DYNAMICS, INTEGRABILITY AND TOPOLOGY FOR LOTKA–VOLTERRA HAMILTONIAN SYSTEMS IN $\mathbb{R}^4$

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Abstract. In this paper first we give the sufficient and necessary conditions in order that two classes of polynomial Kolmogorov systems in $\mathbb{R}^4_+$ are Hamiltonian systems. After we study the integrability of these Hamiltonian systems in the Liouville sense. Finally, we investigate the global dynamics of the completely integrable Lotka–Volterra Hamiltonian systems in $\mathbb{R}^4_+$. As an application of the invariant subsets of these systems, we obtain topological classifications of the 3-submanifolds in $\mathbb{R}^4_+$ defined by the hypersurfaces $axy + bzw + cx^2y + dxy^2 + ez^2w + fzw^2 =$ constant.

1. Introduction and statement of the main results

Kolmogorov systems are defined by the system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = x_i(t)P_i(x_1(t), \cdots, x_n(t)), \qquad i = 1, \cdots, n \tag{1}$$

in the state space

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x = (x_1, \cdots, x_n), \ x_i \geq 0\},$$

which describes the growth rate of populations in a community of n interacting species in population dynamics (cf. [7], [8]), where $x_i(t)$ is the population density or population number of the i species at time t and $P_i(x) : \ \mathbb{R}^n_+ \to \mathbb{R}$ are C^∞ functions. It is well known that system (1) is a *Lotka-Volterra system* if all $P_i(x_1, \cdots, x_n)$ are linear polynomials in the variables $(x_1, \cdots, x_n)$ for $i = 1, \cdots, n$, and system (1) is *competitive (or cooperative) system* if $\partial P_i(x_1, \cdots, x_n)/\partial x_j \leq 0$ ($\partial P_i(x_1, \cdots, x_n)/\partial x_j \geq 0$, respectively) for $i \neq j$.

The dynamics of general systems (1) for $n \geq 3$ are far from being understood, although some dynamics for special classes of systems (1) have been revealed (see [19], [20], [21]). For example, the general theory on competitive or cooperative systems was developed in a series of papers by Hirsch in [9]-[15], which shown that these systems typically have a global attractor which lies on a $(n-1)$-dimensional manifold. And Smale in [18] claimed that the dynamics of competitive system (1) are compatible with any dynamical behavior provided the number n of the species is very large. Comparing with the massive literature devoted to study competitive or cooperative systems, conservative Kolmogorov systems have received little attention although this class of systems goes back to the pioneer work of Volterra and there are empirical and numerical evidences that periodic oscillations often occur in such population dynamics. As it is well known, for two dimensional

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conservative Lotka-Volterra system, there is a family of periodic orbits. However, for higher dimensional Kolmogorov systems, for example, for $n = 4$, we do not even know what conditions can ensure that they are conservative systems or Hamiltonian systems. And if a 4–dimensional Kolmogorov systems are Hamiltonian systems, can we know all solutions of these 4–dimensional Hamiltonian systems and understand the dynamics of their solutions in the phase space? This is a challenging problem.

The goal of this paper is to find the algebraic conditions which ensure that 4–dimensional Kolmogorov systems (1) are Hamiltonian systems if all $P_i(x_1, \cdots, x_4)$ are linear or quadratic polynomials in the variables $(x_1, \cdots, x_4)$ for $i = 1, \dots, 4$. Once the Hamiltonian character of such systems (1) is established, the other goal of this paper is: first, to study those Hamiltonian systems (1) which are integrable in the Liouville sense; second, to characterize the topology of the invariant subsets in the phase space obtained fixing values of the two independent first integrals, and third to describe the global dynamics of the flow on those invariant subsets.

Let (x, y, z, w) be the coordinates of $\mathbb{R}^4$. We consider Kolmogorov systems of the form

$$\begin{aligned} \dot{x} &= xP_1(x, y, z, w), \\ \dot{y} &= yP_2(x, y, z, w), \\ \dot{z} &= zP_3(x, y, z, w), \\ \dot{w} &= wP_4(x, y, z, w), \end{aligned} \tag{2}$$

in $\mathbb{R}^4_+ = \{(x, y, z, w) : \ x \geq 0, y \geq 0, z \geq 0, w \geq 0\}$, where the dot in all this paper means derivative with respect to the independent variable t, and the $P_i(x, y, z, w)$'s are polynomials of (x, y, z, w) with degree at most two for $i = 1, \cdots, 4$.

In the following theorems we characterize the Kolmogorov systems (2) which are Hamiltonian systems, and inside these last systems the ones which are integrable in the Liouville sense. For the definitions of Hamiltonian system, and integrability in the Liouville sense see section 2.

Theorem 1. *The following statements hold.*

(a) *Assume that the $P_i(x, y, z, w)$'s are linear polynomials in the variables (x, y, z, w) for $i = 1, \cdots, 4$, i.e. system (2) is a Lotka-Volterra system in $\mathbb{R}^4_+$. Then system (2) is a Hamiltonian system if and only if its Hamiltonian is of the form*

$$H_3 = axy + bzw + cx^2y + dxy^2 + ez^2w + fzw^2,$$

where a, b, c, d, e and f are real parameters.

(b) *Assume that the $P_i(x, y, z, w)$'s are quadratic polynomials in the variables (x, y, z, w) for $i = 1, \cdots, 4$. Then system (2) is a Hamiltonian system if and only if its Hamiltonian is of the form*

$$\begin{aligned} H_4 = \ & axy + bzw + cx^2y + dxy^2 + ez^2w + fzw^2 + gx^3y \\ & + hx^2y^2 + ixy^3 + jxyzw + kz^3w + \ell z^2w^2 + mzw^3, \end{aligned}$$

where $a, b, c, d, e, f, g, h, i, j, k, \ell$ and m are real parameters.

Theorem 2. *If a Lotka–Volterra system (2) in $\mathbb{R}^4_+$ is a Hamiltonian system, then this system in $\mathbb{R}^4_+$ is integrable in the Liouville sense.*

Theorem 3. *If $j = 0$ in the Hamiltonian H_4, then system* (2) *with quadratic polynomials $P_i(x, y, z, w)$ in $\mathbb{R}^4_+$ is integrable in the Liouville sense.*

If $j \neq 0$ in the Hamiltonian H_4, taking into account that the Hamiltonian H_4 of statement (b) *of Theorem 1 is invariant under interchanging either x and y, or y and z, or z and w, then the essential integrable Hamiltonian systems* (2) *in the Liouville sense with Hamiltonian H_4 are the following ones:*

(i) $H = axy + bzw + ez^2w + fzw^2 + hx^2y^2 + jxyzw + kz^3w + \ell z^2w^2 + mzw^3$, *and the second first integral is $F = xy$.*

(ii) $H = -\frac{b\beta}{\alpha}xy + bzw + cx^2y + ez^2w - \frac{j\beta}{2\alpha}x^2y^2 + jxyzw - \frac{j\alpha}{2\beta}z^2w^2$, *and the second first integral is $F = x^{2\alpha}y^{\alpha}z^{2\beta}w^{\beta}$.*

(iii) $H = -\frac{2b\beta}{\alpha}xy + bzw + cx^2y - \frac{j\beta}{\alpha}x^2y^2 + jxyzw + kz^3w - \frac{j\alpha}{4\beta}z^2w^2$, *and the second first integral is $F = x^{2\alpha}y^{\alpha}z^{3\beta}w^{\beta}$.*

(iv) $H = -\frac{2b\beta}{\alpha}xy + bzw + gx^3y - \frac{j\beta}{2\alpha}x^2y^2 + jxyzw + kz^3w - \frac{j\alpha}{2\beta}z^2w^2$, *and the second first integral is $F = x^{3\alpha}y^{\alpha}z^{3\beta}w^{\beta}$.*

Here α and β are arbitrary non–zero constants.

We then consider the global dynamics of the completely integrable Lotka–Volterra Hamiltonian system

$$
\begin{aligned}
\dot{x} &= -x(a + cx + 2dy),\\
\dot{y} &= y(a + 2cx + dy),\\
\dot{z} &= -z(b + 2fw + ez),\\
\dot{w} &= w(b + fw + 2ez),
\end{aligned}
\tag{3}
$$

with Hamiltonian

$$H(x, y, z, w) = axy + cx^2y + dxy^2 + bwz + fw^2z + ewz^2,$$

and two independent first integrals with H given by

$$F_1(x, y, z, w) = axy + cx^2y + dxy^2,\ F_2(x, y, z, w) = bwz + fw^2z + ewz^2,$$

where a, b, c, d, e and f are real parameters. Without loss of generality, we assume that $a^2 + c^2 + d^2 \neq 0$ and $b^2 + f^2 + e^2 \neq 0$.

