\left(m_1\mathbf{r}_1 + \cdots + m_n\mathbf{r}_n\right),$$

respectively. A configuration $\mathbf{r}$ is a *central configuration* if the acceleration vectors of the bodies satisfy

$$\sum_{j=1,\ j\neq i}^{n} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} + \lambda(\mathbf{r}_i - c) = 0, \quad \text{for} \quad i = 1, \ldots, n, \tag{1}$$

Central configurations started to be studied in the second part of the 18th century, there is an extensive literature concerning these solutions. For a classical background, see the sections on central configurations in the books of Wintner [22] and Hagihara [9]. For a modern background see, for instance, the papers of Albouy and Chenciner [2], Albouy and Kaloshin [3], Hampton and Moeckel [11], Moeckel [14], Palmore [17], Saari [18], Schmidt [19], Xia [23], ... One of the reasons why central configurations are important is that they allow to obtain the unique explicit solutions in function of the time of the n–body problem known until now,

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the *homographic solutions* for which the ratios of the mutual distances between the bodies remain constant. They are also important because the total collision or the total parabolic escape at infinity in the n–body problem is asymptotic to central configurations, see for more details Saari [18]. Also if we fix the total energy h and the angular momentum c of the n–body problem, then some of the bifurcation points (h, c) for the topology of the level sets with energy h and angular momentum c are related with the central configurations, see Meyer [15] and Smale [20] for a full background on these topics.

Moulton [16] proved that for a fixed mass vector $m = (m_1, \ldots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration, up to translation and scaling.

For an arbitrary given set of masses the number of classes of planar non-collinear central configurations of the n–body problem has been only solved for $n = 3$. In this case they are the three collinear and the two equilateral triangle central configurations, due to Euler [7] and Lagrange [13] respectively. Recently, Hampton and Moeckel [11] proved that for any choice of four masses there exist a finite number of classes of central configurations. For five or more masses this result is unproved, but recently an important contribution to the case of five masses has been made by Albouy and Kaloshin [3].

A periodic solution $(\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t))$ of the planar n–body problem of period T and masses $m_1, \ldots, m_n$ is a *choreography* if $(\mathbf{r}_1(t), \mathbf{r}_2(t), \ldots, \mathbf{r}_n(t)) = (\mathbf{r}(t + T/n), \mathbf{r}(t+2T/n), \ldots, \mathbf{r}_n(t+T) = \mathbf{r}(t))$, i.e. all n bodies follow the same curve $\mathbf{r}$(t) with equal time spacing. In 2001 Chenciner [5] trying to answer the question: *Do there exist planar choreographies whose masses are not all equal*? stated another question: *Is the regular n–gon with equal masses the unique central configuration such that all the bodies lie on a circle, and the center of mass coincides with the center of the circle*?

It is not difficult to show that this last question has a positive answer for $n = 3$. In 2003 Hampton [10] proved that also this question has a positive answer for $n = 4$. Up to now this question remained unsolved for $n > 4$. The goal of this paper is to provide a positive answer for $n = 5$.

Our proof is analytic and in one step is a computer assisted proof. More precisely, at some moment of the proof we need to compute the real roots of two polynomials of degrees 70 and 172 in the interval $(0, 2)$. First we detect the exact number of real roots of those polynomials in such interval using the Sturm method (see [12] or [21]). This method is implemented in mathematica and Mapple. After we compute such roots as many precision as we want using these mentioned algebraic manipulators. Only one pair of these roots satisfy the equations of the co–circular central configurations. Moreover, this pair has the exact expression given in (14). On the other hand, there are other ways to justify that the computation of these real roots do not offer any problem, because our polynomials have integer coefficients, and they can be evaluated exactly on rational numbers, for more details see page 2641 of [1].

On the other hand, recently some authors studied in [4, 6] studied the central configurations of the 4– and 5–body problem with all the bodies on a circle.

2. Co-circular central configurations

In this work a central configuration of the n–body problem satisfying that all the masses are on a circle centered at the origin of coordinates and such that its center of mass is located at the origin will be called simply *co–circular.*

It is well known that the set of all central configurations is invariant by rotations and homothecies centered at the center of mass. So we can restricted our study on the co–circular central configurations to the ones which are on the circle of radius one centered at the origin of coordinates. Thus the position of the mass m_k is given by

$$(c_k, s_k) = (\cos\theta_k, \sin\theta_k),$$

with $\theta_i \in [0, 2\pi)$ and $\theta_i \neq \theta_j$ if $i \neq j$. The angles of a such co–circular central configuration will be denoted by

$$\{\theta_1, \ldots, \theta_n\},$$

and without loss of generality we can assume that

$$0 \le \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi.$$

These angles θ are measured in counterclockwise sense with origin at the positive x–axis.

