Periodic orbits and non-integrability in a cosmological scalar field

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We apply the averaging theory of first order to study the periodic orbits of Hamiltonian systems describing a universe filled with a scalar field which possesses three parameters. The main results are the following. First, we provide sufficient conditions on the parameters of these cosmological model, which guarantee that at any positive or negative Hamiltonian level, the Hamiltonian system has periodic orbits, the number of such periodic orbits and their stability change with the values of the parameters. These periodic orbits live in the whole phase space in a continuous family of periodic orbits parameterized by the Hamiltonian level. Second, under convenient assumptions we show the non-integrability of these cosmological systems in the sense of Liouville–Arnol'd, proving that there cannot exist any second first integral of class $\mathcal{C}^1$. It is important to mention that the tools (i.e., the averaging theory for studying the existence of periodic orbits and their kind of stability, and the multipliers of these periodic orbits for studying the integrability of the Hamiltonian system) used here for proving our results on the cosmological scalar field can be applied to Hamiltonian systems with an arbitrary number of degrees of freedom. © *2012 American Institute of Physics*. [doi:10.1063/1.3675493]

I. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

For a good introduction to the cosmological model here studied, we suggest to the reader to look at the paper of Maciejewski *et al.*[14] and references therein for a detailed deduction and implications about the importance of this model.

The foundation of homogeneous and isotopic cosmological models is the Friedmann–Robertson–Walker universe, described by the metric

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + \frac{dr^2}{1 - Kr^2} + r^2 d^2\Omega_2 \right], \tag{1}$$

where a is the scalar factor, $d^2\Omega_2$ is the line element on a two-sphere, and we chose to use the conformal time η. As it is known from the previous metric, the scalar factor represents the relative change in the distance of two points whose spatial coordinates are fixed. It depends only on the time, so that the whole universe is deformed in a homogeneous fashion. Depending on the matter components one obtains various evolutions of the scalar factor a, as given by the general action

$$\mathcal{I} = \frac{c^4}{16\pi G} \int \left[\mathcal{R} - 2\Lambda - \frac{1}{2} \left(\nabla_\alpha \overline{\psi} \nabla^\alpha \psi + V(\psi) + \xi \mathcal{R} |\psi|^2 \right) - \rho \right] \sqrt{-g}\, d^4x, \tag{2}$$

where $\mathcal{R}$ is the Ricci scalar, Λ is the cosmological constant, V is the field's potential, ξ is the coupling constant, and ρ is the density of the perfect fluid. The potential usually includes at least

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