It is clear that system (3) is a decoupled system with respect to the variables (x, y) and (z, w), and that the two subsystems are essentially the same. We only consider the following subsystem of system (3)

$$
\begin{aligned}
\dot{x} &= -x(a + cx + 2dy),\\
\dot{y} &= y(a + 2cx + dy)
\end{aligned}
\tag{4}
$$

in $\mathbb{R}^2_+ = \{(x, y) :\ x \geq 0, y \geq 0\}$. This subsystem is Hamiltonian with Hamiltonian $F_1(x, y) = axy + cx^2y + dxy^2$, where a, c and d are real parameters and $a^2 + c^2 + d^2 \neq 0$.

In section 3 we use the Poincaré compactification for describing the different topological phase portraits of the polynomial differential systems (4) in the compactified quadrant $\mathbb{R}^2_+$, denoted by D^2_+, where we have added to this quadrant a quarter of the circle of infinity corresponding to the different directions that you can reach the infinity from the closed quadrant $\mathbb{R}^2_+$.

Theorem 4. *System* (4) *has* 13 *different topological phase portraits in* $\mathbb{D}_+^2$ *for all parameters* $a^2+c^2+d^2 \neq 0$*, where we have identified the topological phase portraits which differs on the orientation of all their orbits, see figures 1–3.*

Theorem 4 is proved in section 3.

We can also obtain the global dynamics of the other subsystem of system (3)

$$\begin{aligned} \dot{z} &= -z(b+fz+2ew), \\ \dot{w} &= w(b+2fz+ew), \end{aligned} \tag{5}$$

with Hamiltonian $H_2(x,y) = bzw + fz^2w + ezw^2$, where b, f and e are real parameters and $b^2+f^2+e^2 \neq 0$ by interchanging x and z, y and w, a and b, c and f, and d and e in Theorem 4.

In summary, the phase portraits of system (3) in $\mathbb{R}_+^4$ can be obtained doing the product of the different phase portraits in the spaces $(x,y) \in \mathbb{R}_+^2$ and $(z,w) \in \mathbb{R}_+^2$ of systems (5) and (4), respectively. In a similar way to $\mathbb{R}_+^2$ and $\mathbb{D}_+^2$, we denote the Poincaré compactification of $\mathbb{R}_+^4$ by $\mathbb{D}_+^4$. So we have the following result.

Corollary 5. *Systems* (3) *have* $26 \cdot 13 = 338$ *different topological phase portraits in* $\mathbb{D}_+^4$ *having identified the phase portraits which only differs in the orientation of all their orbits.*

As an application of these phase portraits, we can give topological classifications of the 3-submanifolds S_h in $\mathbb{R}_+^4$ defined by the hypersurfaces $axy + bzw + cx^2y + dxy^2 + ez^2w + fzw^2 = h$, h is a constant, where we assume that $a<0$ without loss of generality. The next result is proved in section 4.

Theorem 6. *The hypersurface* S_h *in the interior of* $\mathbb{R}_+^4$ *is a compact orientable* 3*–manifold, which is topological homeomorphic to the sphere* $\mathbb{S}^3$ *if and only if one of the following conditions holds.*

(i) $b<0$, $ac<0$, $ad<0$, $be<0$, $bf<0$, $h = h_1 + h_2 \in (a^3/(27cd) + b^3/(27ef), 0)$, (x,y) *satisfies that* $F_1(x,y) = h_1 \in \big(a^3/(27cd), 0\big)$ *and* (z,w) *satisfies that* $F_2(z,w) = h_2 \in \big(b^3/(27ef), 0\big)$*;*

(ii) $b>0$, $ac<0$, $ad<0$, $be<0$, $bf<0$, $h = h_1+h_2 \in \big(a^3/(27cd), b^3/(27ef)\big)$, (x,y) *satisfies that* $F_1(x,y) \in \big(a^3/(27cd), 0\big)$ *and* (z,w) *satisfies that* $F_2(z,w) \in \big(0, b^3/(27ef)\big)$.

To our knowledge, this is the first completely topological classification on the hypersurface S_h in the interior of $\mathbb{R}_+^4$. The methods used in the paper will provide a tool for studying the topological classification of some hypersurfaces in $\mathbb{R}_+^4$ or $\mathbb{R}^4$.

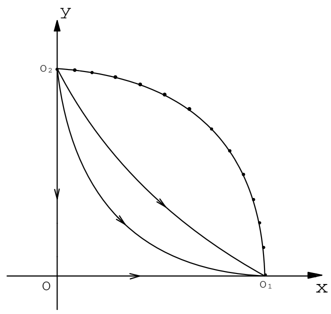
2. Integrability of polynomial Kolmogorov systems

In this section we prove all the results related with the first integrals of polynomial Kolmogorov systems with degree at most three that we consider, i.e. we prove Theorems 1, 2 and 3.

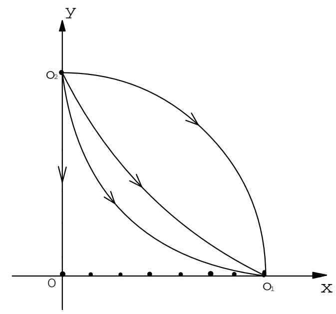
We first recall the definitions of Hamiltonian system and the integrable system in $\mathbb{R}^4$, respectively. More details on these definitions and notions can be found for instance in [1] and in [4].

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a $\mathcal{C}^2$ function, the differential system

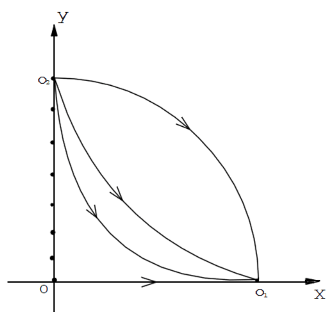
$$\dot{x} = -H_y, \quad \dot{y} = H_x, \quad \dot{z} = -H_w, \quad \dot{w} = H_z, \tag{6}$$



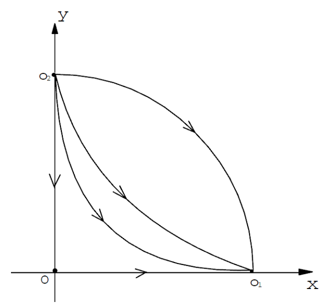
(a) The phase portraits of system (7) with the quarter equator filled by equilibria as $a \neq 0$ and $c = d = 0$, here $a < 0$.



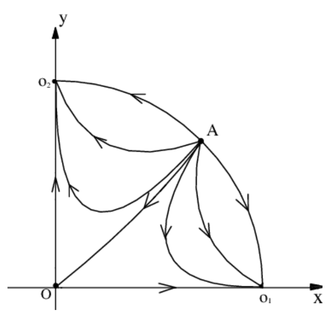
(b) The phase portraits of system (7) with the positive x-axis filled by equilibria as $d \neq 0$ and $a = c = 0$, here $d < 0$.



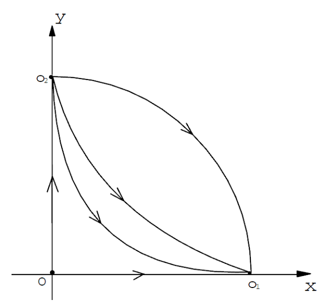
(c) The phase portraits of system (7) with the positive y-axis filled by equilibria as $c \neq 0$ and $a = d = 0$, here $c < 0$.



(d) The phase portraits of system (7) with 3 equilibria, in which O is degenerated, as $cd \neq 0$ and $a = 0$, here $c < 0$ and $d < 0$.



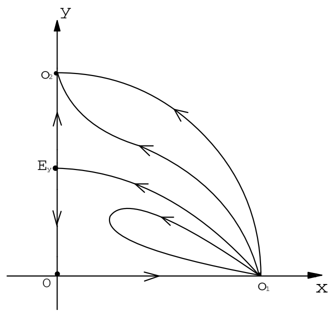
(e) The phase portraits of system (7) with 4 equilibria, in which O is degenerated, as $cd \neq 0$ and $a = 0$, here $c < 0$ and $d > 0$.



