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Global Dynamics of the Kummer–Schwarz Differential Equation

Jaume Llibre and Claudio Vidal

Abstract. This paper studies the Kummer–Schwarz differential equation $2\dot{x}\dddot{x} - 3\ddot{x}^2 = 0$ which is of special interest due to its relationship with the Schwarzian derivative. This differential equation is transformed into a first order differential system in $\mathbb{R}^3$, and we provide a complete description of its global dynamics adding the infinity.

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1. Introduction and Statements of Main Results

The Schwarzian derivative

$$\{x, t\} = \frac{\dddot{x}(t)}{\dot{x}(t)} - \frac{3}{2}\left(\frac{\ddot{x}(t)}{\dot{x}(t)}\right)^2, \tag{1.1}$$

plays an important role in the treatment of univalent functions; see details in [5] and references therein. Here, the dot denotes derivative with respect to the independent variable t. When the right hand in Eq. (1.1) is taken at zero, the resulting equation is the Kummer–Schwarz equation which is given by

$$2\dot{x}\dddot{x} - 3\ddot{x}^2 = 0, \tag{1.2}$$

and is of special interest due to its relationship to the Schwarzian derivative and its exceptional algebraic properties. This equation is also encountered in the study of geodesic curves in spaces of constant curvature; Lie lists the characteristic functions for its contact symmetries, see more results on this differential equation in [1,4–6]. But up to now nobody has described its global qualitative dynamics. This will be the objective of this paper.

To know the global qualitative dynamics of a differential equation is important because, in particular it describes where born and where died all

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