The equations for the central configurations (1) restricted to the co–circular ones become

$$\begin{aligned} e_i = & \sum_{j=1,\ j\neq i}^{n} \frac{m_j(c_j - c_i)}{r_{ij}^3} + \lambda c_i = 0, \\ e_{i+n} = & \sum_{j=1,\ j\neq i}^{n} \frac{m_j(s_j - s_i)}{r_{ij}^3} + \lambda s_i = 0, \end{aligned} \tag{2}$$

for $i = 1, \ldots, n$ where $r_{ij} = \sqrt{(c_i - c_j)^2 + (s_i - s_j)^2}$, and additionally

$$\begin{aligned} e_{2n+1} = & \sum_{j=1}^{n} m_j c_j = 0, \\ e_{2n+2} = & \sum_{j=1}^{n} m_j s_j = 0, \end{aligned}$$

Proposition 1. *Let $cc = \{\theta_1, \ldots, \theta_n\}$ be a co–circular central configuration. Then the following statements hold.*

(a) *The configuration cc_x symmetric with respect to the x–axis of the configuration cc is also a co–circular central configuration. Moreover $cc_x = \{2\pi - \theta_n, \ldots, 2\pi - \theta_1\}$.*

(b) *The configuration cc_y symmetric with respect to the y–axis of the configuration cc is also a co–circular central configuration. Moreover $cc_y = \{\pi - \theta_s, \pi - \theta_{s-1}, \ldots, \pi - \theta_1, 3\pi - \theta_n, 3\pi - \theta_{n-1}, \ldots, 3\pi - \theta_{s+1}\}$ if $0 \le \theta_1 < \ldots < \theta_s \le \pi < \theta_{s+1} < \ldots < \theta_n < 2\pi$.*

Proof. If the configuration cc is $(c_1, s_1, c_2, s_2, \ldots, c_n, s_n)$, then the configuration cc_x is $(c_1, -s_1, c_2, -s_2, \ldots, c_n, -s_n)$. Since cc satisfies the equations (2), then also cc_x

satisfies the equations (2). Therefore, cc_x is a co–circular central configuration. It is easy to check that $cc_x = \{2\pi - \theta_n, \dots, 2\pi - \theta_1\}$. Hence statement (a) is proved.

Now the configuration cc_y is $(-c_1, s_1, -c_2, s_2, \dots, -c_n, s_n)$. Since cc satisfies the equations (2), then also cc_y satisfies the equations (2). Therefore, cc_y is a co–circular central configuration. It follows easily that $cc_y = \{\pi - \theta_s, \pi - \theta_{s-1}, \dots, \pi - \theta_1, 3\pi - \theta_n, 3\pi - \theta_{n-1}, \dots, 3\pi - \theta_{s+1}\}$ if $0 \le \theta_1 < \dots < \theta_s \le \pi < \theta_{s+1} < \dots < \theta_n < 2\pi$. This completes the proof of statement (b). □

3. Co-circular central configurations for $n = 5$

In all this section $n = 5$.

Theorem 2. *For the 5–body problem the unique co–circular central configuration is the regular 5–gon with equal masses.*

Proof. Since the co–circular central configurations are invariant for rotations with respect to the origin of coordinates, and for symmetries with respect to the x–axis and to the y–axis, we can assume without loss of generality that we have a co–circular central configuration $cc = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ such that

$$c_5 = c_2, \quad s_5 = -s_2 < 0, \quad c_1 > 0 \quad \text{and} \quad m_2 \ge m_5.$$

More precisely, first we localize the biggest mass and we call it m_1. After we rename the masses in counterclockwise starting with m_1. We rotate the co–circular central configuration and we put it so that $s_5 = -s_2$ with $s_2 > 0$. If $m_2 < m_5$ we do a symmetry with respect to the x–axis, and the new co–circular central configuration is renamed in counterclockwise starting again with m_1. So we obtain $m_2 \ge m_5$.

Note that if the co-circular central configuration is invariant with respect to the x-axis, then $\theta_1 = 0$ and $\theta_3 = -\theta_4$. Thus $s_1 = 0$, $c_1 = 1$, $s_3 = -s_4$ and $c_3 = c_4$. This will be used later on.

Using that the center of mass is at the origin of the circle we get

$$c_4 = -\frac{m_1 c_1 + (m_2 + m_5) c_2 + m_3 c_3}{m_4} \quad \text{and} \quad s_4 = -\frac{m_1 s_1 + (m_2 - m_5) s_2 + m_3 s_3}{m_4}. \tag{3}$$

The scheme of the proof is the following. We shall divide the proof in two cases, and each one of these cases in some subcases. We shall see that the subcase 1.2 will provide the co–circular central configuration formed by the regular 5–gon with equal masses at the vertices, and that all the other subcases do not provide co–circular central configurations.

Case 1: $m_1 s_1 + (m_2 - m_5) s_2 = 0$. We consider two subcases.

Subcase 1.1: $m_2 > m_5$. Hence

$$s_1 = \frac{m_5 - m_2}{m_1} s_2. \tag{4}$$

Since $s_2 > 0$ and $m_2 > m_5$ we get that $s_1 < 0$. Moreover we have that

$$s_4 = -\frac{m_3}{m_4} s_3.$$

Therefore, again $s_3 > 0$ and consequently $s_4 < 0$. Hence $(m_2 - m_5)s_2 + m_3 s_3 \ne 0$. Now we solve the system

$$c_j^2 + s_j^2 = 1 \quad \text{for} \quad j = 1, 4,$$

with respect to the variables s_1 and c_1. It has two different solutions $R^j = \{c_{1,j}, s_{1,j}\}$ for $j = 1, 2$ with

$$\begin{aligned}
s_{1,1} &= -\frac{m_1}{D_1 D_3}\big((m_1^2 D_1^2 + D_1^2\big(D_1^2 + D_2^2 - m_4^2\big) - D_2 S_1\big),\\
c_{1,1} &= -\frac{m_1}{D_3}\big(D_2\big(m_1^2 + D_1^2 + D_2^2 - m_4^2\big) + S_1\big),\\
s_{1,2} &= -\frac{m_1}{D_1 D_3}\big((m_1^2 D_1^2 + D_1^2\big(D_1^2 + D_2^2 - m_4^2\big) + D_2 S_1\big),\\
c_{1,2} &= -\frac{m_1}{D_3}\big(D_2\big(m_1^2 + D_1^2 + D_2^2 - m_4^2\big) - S_1\big),
\end{aligned}$$

being

$$D_1 = (m_2 - m_5)s_2 + m_3 s_3, \quad D_2 = c_3 m_3 + c_2(m_2 + m_5), \quad D_3 = 2m_1^2\big(D_1^2 + D_2^2\big).$$

and

$$S_1 = \sqrt{D_1^2\big(2m_1^2(D_1^2 + D_2^2 + m_4^2) - (m_1^4 + D_1^2 + D_2^2 - m_4^2)\big)}.$$