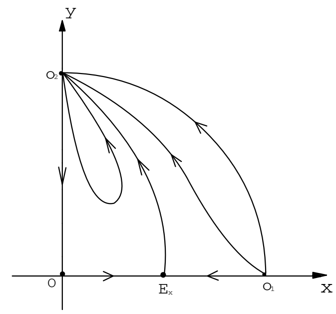
(f) The phase portraits of system (7) with 3 equilibria, in which O_1 is degenerated, as $ad \neq 0$ and $c = 0$, here $a < 0$ and $d < 0$.

FIGURE 1

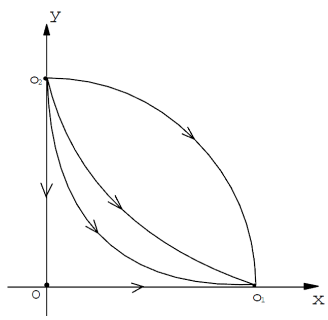
is called a *Hamiltonian system with two degrees of freedom with Hamiltonian* H, where H_x, H_y, H_z and H_w denote the partial derivatives of the function $H(x, y, z, w)$ with respect to the variable x, y, z, w, respectively.



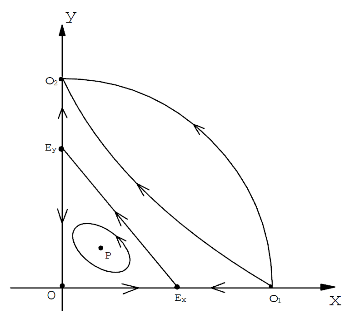
(a) The phase portraits of system (7) with 4 equilibria, in which O_1 is degenerated, as $ad \neq 0$ and $c = 0$, here $a < 0$ and $d > 0$.



(b) The phase portraits of system (7) with 4 equilibria, in which O_2 is degenerated, as $ac \neq 0$ and $d = 0$, here $a < 0$ and $c > 0$.

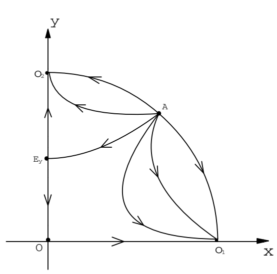


(c) The phase portraits of system (7) with 3 equilibria, in which O_2 is degenerated, as $ac \neq 0$ and $d = 0$, here $a < 0$ and $c < 0$.

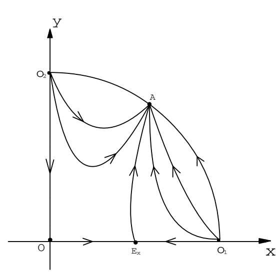


(d) The phase portraits of system (7) with 6 equilibria, in which P is a center, as $ac < 0$ and $ad < 0$, here $a < 0$.

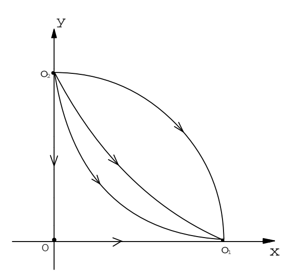
FIGURE 2



(a) The phase portraits of system (7) with 5 hyperbolic equilibria as $ac > 0$ and $ad < 0$, here $a < 0$.



(b) The phase portraits of system (7) with 5 hyperbolic equilibria as $ac < 0$ and $ad > 0$, here $a < 0$.



(c) The phase portraits of system (7) with 3 hyperbolic equilibria as $ac > 0$ and $ad > 0$, here $a < 0$.

FIGURE 3

A non-locally constant $\mathcal{C}^1$ function $F : U \to \mathbb{R}$ defined in an open and dense subset U of $\mathbb{R}^4$ is a *first integral* of the differential system (2) in $\mathbb{R}^4$ if $F(x(t), y(t), z(t),$

$w(t))$ is constant on each solution $(x(t), y(t), z(t), w(t))$ of system (2) contained in U, or equivalently if

$$xF_xP_1 + yF_yP_2 + zF_zP_3 + wF_wP_4 = 0$$

in all the points of U, where $P_i = P_i(x, y, z, w)$.

It is clear that the Hamiltonian H of a Hamiltonian system (6) is a first integral of system (6).

A Hamiltonian system (6) of two degrees of freedom with Hamiltonian H is *(completely) integrable in the Liouville sense in* $\mathbb{R}^4$ if system (6) has a first integral F, defined in $\mathbb{R}^4$ except perhaps in a set of zero Lebesgue measure, which is independent with the Hamiltonian H, i.e. the rank of the 2×4 matrix

$$\begin{pmatrix} H_x & H_y & H_z & H_w \\ F_x & F_y & F_z & F_w \end{pmatrix}$$

is 2, except perhaps in a zero Lebesgue measure set of $\mathbb{R}^4$. Note that the Poisson bracket between H and F is zero if and only if F is a first integral. So, the first integrals H and F are in *involution*.

We shall use the Darboux theory of integrability for finding an independent first integral F with the Hamiltonian H of the Hamiltonian system (6), see for instance Chapter 8 of [5], there it is stated the Darboux theory for polynomial differential systems in $\mathbb{R}^2$, but all the results extend in a natural way to $\mathbb{R}^4$, see also [16] for the Darboux theory of integrability in an arbitrary dimension.

Let $f = f(x, y, z, w)$ be a real polynomial in the variables x, y, z and w. The algebraic hypersurface $f = 0$ is *invariant* by the flow of system (2) if and only if there exists a polynomial $k = k(x, y, z, w)$ such that

$$xf_xP_1 + yf_yP_2 + zf_zP_3 + wf_wP_4 = kf.$$

The polynomial k is called the *cofactor* of the invariant algebraic hypersurface $f = 0$. We note that the degree of the polynomial k is at most the degree of the polynomial differential system (2) minus one.

According to Darboux theorem [4] (see also statement (i) of Theorem 8.7 of [5]), the function $F = f_1^{\alpha_1} \cdots f_p^{\alpha_p}$ is *a first integral of Darboux type* of system (2) if and only if system (2) has p invariant algebraic surfaces $f_i = f_i(x, y, z, w) = 0$ for $i = 1, \ldots, p$ with cofactors $k_i = k_i(x, y, z, w)$ and $\alpha_1 k_1 + \cdots + \alpha_p k_p = 0$.

We are now ready for proving Theorems 1, 2 and 3.

Proof of Theorem 1. We provide the proof of statement (a), the proof of statement (b) is analogous only a little more long due to the computations.

We consider the Lotka–Volterra systems in $\mathbb{R}^4_+$ given in (2), and a general polynomial $H = H(x, y, z, w)$ of degree 3 in the four variables x, y, z and w without constant term, i.e.

$$H = \sum_{i+j+k+l=1}^{3} h_{ijkl} x^i y^j z^k w^l, \tag{7}$$

where i, j, k and l are non-negative integers. Now we consider the Hamiltonian system (6) defined by (7). Substituting the left hand side of system (6) by the right hand side of system (2), and substituting the H of the right hand side of (6) by (7), we have four equations between quadratic polynomials. These polynomials are equal if their coefficients are equal. Equating such coefficients we get a system of equations which relates the coefficients of the Lotka–Volterra system (2) with the coefficients of the polynomial H. After a tedious computation, but easy to do with the help of an algebraic manipulator as mathematica or mapple, this system has a unique solution for the polynomial H, namely

$$H_3 = h_{1100}xy + h_{0011}zw + h_{2100}x^2y + h_{1200}xy^2 + h_{0021}z^2w + h_{0012}zw^2.$$

Changing the name of the coefficients of H_3, we get the polynomial H_3 of the statement (a) of Theorem 1. □

Proof of Theorem 2. The Hamiltonian system defined by the Hamiltonian H_3 given in the statement (a) of Theorem 1 is

$$\begin{aligned} \dot{x} &= -x(a + cx + 2dy), \\ \dot{y} &= y(a + 2cx + dy), \\ \dot{z} &= -z(b + 2fw + ez), \\ \dot{w} &= w(b + fw + 2ez). \end{aligned} \tag{8}$$

It is immediate to check that the Hamiltonian system (8) has two independent first integrals with $H = H_3$, namely

$$F_1 = axy + cx^2y + dxy^2, \quad \text{and} \quad F_2 = bzw + ez^2w + fzw^2.$$

By the definition of integrable Hamiltonian system, the system (3) is integrable in the Liouville sense. This completes the proof of the theorem. □

Proof of Theorem 3. The Kolmogorov system defined by the Hamiltonian H_4 given in statement (b) of Theorem 1 is

$$\begin{aligned} \dot{x} &= -x(a + cx + 2dy + gx^2 + 2hxy + 3iy^2 + jzw), \\ \dot{y} &= y(a + 2cx + dy + 3gx^2 + 2hxy + iy^2 + jzw), \\ \dot{z} &= -z(b + ez + 2fw + jxy + kz^2 + 2\ell zw + 3mw^2), \\ \dot{w} &= w(b + 2ez + fw + jxy + 3kz^2 + 2\ell zw + mw^2). \end{aligned} \tag{9}$$

For proving its integrability in the Liouville sense, we look for an independent first integral F with the Hamiltonian H_4 of system (9).

If $j = 0$, then it is easy to check that system (9) has the following two independent first integrals with H_4:

$$\begin{aligned} F_1 &= axy + cx^2y + dxy^2 + gx^3y + hx^2y^2 + ixy^3, \\ F_2 &= bzw + ez^2w + fzw^2 + kz^3w + \ell z^2w^2 + mzw^3. \end{aligned}$$

Thus, system (9) is integrable in the Liouville sense if $j = 0$.

If $j \neq 0$, then we look for a first integrals of Darboux type for system (9), which is independent with the Hamiltonian H_4.

It can be checked that system (9) has four invariant hyperplanes $f_1 = x = 0$, $f_2 = y = 0$, $f_3 = z = 0$ and $f_4 = w = 0$, with cofactors

$$\begin{aligned}
k_1 &= -(a + cx + 2dy + gx^2 + 2hxy + 3iy^2 + jzw),\\
k_2 &= a + 2cx + dy + 3gx^2 + 2hxy + iy^2 + jzw,\\
k_3 &= -(b + ez + 2fw + jxy + kz^2 + 2\ell zw + 3mw^2),\\
k_4 &= b + 2ez + fw + jxy + 3kz^2 + 2\ell zw + mw^2,
\end{aligned}$$

respectively. Looking for the solutions of the system

$$\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 = 0, \tag{10}$$

in the variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4, a, b, c, d, e, f, g, h, i, j, k, \ell$ and m, and without considering the repetitive solutions in the sense that produce Hamiltonians H_4 that have been already obtained interchanging either x and y, or y and z, or z and w, or by compositions of these interchanges, we obtain only the following four different solutions:

(i) $c = d = g = i = 0$, $\alpha_1 = \alpha_2 \neq 0$ and $\alpha_3 = \alpha_4 = 0$.

(ii) $d = f = g = i = k = m = 0$, $a = -b\alpha_3/(2\alpha_2)$, $h = -j\alpha_3/(4\alpha_2)$, $\ell = -j\alpha_2/\alpha_3$, $\alpha_1 = 2\alpha_2 \neq 0$ and $\alpha_4 = \alpha_3/2 \neq 0$.

(iii) $d = e = f = g = i = m = 0$, $a = -2b\alpha_3/(3\alpha_2)$, $h = -j\alpha_3/(3\alpha_2)$, $\ell = -3j\alpha_2/(4\alpha_3)$, $\alpha_1 = 2\alpha_2 \neq 0$ and $\alpha_4 = \alpha_3/3 \neq 0$.

(iv) $c = d = e = f = i = m = 0$, $a = -b\alpha_3/(3\alpha_2)$, $h = -j\alpha_3/(6\alpha_2)$, $\ell = -3j\alpha_2/(2\alpha_3)$, $\alpha_1 = 3\alpha_2 \neq 0$ and $\alpha_4 = \alpha_3/3 \neq 0$.

Hence, by the definition of Darboux integral, $f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} f_4^{\alpha_4}$ is a first integral of Darboux type of the Hamiltonian system (9) for each one of the previous four solutions of system (10). More precisely, we have the following results.

For case (i) we take $\alpha_1 = \alpha_2 = 1 \neq 0$ and obtain the two independent first integrals, namely

$$\begin{aligned}
H &= axy + bzw + ez^2w + fzw^2 + hx^2y^2 + jxyzw + kz^3w + \ell z^2w^2 + mzw^3,\\
F &= xy.
\end{aligned}$$

For case (ii) we take $\alpha_2 = \alpha \neq 0$ and $\alpha_3 = 2\beta \neq 0$ and obtain the two independent first integrals, namely

$$\begin{aligned}
H &= -\frac{b\beta}{\alpha}xy + bzw + cx^2y + ez^2w - \frac{j\beta}{2\alpha}x^2y^2 + jxyzw - \frac{j\alpha}{2\beta}z^2w^2,\\
F &= x^{2\alpha}y^{\alpha}z^{2\beta}w^{\beta}.
\end{aligned}$$

For case (iii) we take $\alpha_2 = \alpha \neq 0$ and $\alpha_3 = 3\beta \neq 0$ and obtain the two independent first integrals, namely

$$\begin{aligned}
H &= -\frac{2b\beta}{\alpha}xy + bzw + cx^2y - \frac{j\beta}{\alpha}x^2y^2 + jxyzw + kz^3w - \frac{j\alpha}{4\beta}z^2w^2,\\
F &= x^{2\alpha}y^{\alpha}z^{3\beta}w^{\beta}.
\end{aligned}$$

For case (iv) we take $\alpha_2 = \alpha \neq 0$ and $\alpha_3 = 3\beta \neq 0$ and obtain the two independent first integrals, namely

$$H = -\frac{2b\beta}{\alpha}xy + bzw + gx^3y - \frac{j\beta}{2\alpha}x^2y^2 + jxyzw + kz^3w - \frac{j\alpha}{2\beta}z^2w^2,$$
$$F = x^{3\alpha}y^{\alpha}z^{3\beta}w^{\beta}.$$

This completes the proof of the theorem. □

3. Global dynamics of the completely integrable Lotka-Volterra Hamiltonian systems

In this section first we study the equilibrium points and their local phase portraits of systems (4) depending on the values of their parameters.

Lemma 7. *System (4) has at most one positive equilibrium in the interior of $\mathbb{R}^2_+$ for all values of parameters. More precise, the existence and topological classification of equilibria of system (4) is the following.*

- (i) *If $a \neq 0$ and $c = d = 0$, then system (4) has a unique equilibrium $O = (0, 0)$ in $\mathbb{R}^2_+$, which is a hyperbolic saddle.*
- (ii) *If $d \neq 0$ and $a = c = 0$, then system (4) has a continuum of equilibria which fill the positive x-axis.*
- (iii) *If $c \neq 0$ and $a = d = 0$, then system (4) has a continuum of equilibria which fill the positive y-axis.*
- (iv) *If $cd \neq 0$ and $a = 0$, then system (4) has a unique equilibrium O in $\mathbb{R}^2_+$, which is degenerated. Further, the positive x-axis and y-axis are orbits of system (4), $\mathbb{R}^2_+$ is a hyperbolic sector if $cd > 0$, and if $cd < 0$, then $\mathbb{R}^2_+$ is divided into two hyperbolic sectors by the orbit $cx + dy = 0$ of system (4) in $\mathbb{R}^2_+$.*
- (v) *If $ad > 0$ and $c = 0$, then system (4) has a unique equilibrium O in $\mathbb{R}^2_+$, which is a hyperbolic saddle; and if $ad < 0$ and $c = 0$, then system (4) has two equilibria in $\mathbb{R}^2_+$: O and $E_y = (0, -a/d)$, and both of them are hyperbolic saddles.*
- (vi) *If $ac > 0$ and $d = 0$, then system (4) has a unique equilibrium O in $\mathbb{R}^2_+$, which is a hyperbolic saddle; and if $ac < 0$ and $d = 0$, then system (4) has two equilibria O and $E_x = (-a/c, 0)$ in $\mathbb{R}^2_+$, and both of them are hyperbolic saddles.*
- (vii) *If $acd \neq 0$, then there are four cases:*
  - (vii.1) *If $ac < 0$ and $ad < 0$, then system (4) has four equilibria in $\mathbb{R}^2_+$: O, E_y, E_x and $P = (-a/(3c), -a/(3d))$; P is a center and the others are hyperbolic saddles. There is a one–parameter family of closed orbits around P, $F_1(x, y) = axy + cx^2y + dxy^2 = h_1$ for $h_1 \in (0, a^3/(27cd))$ (or for $h_1 \in (a^3/(27cd), 0)$) if $a > 0$ (or $a < 0$). Moreover, the orbits in $F_1(x, y) = 0$ correspond to the triangle polycycle OE_xE_y and $F_1(x, y) = a^3/(27cd)$ corresponds to the center P, and $F_1(x, y) = h_1$ is a non-compact orbit of system (4) in $\mathbb{R}^2_+$ if $h_1 \in (-\infty, 0)$ (or $(0, \infty)$) if $a > 0$ (or $a < 0$).*
  - (vii.2) *If $ac > 0$ and $ad < 0$, then system (4) has two equilibria in $\mathbb{R}^2_+$: O and E_y. Both of them are hyperbolic saddles.*