It follows from Proposition 1(a) that the configuration cc_x symmetric with respect to the x–axis of the co–circular central configuration cc is also a co–circular central configuration. Then either the solution cc is invariant with respect to the x-axis, or

$$c_{1,1} = c_{1,2}\big|_{s_2 \to -s_2,\ s_3 \to -s_3} \quad \text{and} \quad s_{1,1} = -s_{1,2}\big|_{s_2 \to -s_2,\ s_3 \to -s_3}. \tag{5}$$

In the first case as it was before mentioned this implies in particular that $s_1 = 0$, in contradiction with the fact that we are under the assumptions of Subcase 1.1. Hence (5) holds. Then we get the conditions $S_1 = 0$ and $D_2 S_1 = 0$, respectively. So $S_1 = 0$. Since $D_1 > 0$ we get that

$$2m_1^2(D_1^2 + D_2^2 + m_4^2) - (m_1^4 + D_1^2 + D_2^2 - m_4^2) = 0.$$

Solving with respect to m_1 we get four possible solutions that we call them $m_{1,j}$ for $j = 1, 2, 3, 4$:

$$m_{1,j} = (-1)^j m_4 - \sqrt{D_1^2 + D_2^2} \quad \text{and} \quad m_{1,j+2} = (-1)^j m_4 + \sqrt{D_1^2 + D_2^2},\ j = 1, 2.$$

Since m_1 is positive, solution $m_{1,1}$ is never satisfied. So, we consider only $m_{1,2}$, $m_{1,3}$ and $m_{1,4}$.

If $m_1 = m_{1,2}$ we have that

$$s_1 = -s_4 = \frac{(m_2 - m_5)s_2 + m_3 s_3}{\sqrt{D_1^2 + D_2^2}}.$$

Note that due to (4) we have that $s_1 < 0$. Then $s_4 > 0$ in contradiction with the fact that in subcase 1.1 we have that $s_4 < 0$. So the solution $m_1 = m_{1,2}$ is not possible.

If $m_1 = m_{1,3}$ we obtain that

$$s_1 = s_4 = -\frac{(m_2 - m_5)s_2 + m_3 s_3}{\sqrt{D_1^2 + D_2^2}}, \quad c_1 = c_4 = -\frac{c_3 m_3 + (m_2 + m_5)c_2}{\sqrt{D_1^2 + D_2^2}}.$$

This implies that there is a collision between the masses m_1 and m_4, a contradiction. Hence the solution $m_1 = m_{1,3}$ is not possible.

If $m_1 = m_{1,4}$ we get that

$$s_1 = -s_4 = -\frac{(m_2 - m_5)s_2 + m_3 s_3}{\sqrt{D_1^2 + D_2^2}}.$$

Therefore the solution $m_1 = m_{1,4}$ is not possible following the same arguments of the solution $m_1 = m_{1,2}$. This completes the proof of subcase 1.1 showing that under the assumptions of this subcase there are no co–circular central configurations.

Subcase 1.2: $m_2 = m_5$. Then, since $m_1 s_1 + (m_2 - m_5)s_2 = 0$ we have that $s_1 = 0$. Then $c_1 = 1$. Then $r_{15} = r_{12}$. Moreover, we have that

$$c_4 = -\frac{m_1 + 2m_2 c_2 + m_3 c_3}{m_4} \quad \text{and} \quad s_4 = -\frac{m_3}{m_4} s_3. \tag{6}$$

This last equality implies that $s_3 > 0$ and $s_4 < 0$.

Now equation $e_6 = 0$ of (2) reduces to

$$m_3 s_3 \left(\frac{1}{r_{13}^3} - \frac{1}{r_{14}^3} \right) = 0.$$

Therefore $r_{14} = r_{13}$. Consequently $c_4 = c_3$ and $s_4 = -s_3$. So, from (6) we get that $m_4 = m_3$ and $m_1 = -2(c_2 m_2 + c_3 m_3)$. Note that $c_3 < 0$ due to the fact that the center of mass is at the center of the circle, i.e. at the origin of coordinates.