(vii.3) *If $ac < 0$ and $ad > 0$, then system* (4) *has two equilibria in $\mathbb{R}^2_+$: O and E_x. Both of them are hyperbolic saddles.*

(vii.4) *If $ac > 0$ and $ad > 0$, then system* (4) *has a unique equilibrium O in $\mathbb{R}^2_+$, which is a hyperbolic saddle.*

Proof. All statements except statements (iv) and (vii.1) follows directly studying the existence of equilibria of system (4) and their linear analysis. So only we shall prove statements (iv) and (vii.1).

When $cd \neq 0$ and $a = 0$ system (4) becomes

$$\dot{x} = -x(cx + 2dy), \qquad \dot{y} = y(2cx + dy), \tag{11}$$

in $\mathbb{R}^2_+$. This system has a unique equilibrium $O = (0, 0)$ in $\mathbb{R}^2_+$, which is degenerated. To determine the topological type of O we blow up O using polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, and changing the time $\tau = rt$, where $r > 0$ is a very small positive number. Hence, system (11) in $\mathbb{R}^2_+$ can be transferred to

$$\begin{aligned} \frac{dr(\tau)}{d\tau} &= -r\left(c(\cos^3\theta - 2\cos\theta\sin^2\theta) + d(2\cos^2\theta\sin\theta - \sin^3\theta)\right), \\ \frac{d\theta(\tau)}{d\tau} &= 3\cos\theta\sin\theta(c\cos\theta + d\sin\theta), \end{aligned} \tag{12}$$

where $0 \leq \theta \leq \pi/2$ and $0 < r \ll 1$.

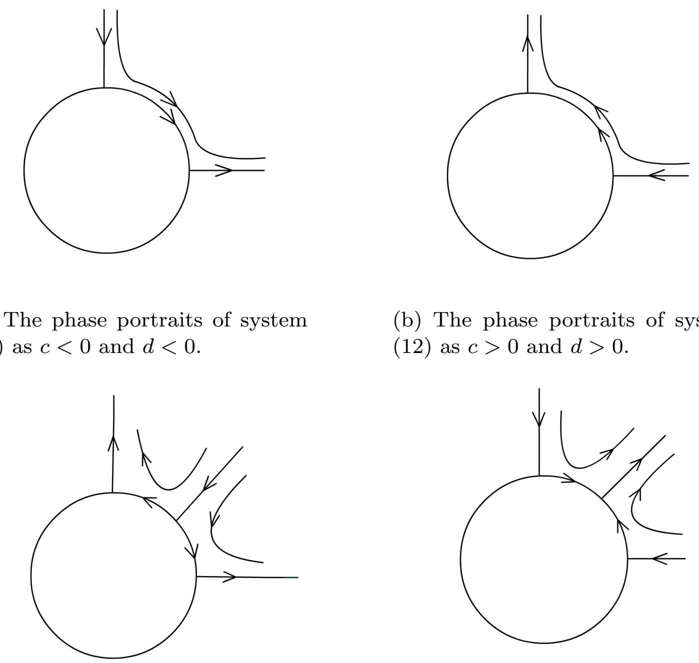
From system (12) we can see that there are at most two directions $\theta_1 = 0$ and $\theta_2 = \pi/2$ in $\mathbb{R}^2_+$ where a trajectory or many trajectories of system (11) may approach the origin O as t tends to $+\infty$ or $-\infty$ if $cd > 0$, and that there are at most three directions $\theta_1 = 0$, $0 < \theta_3 = \arctan(-c/d) < \pi/2$, and $\theta_2 = \pi/2$ in $\mathbb{R}^2_+$ where a trajectory or many trajectories of system (11) may approach the origin O as t tends to $+\infty$ or $-\infty$ if $cd < 0$. And it is easy to check that $cx + dy = 0$ is an invariant straight line of system (11) if $cd < 0$.

We now calculate the eigenvalues of the equilibria: $Q_1 = (0, \theta_1)$, $Q_2 = (0, \theta_2)$ and $Q_3 = (0, \theta_3)$ of system (12), and obtain the corresponding eigenvalues $-c$ and $3c$ for Q_1, d and $-3d$ for Q_2, and $c - d + 2c^3/d^2$ and $-3c(c^2 + d^2)/d^2$ for Q_3. The phase portraits of system (12) in $\mathbb{R}^2_+$ are showed in figure 4.

This leads that system (11) has a unique orbit, the positive x–axis (resp. the positive y–axis) approaching the origin O as t tends to $+\infty$, and another orbit, the positive y–axis (resp. the positive x–axis) approaching to the origin O as t tends to $-\infty$ if $c > 0$ and $d > 0$ (resp. $c < 0$ and $d < 0$). This implies that $\mathbb{R}^2_+$ is a hyperbolic sector if $cd > 0$. And system (11) has two orbits: the positive x–axis and the positive y–axis (resp. a unique orbit: $cx + dy = 0$ in $\mathbb{R}^2_+$) approaching to the origin O as t tends to $+\infty$, and a unique orbit $cx + dy = 0$ in $\mathbb{R}^2_+$ (resp. two orbits x–axis and y–axis) approaching to the origin O as t tends to $-\infty$ if $c > 0$ and $d < 0$ (resp. $c < 0$ and $d > 0$). This implies that $\mathbb{R}^2_+$ is divided into two hyperbolic sectors by the orbit $cx + dy = 0$ of system (4) if $cd < 0$. Hence, the conclusion (iv) holds.

When $acd \neq 0$ we consider case (vii.1): $ac < 0$ and $ad < 0$. By direct computations we obtain that system (4) has four equilibria in $\mathbb{R}^2_+$: O, E_y, E_x and P, and P is a center and the others are hyperbolic saddles.

Note that $F_1(x, y) = axy + cx^2y + dxy^2$ is Hamiltonian function of system (4), and $F_1(x, y) = h_1$ are the level curves of this Hamiltonian function.



(a) The phase portraits of system (12) as $c<0$ and $d<0$.

(b) The phase portraits of system (12) as $c>0$ and $d>0$.

(c) The phase portraits of system (12) as $c<0$ and $d>0$.

(d) The phase portraits of system (12) as $c>0$ and $d<0$.

FIGURE 4

Consider the ray

$$L: \; y = \frac{c}{d}x, \quad \forall x \geq -\frac{a}{3c},$$

starting at P, it crosses the level curve of this Hamiltonian function at a point, and

$$\left.\frac{\partial h_1}{\partial x}\right|_L = \frac{2c}{d}x\,(a+3cx)\begin{cases} >0 \text{ (resp. } <0) & \text{as } x > -\dfrac{a}{3c},\ a<0 \text{ (resp. } a>0); \\ =0 & \text{as } x = -\dfrac{a}{3c}. \end{cases}$$

Hence, when $a<0$, $F_1(x,y)$ takes the minimal value $a^3/(27cd)$ at P and increases for $x>-a/(3c)$, and $F_1(x,y)=0$ corresponds to the polycycle OE_xE_y: $x=0$, $y=0$ and $y=-cx/d-a/d$. This implies that if $a<0$ then the level curve $F_1(x,y)=h_1$ is a closed orbit of system (4) in $\mathbb{R}^2_+$ if $h_1 \in (a^3/(27cd), 0)$, and it is not a closed orbit if $h_1 \in (0,+\infty)$. When $a>0$, $F_1(x,y)$ takes the maximum value $a^3/(27cd)$ at P and decreases for $x>-a/(3c)$, and $F_1(x,y)=0$ also corresponds to the polycycle OE_xE_y. This implies that the level curve $F_1(x,y)=h_1$ is a closed orbit of system (4) in $\mathbb{R}^2_+$ if $h_1 \in (0, a^3/(27cd))$ and it is not a closed orbit if $h_1 \in (0,-\infty)$ and $a>0$. Thus, statement (vii.1) holds. □

In order to study the global dynamics of system (4) in $\mathbb{R}^2_+$, we need to investigate the behavior of trajectories in a neighborhood of infinity for system (4) by using the Poincaré compactification of $\mathbb{R}^2$. That is, we project from the center of the unit sphere $\mathbb{S}^2 = \{y=(y_1,y_2,y_3) \in \mathbb{R}^3 : y_1^2+y_2^2+y_3^2=1\}$ (the *Poincaré sphere*) onto the (x, y)-plane tangent to $\mathbb{S}^2$ at either the north or south pole. For example, we project the upper hemisphere of $\mathbb{S}^2$, then the equations defining (x, y) in terms of (y_1, y_2, y_3) are given by

$$x = \frac{y_1}{y_3},\ y = \frac{y_2}{y_3}.$$

And the equations defining (y_1, y_2, y_3) in terms of (x, y) are given by

$$y_1 = \frac{x}{\sqrt{x^2 + y^2 + 1}},\ y_2 = \frac{y}{\sqrt{x^2 + y^2 + 1}},\ y_3 = \frac{1}{\sqrt{x^2 + y^2 + 1}}.$$

Clearly, the above equations define an one-to-one correspondence between points (y_1, y_2, y_3) on the upper hemisphere of $\mathbb{S}^2$ with $y_3 > 0$ and points (x, y) in $\mathbb{R}^2$. The origin $(0, 0) \in \mathbb{R}^2$ corresponds to the north pole $(0, 0, 1) \in \mathbb{S}^2$ and the equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$, of the Poincaré sphere $\mathbb{S}^2$ is identified to the *infinity* of $\mathbb{R}^2$. Any two antipodal points on the equator $\mathbb{S}^1$ is belonging to the same point at infinity. Hence, system (4) in $\mathbb{R}^2_+$ can define a family of solution curves or a flow on $\mathbb{S}^2$ with $y_1 \geq 0$ and $y_2 \geq 0$, which allows to extend the dynamics of system (4) to infinity and study its dynamics in its neigborhood. The equator $\mathbb{S}^1$ with $y_1 \geq 0$ and $y_2 \geq 0$ consists of trajectories and equilibria of the reduced system (4) on $\mathbb{S}^2$ with $y_1 \geq 0$ and $y_2 \geq 0$.

As $\mathbb{S}^2$ with $y_1 \geq 0$ and $y_2 \geq 0$ is a differentiable manifold, for studying the flow near the equator $\mathbb{S}^1$ with $y_1 \geq 0$ and $y_2 \geq 0$ of system (4) in $\mathbb{R}^2_+$, we consider the two local charts $U_1 = \{y \in \mathbb{S}^2 : y_1 > 0\}$, $U_2 = \{y \in \mathbb{S}^2 : y_2 > 0\}$, and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ for $i = 1, 2$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$ and $(0, 1, 0)$, respectively. That is, the Poincaré transformation in the local chart U_1 is

$$x = \frac{1}{Z},\ y = \frac{u}{Z},\ Z \neq 0$$

and in the local chart U_2 is

$$x = \frac{v}{Z},\ y = \frac{1}{Z},\ Z \neq 0.$$

More details about the Poincaré compactification can be found in many books, see for example the paper [6], or the Chapter 5 of [5], or the book [22].

In the local chart U_1 system (4) becomes

$$\begin{aligned} \frac{du(t)}{dt} &= 2au + 3c\frac{u}{Z} + 3d\frac{u^2}{Z}, \\ \frac{dZ(t)}{dt} &= c + 2du + aZ, \end{aligned} \tag{13}$$

where $u \geq 0$ and $Z \geq 0$. When $Z \neq 0$, system (4) and system (13) are topologically equivalent. And $Z = 0$ always denotes the points of $\mathbb{S}^1$, which corresponds to the infinity of $\mathbb{R}^2_+$.

Changing time t by the time τ through $d\tau = dt/Z$, system (13) becomes

$$\begin{aligned} \frac{du(\tau)}{d\tau} &= 2auZ + 3cu + 3du^2, \\ \frac{dZ(\tau)}{d\tau} &= cZ + 2duZ + aZ^2. \end{aligned} \tag{14}$$

Thus system (14) is defined at $Z = 0$.

System (14) has always the equilibrium $O_1 = (0, 0)$ for all values of parameters, which corresponds to the endpoint at infinity of the positive x-axis in $\mathbb{R}^2_+$. And system (14) has the other equilibrium $A = (-c/d, 0)$ in $\mathbb{R}^2_+$ if $cd < 0$, which corresponds to the endpoint at infinity of the invariant straight line $y = -cx/d$ in $\mathbb{R}^2_+$. Otherwise, that is the case that $cd \geq 0$, system (14) has a unique equilibrium O_1.

Since the linear part of system (14) at O_1 has eigenvalues c and $3c$ and the eigenvalues of the linearized matrix at A are $-3c$ and $-c$ if $cd < 0$, O_1 and A are hyperbolic nodes, and they have converse stability depending on the sign of c.

We now discuss the local phase portrait of O_1 if $cd \geq 0$. We divide its study in four cases: (i) $cd > 0$, (ii) $c \neq 0, d = 0$, (iii) $c = 0, d \neq 0$, and (iv) $c = d = 0$.

It is clear that system (14) has a unique equilibrium O_1, which is hyperbolic stable (unstable) node if $c < 0$ (resp. $c > 0$) in cases (i) and (ii).

In case (iii) the unique equilibrium O_1 is degenerated. In this case, system (14) becomes

$$
\begin{aligned}
\frac{du(\tau)}{d\tau} &= 2auZ + 3du^2, \\
\frac{dZ(\tau)}{d\tau} &= 2duZ + aZ^2,
\end{aligned}
\tag{15}
$$

where $Z \geq 0$ and $u \geq 0$.

We now distinguish three cases: $a = 0$, $ad > 0$ and $ad < 0$, in order to describe the local phase portrait at the degenerated equilibrium O_1 in $\mathbb{R}^2_+$.

(iii.a) $c = a = 0$ but $d \neq 0$. Then it is clear that the positive x–axis of system (4) is filled with equilibria, and that O_1 is stable (unstable) if $d < 0$ (resp. $d > 0$) in $\mathbb{R}^2_+$.

(iii.b) $c = 0$ but $ad > 0$. We do a polar blow up as follows $u = r\cos\theta$, $Z = r\sin\theta$. It transforms system (15) into the system

$$
\begin{aligned}
\frac{dr(\tau)}{d\tau} &= r^2(a\sin\theta + 2d\cos\theta + a\cos^2\theta\sin\theta + d\cos^3\theta), \\
\frac{d\theta(\tau)}{d\tau} &= -r\sin\theta\cos\theta(a\sin\theta + d\cos\theta).
\end{aligned}
\tag{16}
$$

Hence, system (16) becomes

$$
r\frac{d\theta}{dr} = -\frac{\sin\theta\cos\theta(a\sin\theta + d\cos\theta)}{a\sin\theta + 2d\cos\theta + a\cos^2\theta\sin\theta + d\cos^3\theta}.
\tag{17}
$$

In a neighborhood V_1 of O_1 in $\mathbb{R}^2_+$ we have $Z \geq 0$ and $u \geq 0$. It can be checked that there are only two characteristic directions of system (15): $\theta_1 = 0$ and $\theta_2 = \pi/2$ in V_1, which form the boundary of a parabolic sector of the equilibrium O_1. All orbits of system (15) in this sector are asymptotic to O_1 as $t \to +\infty$ (resp. $t \to -\infty$) if $a > 0$ and $d > 0$ (resp. $a < 0$ and $d < 0$).