Clearly we have

$$r_{45} = r_{23}, \quad r_{35} = r_{24}, \quad r_{25} = 2s_2, \quad r_{34} = 2s_3.$$

Now from the ten equations (2) only equations $e_k = 0$ for $k = 1, 3, 7, 8$ remain independent, because $e_2 = 0$ can be obtained from the linear combination $m_1 e_1 + 2m_3 e_3 + 2m_2 e_2 = 0$, $e_4 = e_3$, $e_5 = e_2$, $e_6 = 0$, $e_9 = -e_8$ and $e_{10} = -e_7$. From $e_1 = 0$ we obtain

$$\lambda = \frac{2(\mathrm{c}2 - 1)m_2}{r_{12}^3} + \frac{2(\mathrm{c}3 - 1)m_3}{r_{13}^3}.$$

Substituting λ in $e_k = 0$ for $k = 3, 7, 8$ we obtain the equations

$$\begin{aligned}
f_1 = {} & -r_{12}^2 r_{24}^3 r_{23}^3 - r_{13}^2 r_{24}^3 r_{23}^3 - r_{12} r_{13} r_{24}^3 r_{23}^3 + 2r_{24}^3 r_{23}^3 + r_{12} r_{13}^2 r_{23}^3 + r_{12}^2 r_{13} r_{23}^3 \\
& + r_{12} r_{13}^2 r_{24}^3 + r_{12}^2 r_{13} r_{24}^3, \\
f_2 = {} & \frac{1}{2(r_{12} - 2) r_{12}^3 (r_{12} + 2) r_{13} r_{23}^3 r_{24}^3} \Big(2m_2 r_{13} r_{23}^3 r_{24}^3 \\
& - m_3 (r_{12} - 2) r_{12}^2 (r_{12} + 2) r_{13}^2 \sqrt{4 - r_{13}^2} (r_{23} - r_{24}) \left(r_{23}^2 + r_{24} r_{23} + r_{24}^2 \right) \\
& - (r_{12} - 2)(r_{12} + 2) \sqrt{4 - r_{12}^2} \big(- m_3 r_{12}^3 r_{24}^3 r_{23}^3 + m_3 r_{13}^3 r_{24}^3 r_{23}^3 \\
& - 2m_2 r_{13} r_{24}^3 r_{23}^3 - 2m_3 r_{13} r_{24}^3 r_{23}^3 + m_3 r_{12}^3 r_{13} r_{23}^3 + m_3 r_{12}^3 r_{13} r_{24}^3 \big) \Big), \\
f_3 = {} & \frac{1}{2r_{12} (r_{13} - 2) r_{13}^3 (r_{13} + 2) r_{23}^3 r_{24}^3} \Big(2m_3 r_{12} r_{23}^3 r_{24}^3 r_{13} \\
& - m_2 r_{12}^2 \sqrt{4 - r_{12}^2} (r_{13} - 2)(r_{13} + 2)(r_{23} - r_{24})(r_{23}^2 + r_{24} r_{23} + r_{24}^2) r_{13}^3 \\
& - (r_{13} - 2)(r_{13} + 2) \sqrt{4 - r_{13}^2} \big(m_2 r_{12} r_{13}^3 r_{23}^3 + m_2 r_{12}^3 r_{24}^3 r_{23}^3 - m_2 r_{13}^3 r_{24}^3 r_{23}^3 \\
& - 2m_2 r_{12} r_{24}^3 r_{23}^3 - 2m_3 r_{12} r_{24}^3 r_{23}^3 + m_2 r_{12} r_{13}^3 r_{24}^3 \big) r_{13} \Big),
\end{aligned}$$

respectively; where we have used that

$$c_2 = \frac{1}{2}\left(2 - r_{12}^2\right), \quad c_3 = \frac{1}{2}\left(2 - r_{13}^2\right).$$

In what follows we shall omit the denominators from f_2 and f_3 because they cannot be zero in a co–circular central configuration of the 5–body problem. For instance, r_{13} cannot be 2, otherwise r_{14} would be also equal to 2, and we will have a collision between the masses m_3 and m_4.

The system formed by the two equations $f_2 = 0$ and $f_3 = 0$ is a homogeneous linear system in the variables m_2 and m_3. Since we are interested in positive solutions for m_2 and m_3, the determinant of this homogeneous linear system must be zero, obtaining the equation