(iii.c) $c = 0$ but $ad < 0$. From (17) we can see that there exist three characteristic directions of system (15): $\theta_1 = 0$, $0 < \theta_2 = \arctan(-d/a) < \pi/2$ and $\theta_3 = \pi/2$ in V_1, which bound two sectors of the equilibrium O_1. θ_1 and θ_2 forms the boundary of a parabolic sector and all orbits of system (15) in this sector are asymptotic to O_1 as $t \to -\infty$ (resp. $t \to +\infty$) if $a < 0$ and $d > 0$ (resp. $a > 0$ and $d < 0$). θ_2 and θ_3 forms the boundary of an elliptic sector and all orbits of system (15) in this sector are asymptotic to O_1 as $t \to \pm\infty$ if $a < 0$ and $d > 0$, or $a > 0$ and $d < 0$.

In case (iv), i.e. $c = d = 0$, then system (14) becomes

$$\frac{du(\tau)}{d\tau} = 2auZ,$$
$$\frac{dZ(\tau)}{d\tau} = aZ^2,$$

which has the straight line $Z = 0$ filled of equilibria, these equilibria correspond to the endpoints of the invariant straight lines $y = kx$ in $\mathbb{R}^2_+$ for all $k \geq 0$. Hence, the quarter of the equator in $\mathbb{R}^2_+$ is filled with equilibria and the positive x–axis of system (4) is a stable (resp. unstable) manifold of O_1 if $a < 0$ (resp. $a > 0$).

Since the local chart U_1 does not cover the endpoint at infinity of the positive y–axis of $\mathbb{R}^2_+$, we must consider system (4) in the local chart U_2 with coordinates

$$x = \frac{v}{Z},\ y = \frac{1}{Z},\ Z \neq 0.$$

Thus in such a local chart system (4) becomes

$$\begin{aligned} \frac{dv(t)}{dt} &= -2av - 3d\frac{v}{Z} - 3c\frac{v^2}{Z}, \\ \frac{dZ(t)}{dt} &= -d - 2cv - aZ, \end{aligned} \tag{18}$$

where $v \geq 0$ and $Z \geq 0$. Doing the change of time $d\tau = dt/Z$ system (18) writes

$$\begin{aligned} \frac{dv(\tau)}{d\tau} &= -2avZ - 3dv - 3cv^2, \\ \frac{dZ(\tau)}{d\tau} &= -dZ - 2cvZ - aZ^2. \end{aligned} \tag{19}$$

We are only interested in the equilibria with $v = 0$ and $Z = 0$ of system (19), which corresponds to the endpoint at infinity of the positive y–axis. Clearly $O_2 = (0, 0)$ is an equilibrium of system (19) having eigenvalues $-3d$ and $-d$. O_2 is a hyperbolic stable (resp. unstable) node if $d > 0$ (resp. $d < 0$). If $d = 0$, then we divide the study of the local phase portrait at O_2 into four cases: (i) $ac \neq 0$, (ii) $c \neq 0, a = 0$, (iii) $c = 0, a \neq 0$, and (iv) $a = c = 0$. Using similar arguments to the ones in the study of the degenerated equilibrium O_1, we can obtain the topological classification O_2.

Note that both local charts U_1 and U_2 together cover all equilibria at the infinity of system (4) in $\mathbb{R}^2_+$. Hence, we can summarize the above analysis as follows.

Lemma 8. *The following statements hold for the infinite equilibria of system* (4).

(i) *System* (4) *has infinitely many equilibria filling the boundary at infinity of* $\mathbb{R}^2_+$ *when* $c = d = 0$, *which are the endpoints at infinity of the invariant straight lines* $y = kx$ *for all* $k \geq 0$ *and* $x = 0$.

(ii) *System* (4) *has at most three isolated equilibria at the infinity of* $\mathbb{R}^2_+$ *when* $c^2 + d^2 \neq 0$. *More precisely:*

(ii.1) *If* $cd < 0$, *then system* (4) *has three equilibria at infinity:* O_1, *which corresponds to the endpoint at infinity of the invariant positive* x*–axis,* O_2, *which corresponds to the endpoint at infinity of the invariant positive* y*–axis, and* A *which corresponds to the endpoint at infinity of the invariant straight line* $y = -cx/d$ *in* $\mathbb{R}^2_+$. *Moreover,* O_1 *is an unstable (resp. stable) hyperbolic node,* A *is a stable (resp. unstable) hyperbolic* *node, and O_2 is an unstable (resp. stable) hyperbolic node if $c > 0$ and $d < 0$ (resp. $c < 0$ and $d > 0$).*

(ii.2) *If $cd > 0$, then system (4) has two equilibria at infinity: O_1 and O_2. Moreover, O_1 is an unstable (resp. stable) hyperbolic node, and O_2 is a stable (resp. unstable) hyperbolic node if $c > 0$ and $d > 0$ (resp. $c < 0$ and $d < 0$).*

(ii.3) *If $c = 0$ and $d \neq 0$, then system (4) has two equilibria at infinity: O_1 and O_2. O_2 is an unstable (resp. stable) hyperbolic node as $d < 0$ (resp. $d > 0$), and O_1 is degenerated. Further, O_1 in $\mathbb{R}^2_+$ has a stable (resp. unstable) parabolic sector if $a = 0$ and $d < 0$ (resp. $a = 0$ and $d > 0$); O_1 in $\mathbb{R}^2_+$ has an unstable parabolic sector and a hyperbolic sector if $ad > 0$; and O_1 in $\mathbb{R}^2_+$ has a parabolic sector and an elliptic sector if $ad < 0$. In these last two cases the two sectors are separated by the separatrix $du + az = 0$.*

(ii.4) *If $c \neq 0$ and $d = 0$, then system (4) has two equilibria at infinity: O_1 and O_2. O_1 is unstable (asymptotically stable) hyperbolic node as $c > 0$ ($c < 0$, resp.), and O_2 is degenerated. Further, O_2 is stable (unstable) with parabolic sector if $a = 0$ and $c > 0$ ($a = 0$ and $c < 0$, resp.); O_2 is unstable with hyperbolic sector if $ac > 0$; and O_1 has a parabolic sector and an elliptic sector with the separatrix $cv + az = 0$ if $ac < 0$.*

The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \to (y_1, y_2)$ is called the *Poincaré disc,* denoted by $\mathbb{D}^2$. And we denote by $\mathbb{D}^2_+$ the projection of the compactified first quadrant $\mathbb{R}^2_+$ projected to $y_3 = 0$.

Proof of Theorem 4. From Lemmas 7 and 8, we obtain the global dynamics of system (4) in $\mathbb{D}^2_+$ described in figure 1 - 3. This leads to Theorem 4. So this theorem is proved. □

Note that when $ac < 0$ and $ad < 0$, system (4) has a one–parameter family of periodic solutions around the equilibrium point P, $F_1(x, y) = axy + cx^2y + dxy^2 = h_1$, $h_1 \in (a^3/(27cd), 0)$ (resp. $h_1 \in (0, a^3/(27cd))$) as $a < 0$ (resp. $a > 0$). Let $T_1(h_1)$ be the minimum period of the one–parameter family of periodic solutions $F_1(x, y) = axy + cx^2y + dxy^2 = h_1$.

On the other hand, when $bf < 0$ and $be < 0$, system (5) also has a one–parameter family of periodic solutions around the equilibrium point $Q = (-b/(3f), -b/(3e))$, $F_2(z, w) = bzw + fz^2w + ezw^2 = h_2$, $h_2 \in (b^3/(27ef), 0)$ (resp. $h_2 \in (0, b^3/(27ef))$) as $b < 0$ (resp. $b > 0$). Let $T_2(h_2)$ be the minimum period of the one–parameter family of periodic solutions $F_2(z, w) = bzw + fz^2w + ezw^2 = h_2$. Then we have the following property on the period functions $T_1(h_1)$ and $T_2(h_2)$. The proof can be found in [2].