$$\begin{aligned}
f_4 = \;& 4r_{12}^2(r_{13}-2)r_{13}^2(r_{13}+2)\sqrt{4-r_{13}^2}\,r_{23}^6r_{24}^6 + r_{12}^2r_{13}^2\Big(r_{13}^4r_{23}^6r_{12}^8 - 4r_{13}^2r_{23}^6r_{12}^8 \\
& -r_{13}^4r_{24}^6r_{12}^8 + r_{13}^3r_{23}^3r_{24}^6r_{12}^8 - 4r_{13}r_{23}^3r_{24}^6r_{12}^8 + 4r_{13}^2r_{24}^6r_{12}^8 - r_{13}^3r_{23}^6r_{24}^3r_{12}^8 \\
& +4r_{13}r_{23}^6r_{24}^3r_{12}^8 - 8r_{13}^4r_{23}^6r_{12}^6 + 32r_{13}^2r_{23}^6r_{12}^6 + 8r_{13}^4r_{24}^6r_{12}^6 - r_{13}^5r_{23}^3r_{24}^6r_{12}^6 \\
& +16r_{13}r_{23}^3r_{24}^6r_{12}^6 - 32r_{13}^2r_{24}^6r_{12}^6 + r_{13}^5r_{23}^6r_{24}^3r_{12}^6 - 16r_{13}r_{23}^6r_{24}^3r_{12}^6 \\
& -r_{13}^6r_{23}^3r_{24}^6r_{12}^5 + 6r_{13}^4r_{23}^3r_{24}^6r_{12}^5 - 8r_{13}^2r_{23}^3r_{24}^6r_{12}^5 + r_{13}^6r_{23}^6r_{24}^3r_{12}^5 \\
& -6r_{13}^4r_{23}^6r_{24}^3r_{12}^5 + 8r_{13}^2r_{23}^6r_{24}^3r_{12}^5 + r_{13}^8r_{23}^6r_{12}^4 - 8r_{13}^6r_{23}^6r_{12}^4 + 32r_{13}^4r_{23}^6r_{12}^4 \\
& -64r_{13}^2r_{23}^6r_{12}^4 - r_{13}^8r_{24}^6r_{12}^4 + 8r_{13}^6r_{24}^6r_{12}^4 - 32r_{13}^4r_{24}^6r_{12}^4 + 6r_{13}^5r_{23}^3r_{24}^6r_{12}^4 \\
& -32r_{13}^3r_{23}^3r_{24}^6r_{12}^4 + 32r_{13}r_{23}^3r_{24}^6r_{12}^4 + 64r_{13}^2r_{24}^6r_{12}^4 - 6r_{13}^5r_{23}^6r_{24}^3r_{12}^4 \\
& +32r_{13}^3r_{23}^6r_{24}^3r_{12}^4 - 32r_{13}r_{23}^6r_{24}^3r_{12}^4 + r_{13}^8r_{23}^3r_{24}^6r_{12}^3 - 32r_{13}^4r_{23}^3r_{24}^6r_{12}^3 \\
& +64r_{13}^2r_{23}^3r_{24}^6r_{12}^3 - r_{13}^8r_{23}^6r_{24}^3r_{12}^3 + 32r_{13}^4r_{23}^6r_{24}^3r_{12}^3 - 64r_{13}^2r_{23}^6r_{24}^3r_{12}^3 \\
& -4r_{13}^8r_{23}^6r_{12}^2 + 32r_{13}^6r_{23}^6r_{12}^2 - 64r_{13}^4r_{23}^6r_{12}^2 + 4r_{13}^8r_{24}^6r_{12}^2 - 32r_{13}^6r_{24}^6r_{12}^2 \\
& +64r_{13}^4r_{24}^6r_{12}^2 - 8r_{13}^5r_{23}^3r_{24}^6r_{12}^2 + 64r_{13}^3r_{23}^3r_{24}^6r_{12}^2 - 128r_{13}r_{23}^3r_{24}^6r_{12}^2 \\
& +8r_{13}^5r_{23}^6r_{24}^3r_{12}^2 - 64r_{13}^3r_{23}^6r_{24}^3r_{12}^2 + 128r_{13}r_{23}^6r_{24}^3r_{12}^2 - 4r_{13}^8r_{23}^3r_{24}^6r_{12} \\
& +16r_{13}^6r_{23}^3r_{24}^6r_{12} + 32r_{13}^4r_{23}^3r_{24}^6r_{12} - 128r_{13}^2r_{23}^3r_{24}^6r_{12} + 4r_{13}^8r_{23}^6r_{24}^3r_{12} \\
& -16r_{13}^6r_{23}^6r_{24}^3r_{12} - 32r_{13}^4r_{23}^6r_{24}^3r_{12} + 128r_{13}^2r_{23}^6r_{24}^3r_{12} + 4r_{23}^6r_{24}^6\Big) \\
& +\sqrt{4-r_{12}^2}\Big(4(r_{12}-2)r_{12}^2(r_{12}+2)r_{13}^2r_{23}^6r_{24}^6 \\
& -(r_{12}-2)r_{12}(r_{12}+2)(r_{13}-2)r_{13}(r_{13}+2)\sqrt{4-r_{13}^2}\big(-r_{12}^6r_{24}^6r_{23}^6 \\
& -r_{13}^6r_{24}^6r_{23}^6 + 2r_{12}^4r_{24}^6r_{23}^6 + 2r_{13}^4r_{24}^6r_{23}^6 + 2r_{12}^3r_{13}^3r_{24}^6r_{23}^6 - 2r_{12}r_{13}^3r_{24}^6r_{23}^6 \\
& -2r_{12}^3r_{13}r_{24}^6r_{23}^6 + 2r_{12}^4r_{13}^4r_{23}^6 + r_{12}r_{13}^6r_{24}^3r_{23}^6 - r_{12}^3r_{13}^4r_{24}^3r_{23}^6 - 2r_{12}r_{13}^4r_{24}^3r_{23}^6 \\
& -r_{12}^4r_{13}^3r_{24}^3r_{23}^6 + r_{12}^6r_{13}r_{24}^3r_{23}^6 - 2r_{12}^4r_{13}r_{24}^3r_{23}^6 + r_{12}r_{13}^6r_{24}^6r_{23}^3 - r_{12}^3r_{13}^4r_{24}^6r_{23}^3 \\
& -2r_{12}r_{13}^4r_{24}^6r_{23}^3 - r_{12}^4r_{13}^3r_{24}^6r_{23}^3 + r_{12}^6r_{13}r_{24}^6r_{23}^3 - 2r_{12}^4r_{13}r_{24}^6r_{23}^3 + 2r_{12}^4r_{13}^4r_{24}^6\big)\Big).
\end{aligned}$$

Note that no mass appears in the equations $f_1 = 0$ and $f_4 = 0$ since they only depend on the distances r_{12}, r_{13}, r_{23} and r_{24}. Now we shall compute the distances r_{23} and r_{24} in function of the distances r_{12} and r_{13} using the Ptolemy's Theorem, which says that if four masses m_1, m_2, m_3 and m_4 lie on a circle and are ordered sequentially then

$$r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24} = 0.$$

So we obtain that

$$r_{24} = r_{12}\sqrt{4 - r_{13}^2} + r_{23}. \tag{7}$$

Applying Ptolemy's Theorem to the masses m_1, m_2, m_3 and m_5 we get

$$r_{12}r_{35} + r_{15}r_{23} - r_{13}r_{25} = 0.$$

Therefore

$$r_{23} = \sqrt{4 - r_{12}^2}r_{13} - \sqrt{r_{13}^2 + \frac{1}{2}r_{12}\left(\sqrt{4 - r_{12}^2}r_{13}\sqrt{4 - r_{13}^2} - r_{12}\left(r_{13}^2 - 2\right)\right)}. \tag{8}$$

Now we substitute r_{23} and r_{24} in the equations $f_1 = 0$ and $f_4 = 0$. Elevating these two equations three times to the square we can eliminate all the squareroots, obtaining two new equations $g_1 = 0$ and $g_4 = 0$ having the solutions of $f_1 = 0$ and $f_4 = 0$ and some additional solutions which are not solution of $f_1 = 0$ and $f_4 = 0$. Thus, we have

$$g_1 = -(r_{12} + r_{13})^6 g_{11}^2 g_{12},$$

where

$$\begin{aligned} g_{11} = \; & r_{12}^{14} - 4r_{13}^2r_{12}^{12} - 4r_{12}^{12} - 2r_{13}^3r_{12}^{11} + 4r_{13}r_{12}^{11} + 6r_{13}^4r_{12}^{10} + 16r_{13}^2r_{12}^{10} + 4r_{12}^{10} \\ & + 8r_{13}^5r_{12}^9 - 12r_{13}^3r_{12}^9 - 8r_{13}r_{12}^9 + r_{13}^8r_{12}^8 - 9r_{13}^6r_{12}^8 - 19r_{13}^4r_{12}^8 - 12r_{13}^2r_{12}^8 \\ & - 12r_{13}^7r_{12}^7 + 8r_{13}^5r_{12}^7 + 32r_{13}^3r_{12}^7 - 9r_{13}^8r_{12}^6 + 62r_{13}^6r_{12}^6 - 28r_{13}^4r_{12}^6 \\ & + 8r_{13}^9r_{12}^5 + 8r_{13}^7r_{12}^5 - 48r_{13}^5r_{12}^5 + 6r_{13}^{10}r_{12}^4 - 19r_{13}^8r_{12}^4 - 16r_{13}^6r_{12}^4 \\ & - 2r_{13}^{11}r_{12}^3 - 12r_{13}^9r_{12}^3 + 32r_{13}^7r_{12}^3 - 4r_{13}^{12}r_{12}^2 + 16r_{13}^{10}r_{12}^2 - 16r_{13}^8r_{12}^2 \\ & + 4r_{13}^{11}r_{12} - 8r_{13}^9r_{12} + r_{13}^{14} - 4r_{13}^{12} + 4r_{13}^{10}. \end{aligned}$$

The expression of the polynomial g_{12} is more than ten times longer than the polynomial g_{11}, and since it will not provide any solution of the system $f_1 = 0$ and $f_4 = 0$, we do not write it. Moreover

$$g_4 = -r_{12}^8(r_{12} - r_{13})^{12}r_{13}^8(r_{12} + r_{13})^{12}g_{41}g_{42},$$

where the expression of the polynomial g_{41} is approximately two hundred times longer than the expression of g_{11}, and the expression of g_{42} is approximately six hundred times longer than the expression of g_{11}. We do not provide these expressions here. They are easy to obtain with the help of an algebraic manipulator as mathematica or mapple.

Looking at the expressions of g_1 and g_4, for computing the co–circular central configurations we are only interested in the solutions of the system

$$g_{11}g_{12} = 0 \quad g_{41}g_{42} = 0,$$

or equivalently in the solutions of the four systems

$$g_{11} = 0, \quad g_{41} = 0; \tag{9}$$

$$g_{11} = 0, \quad g_{42} = 0; \tag{10}$$

$$g_{12} = 0, \quad g_{41} = 0; \tag{11}$$

$$g_{12} = 0, \quad g_{42} = 0. \tag{12}$$

For solving each one of these system we do the following. Every g_{ij} is a polynomial in the variables r_{12} and r_{13}.

We restrict now our attention to solving the system (9). We define the polynomials in one variable

$$p(r_{12}) = \text{Resultant}[g_{11}, g_{41}, r_{13}],$$
$$q(r_{13}) = \text{Resultant}[g_{11}, g_{41}, r_{12}],$$

where Resultant$[g_{11}, g_{41}, r_{13}]$ denotes the resultant of the polynomials g_{11} and g_{41} with respect to the variable r_{13}. This resultant is a polynomial in the variable r_{12}. By the properties of the resultant we have that if (r_{12}^*, r_{13}^*) is a solution of system (9), then r_{12}^* is a root of the polynomial $p(r_{12})$, and r_{13}^* is a root of the polynomial $q(r_{13})$. For more details on the resultant see for instance the book [8]. We have

$$p(r_{12}) = a(r_{12}-2)^{96} r_{12}^{416}(r_{12}+2)^{96}(r_{12}^2-2)^8(r_{12}^4-5r_{12}^2+5)p_{140}(r_{12})p_{304}(r_{12}),$$
$$q(r_{13}) = b(r_{13}-2)^{96} r_{13}^{416}(r_{13}+2)^{96}(r_{13}^2-2)^8(r_{13}^4-5r_{13}^2+5)q_{140}(r_{13})q_{304}(r_{13}),$$

where a and b are some positive integers, $p_k(r_{12})$ denotes a polynomial with integer coefficients in the variable r_{12} of degree k, and $q_l(r_{13})$ denotes a polynomial with integer coefficients in the variable r_{13} of degree l. We note that $p_{140}(r_{12}) \neq q_{140}(r_{12})$ and that $p_{304}(r_{12}) \neq q_{304}(r_{12})$, but the polynomials $p_{140}(x)$, $p_{304}(x)$, $q_{140}(x)$, $q_{304}(x)$ depend on x through x^2, i.e. are polynomials in the variable x^2 of degrees 70 and 152.

Our co–circular central configurations satisfy that

$$0 < r_{12} < r_{13} < 2. \tag{13}$$

So we only are interested in the real roots r_{12}^* and r_{13}^* of the polynomials $p(r_{12})$ and $q(r_{13})$ which are in the interval $(0, 2)$. Then we take all the pairs (r_{12}^*, r_{13}^*) with $r_{12}^* < r_{13}^*$ and we check if they are solutions of the system $f_1 = 0$ and $f_4 = 0$. Only one of such pairs is solution of the mentioned system, namely the pair

$$(r_{12}^*, r_{13}^*) = \left(\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}, \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}\right). \tag{14}$$

In short, this is the unique solution of system (9) which is solution of the system $f_1 = 0$ and $f_4 = 0$.

Now we study the solutions of systems (10), (11) and (12) in the same way that we have studied the solutions of the system (9), but these systems do not provide any solution satisfying $f_1 = 0$, $f_4 = 0$ and (13). Hence the unique solution of the system $f_1 = 0$ and $f_4 = 0$ satisfying (13) is the solution (14).

Finally we substitute the solution (14) in the equations $f_2 = 0$ and $f_3 = 0$, where previously we have substituted r_{24} and r_{23} by their expressions (7) and (8), and we obtain

$$-50\left(1+\sqrt{5}\right)(m_2-m_3) = 0 \quad \text{and} \quad 50\left(-3+\sqrt{5}\right)(m_2-m_3) = 0,$$

respectively. So $m_2 = m_3$, and it follows that the five masses are all equal. This completes the proof of subcase 1.2 showing that under the assumptions of this subcase there is a unique co–circular central configuration given by the regular 5–gon with equal masses.

Case 2: $m_1s_1 + (m_2 - m_5)s_2 \neq 0$. Now we shall solve the system

$$c_j^2 + s_j^2 = 1 \quad \text{for} \quad j = 3, 4,$$

with respect to the variables s_3 and c_3. It has two different solutions $T^j = \{c_{3,j}, s_{3,j}\}$ for $j = 1, 2$ with

$$\begin{aligned}
s_{3,1} &= -\frac{m_3}{D_4D_6}\big(D_4^2\big(D_5^2 + m_3^2 - m_4^2 + D_4^2\big) - D_5S_2\big)\\
c_{3,1} &= -\frac{m_3}{D_6}\big(D_5\big(D_5^2 + m_3^2 - m_4^2 + D_4^2\big) + S_2\big)\\
s_{3,2} &= -\frac{m_3}{D_4D_6}\big(D_4^2\big(D_5^2 + m_3^2 - m_4^2 + D_4^2\big) + D_5S_2\big)\\
c_{3,2} &= -\frac{m_3}{D_6}\big(D_5\big(D_5^2 + m_3^2 - m_4^2 + D_4^2\big) - S_2\big)
\end{aligned}$$

being

$$D_4 = m_1s_1 + (m_2 - m_5)s_2, \quad D_5 = c_1m_1 + c_2(m_2 + m_5), \quad D_6 = 2m_3^2(D_4^2 + D_5^2)$$

and

$$S_2 = \sqrt{-D_4^2(D_5^2 - (m_3 - m_4)^2 + D_4^2)(D_5^2 - (m_3 + m_4)^2 + D_4^2)}.$$

It follows from Proposition 1(a) that the configuration cc_x symmetric with respect to the x–axis of the co–circular central configuration cc is also a co–circular central configuration. Then, either the cc is invariant with respect to the x-axis, or

$$c_{3,1} = c_{3,2}\big|_{s_1\to -s_1,\ s_2\to -s_2} \quad \text{and} \quad s_{3,1} = -s_{3,2}\big|_{s_1\to -s_1,\ s_2\to -s_2}. \tag{15}$$

In the first case as it was before mentioned this implies that $s_1 = 0$, $c_1 = 1$, $s_4 = -s_3$ and $c_4 = -c_3$. Then $s_3 > 0$ and $s_4 < 0$. Moreover, since we are under the assumptions of Case 2 and $m_2 \geq m_5$, we must have $m_2 > m_5$. Using (3) we get that

$$s_2 = \frac{m_4 - m_3}{m_2 - m_5}s_3.$$

Since $s_2 > 0$ we must have $m_4 > m_3$. Note that now $r_{14} = r_{13}$ and $r_{15} = r_{12}$. Now equation $e_6 = 0$ of (2) reduces to

$$(m_3 - m_4)s_3\left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right) = 0.$$

Therefore $r_{13} = r_{12}$, which is not possible because we would have a collision between the two masses m_2 and m_3.

In short (15) must hold. Then we get the conditions $S_2 = 0$ and $D_5S_2 = 0$, respectively. So $S_2 = 0$. Since in this case $D_4 \neq 0$ we get that

$$(D_5^2 - (m_3 - m_4)^2 + D_4^2)(D_5^2 - (m_3 + m_4)^2 + D_4^2) = 0.$$

Solving $D_5^2 - (m_3 + m_4)^2 + D_4^2 = 0$ with respect to m_1 we get two possible solutions that we call them $M_{1,j}$ for $j = 1, 2$:

$$M_{1,j} = -((m_2 + m_5)c_1c_2 + (m_2 - m_5)s_1s_2 + (-1)^{j+1}\sqrt{N}),$$

for $j = 1, 2$ where

$$\begin{aligned}
N = \big(&(m_3 + m_4)^2 - (m_2 + m_5)^2c_2^2\big)s_1^2 + 2(m_2^2 - m_5^2)c_1c_2s_1s_2 + ((m_3 + m_4)^2\\
&- (m_2 - m_5)^2s_2^2)c_1^2.
\end{aligned}$$

Solving $D_5^2-(m_3-m_4)^2+D_4^2=0$ with respect to m_1 we get two possible solutions that we call them $M_{1,j}$ for $j=3,4$:

$$M_{1,j+2}=-((m_2+m_5)c_1c_2+(m_2-m_5)s_1s_2+(-1)^{j+1}\sqrt{N_1}),$$

for $j=1,2$ where

$$\begin{aligned}N_1=&\big((m_3-m_4)^2-(m_2+m_5)^2c_2^2\big)s_1^2+2(m_2^2-m_5^2)c_1c_2s_1s_2+((m_3-m_4)^2\\&-(m_2-m_5)^2s_2^2)c_1^2.\end{aligned}$$

Note that $m_3\neq m_4$ otherwise $D_5^2+D_4^2$ cannot be zero, because $D_4\neq 0$.

We consider the four possible solutions $M_{1,1}$, $M_{1,2}$, $M_{1,3}$ and $M_{1,4}$.

If $m_1=M_{1,1}$ we have that

$$\begin{aligned}s_3=s_4=&\frac{1}{(m_3+m_4)}\big((m_2+m_5)c_1c_2s_1+(m_5-m_2)c_1^2s_2+s_1\sqrt{N}\big),\\c_3=c_4=&\frac{1}{(m_3+m_4)}\big(-(m_2+m_5)c_2s_1^2+\big((m_2-m_5)s_1s_2+\sqrt{N}\big)c_1\big).\end{aligned}$$

This implies that there is a collision between the masses m_3 and m_4, a contradiction. Hence this solution is not possible.

If $m_1=M_{1,2}$ we obtain that

$$\begin{aligned}s_3=s_4=&\frac{1}{(m_3+m_4)}\big((m_2+m_5)c_1c_2s_1+(m_5-m_2)c_1^2s_2-s_1\sqrt{N}\big),\\c_3=c_4=&-\frac{1}{(m_3+m_4)}\big((m_2+m_5)c_2s_1^2+\big((m_5-m_2)s_1s_2+\sqrt{N}\big)c_1\big).\end{aligned}$$

As before this solution is not possible.

If $m_1=M_{1,3}$ we get that

$$\begin{aligned}s_3=-s_4=&\frac{1}{(m_3-m_4)}\big((m_2+m_5)c_1c_2s_1+(m_5-m_2)c_1^2s_2+s_1\sqrt{N_1}\big),\\c_3=-c_4=&\frac{1}{(m_3-m_4)}\big(-(m_2+m_5)c_2s_1^2+c_1\big((m_2-m_5)s_1s_2+\sqrt{N_1}\big)\big).\end{aligned}$$

If $(m_2-m_5)s_2+m_3s_3=0$ then $m_2>m_5$, otherwise $s_3=0$ because $m_2\geq m_5$. So $s_4=0$ and $c_3=-1$ and $c_4=1$ in contradiction with the fact that $\theta_4<\theta_5$. So

$$s_3=-\frac{m_2-m_5}{m_3}s_2.$$

Since $m_2>m_5$ and $s_2>0$ we get that $s_3<0$. Then $s_4>0$, in contradiction with the fact that $\theta_4>\theta_3$. Hence $(m_2-m_5)s_2+m_3s_3\neq 0$. Now, using the same arguments than in subcase 1.1 it follows that m_1 must be one of the three solutions $m_{1,j}$ for $j=2,3,4$. Note that $m_1=m_{1,3}$ is not possible because it implies collision between m_1 and m_4. When m_1 is equal to either $m_{1,2}$ or $m_{1,4}$ we have that $s_1=-s_4$ and $c_1=-c_4$. Then, since $s_3=-s_4$ and $c_3=-c_4$ we have a collision between the masses m_1 and m_3, a contradiction. Hence the solution $m_1=M_{1,3}$ is not possible.

If $m_1 = M_{1,4}$ we have that

$$s_3 = -s_4 = \frac{1}{(m_3 - m_4)}\big((m_2 + m_5)c_1c_2s_1 + (m_5 - m_2)c_1^2 s_2 - s_1\sqrt{N_1}\big),$$
$$c_3 = -c_4 = \frac{1}{(m_3 - m_4)}\big(-(m_2 + m_5)c_2s_1^2 + c_1\big((m_2 - m_5)s_1s_2 - \sqrt{N_1}\big)\big).$$

Now the same arguments used in the solution $m_1 = M_{1,3}$ can be applied for the solution $m_1 = M_{1,4}$, obtaining that this last solution is not possible. This completes the proof of the theorem. □

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