Proposition 9. *If $ac < 0$ and $ad < 0$, the minimum period $T_1(h_1)$ of periodic solutions $F_1(x, y) = h_1$ of system (4) is monotone increasing (resp. decreasing) with respect to h_1 as $a < 0$ (resp. $a > 0$). Moreover,*

$$\lim_{h_1 \to \frac{a^3}{7cd}} T_1(h_1) = \frac{2\sqrt{3}\pi}{|a|} \quad \textit{and} \quad \lim_{h_1 \to 0} T_1(h_1) = +\infty.$$

If $bf<0$ and $be<0$, the minimum period $T_2(h_2)$ of periodic solutions $F_2(z,w)=h_2$ of system (5) is monotone increasing (resp. decreasing) with respect to h_2 as $b<0$ (resp. $b>0$). Moreover,

$$\lim_{h_2\to\frac{b^3}{27ef}} T_2(h_2)=\frac{2\sqrt{3}\pi}{|b|} \quad \textit{and} \quad \lim_{h_2\to 0} T_2(h_2)=+\infty.$$

We now consider the bounded solution $\phi(t,x_0,y_0,z_0,w_0)$ of system (3) with the initial conditions $(x_0,y_0,z_0,w_0)\in\mathbb{R}^4_+$

$$\begin{aligned}
\dot{x}&=-x(a+cx+2dy),\\
\dot{y}&=y(a+2cx+dy),\\
\dot{z}&=-z(b+2fw+ez),\\
\dot{w}&=w(b+fw+2ez),\\
x(0)&=x_0,\ y(0)=y_0,\\
z(0)&=z_0,\ w(0)=w_0.
\end{aligned} \tag{20}$$

Based on the global dynamics of system (3) (see figure 1 - 3), and Proposition 9, we can obtain

Theorem 10. *The solution $\phi(t,x_0,y_0,z_0,w_0)$ of system (20) is a periodic solution if and only if one of the following conditions holds.*

(i) *$ac<0$ and $ad<0$, $F_1(x_0,y_0)\in\big(a^3/(27cd),0\big)$ (resp. $F_1(x_0,y_0)\in\big(0,a^3/(27cd)\big)$) as $a<0$ (resp. $a>0$), and $z_0=w_0=0$;*

(ii) *$be<0$ and $bf<0$, $F_2(z_0,w_0)\in\big(b^3/(27ef),0\big)$ (resp. $F_2(z_0,w_0)\in\big(0,b^3/(27ef)\big)$) as $b<0$ (resp. $b>0$), and $x_0=y_0=0$;*

(iii) *$ac<0$, $ad<0$, $be<0$ and $bf<0$, $F_1(x_0,y_0)\in\big(a^3/(27cd),0\big)$ (resp. $F_1(x_0,y_0)\in\big(0,a^3/(27cd)\big)$) as $a<0$ (resp. $a>0$), $F_2(z_0,w_0)\in\big(b^3/(27ef),0\big)$ (resp. $F_2(z_0,w_0)\in\big(0,b^3/(27ef)\big)$) as $b<0$ (resp. $b>0$), and $T_1(F_1(x_0,y_0))/T_2(F_2(z_0,w_0))$ is rational.*

Except periodic solutions and equilibria, the solution $\phi(t,x_0,y_0,z_0,w_0)$ of system (20) is bounded if and only if one of the following conditions holds.

(i) *$ac<0$ and $ad<0$, $0<x_0<-a/c$ and $y_0=z_0=w_0=0$;*

(ii) *$ac<0$ and $ad<0$, $0<y_0<-a/d$ and $x_0=z_0=w_0=0$;*

(iii) *$ac<0$ and $ad<0$, $0<x_0$, $0<y_0$, $y_0+cx_0/d+a/d=0$ and $z_0=w_0=0$;*

(iv) *$ac>0$ and $ad<0$, $0<y_0<-a/d$ and $x_0=z_0=w_0=0$;*

(v) *$ac<0$ and $ad>0$, $0<x_0<-a/c$ and $y_0=z_0=w_0=0$;*

(vi) *$bf<0$ and $be<0$, $0<z_0<-b/f$ and $x_0=y_0=w_0=0$;*

(vii) *$bf<0$ and $be<0$, $0<w_0<-b/e$ and $x_0=y_0=z_0=0$;*

(viii) *$bf<0$ and $be<0$, $0<z_0$, $0<w_0$, $w_0+fz_0/e+b/e=0$ and $x_0=y_0=0$;*

(ix) *$bf>0$ and $be<0$, $0<w_0<-b/e$ and $z_0=x_0=y_0=0$;*

(x) *$bf<0$ and $be>0$, $0<z_0<-b/f$ and $w_0=x_0=y_0=0$;*

(xi) *$ac<0$, $ad<0$, $be<0$ and $bf<0$, $F_1(x_0,y_0)\in\big(a^3/(27cd),0\big)$ (resp. $F_1(x_0,y_0)\in\big(0,a^3/(27cd)\big)$) as $a<0$ (resp. $a>0$), $F_2(z_0,w_0)\in\big(b^3/(27ef),0\big)$ (resp. $F_2(z_0,w_0)\in\big(0,b^3/(27ef)\big)$) as $b<0$ (resp. $b>0$), and $T_1(F_1(x_0,y_0))/T_2(F_2(z_0,w_0))$ is irrational.*

4. An application

As an application of the global dynamics of system (3) studied in section 3, we consider the topological classification of the hypersurfaces

$$S_h = \{(x, y, z, w) \in \mathbb{R}^4_+ : axy + cx^2y + dxy^2 + bwz + fw^2z + ewz^2 = h\},$$

where a, b, c, d, e and f are real parameters, and $a^2 + c^2 + d^2 + b^2 + f^2 + e^2 \neq 0$. Without loss of generality we assume that $a < 0$ throughout this section.

Note that system (3) has Hamiltonian function

$$H = H(x, y, z, w) = axy + cx^2y + dxy^2 + bwz + fw^2z + ewz^2,$$

and two independent first integrals, each one with H, given by

$$F_1(x, y, z, w) = axy + cx^2y + dxy^2, \; F_2(x, y, z, w) = bwz + fw^2z + ewz^2.$$

It is well known that the sets

$$F_i^{-1}(h_i) = \{(x, y, z, w) : \; F_i(x, y, z, w) = h_i, \; (x, y, z, w) \in \mathbb{R}^4_+, \; h_i \in \mathbb{R}\} \triangleq I_{h_i},$$

defined using the first integral F_i are invariant under the flow of system (3), for $i = 1, 2$.

The sets $I_{h_1h_2}$ are the intersections of I_{h_1} and I_{h_2}, which are also invariant and are submanifolds of the phase space $\mathbb{R}^4_+$ if (h_1, h_2) are regular values of the function (F_1, F_2). The submanifold $I_{h_1h_2}$ is characterized by the global flow of the system (3) and the phase space $\mathbb{R}^4_+$ is foliated by $I_{h_1h_2}$. Then we use $I_{h_1h_2}$ to characterize the topological classification of the hypersurfaces S_h.

In order to study the foliation of the phase space by the invariant sets $I_{h_1h_2}$, we have to consider the critical and the regular values of the map

$$(F_1, \; F_2) : \; \mathbb{R}^4_+ \to \mathbb{R}^2.$$

The point $(x, y, z, w) \in \mathbb{R}^4_+$ is *regular* of $(F_1, \; F_2)$ if the linear map

$$D(F_1, F_2)(x, y, z, w) = \begin{pmatrix} F_{1x} & F_{1y} & F_{1z} & F_{1w} \\ F_{2x} & F_{2y} & F_{2z} & F_{2w} \end{pmatrix}$$

is surjective. And a point $(x, y, z, w) \in \mathbb{R}^4_+$ is *critical* if it is not regular. Then the rank of the matrix $D(F_1, F_2)(x, y, z, w)$ is less than two if $(x, y, z, w) \in \mathbb{R}^4_+$ is a critical point. A point $(h_1, h_2) \in \mathbb{R}^2$ is a *regular value* for (F_1, F_2) if the linear map $D(F_1, F_2)(x, y, z, w)$ is surjective for any a point $(x, y, z, w) \in \mathbb{R}^4_+$ satisfying

$$(F_1(x, y, z, w), \; F_2(x, y, z, w)) = (h_1, h_2).$$

If $(h_1, h_2) \in \mathbb{R}^2$ is not a regular value it is called a *critical value*. Thus, the regular values are those whose inverse image by $(F_1, \; F_2)$ is formed by regular points or it is empty. And the critical values are the image of some critical points.

It is clear that the matrix $D(F_1, F_2)(x, y, z, w)$ has rank less than 2 if and only if

$$ay + 2cxy + dy^2 = 0, \qquad x(a + cx + 2dy) = 0, \tag{21}$$

or

$$bw + 2ezw + fw^2 = 0, \qquad z(b + ez + 2fw) = 0. \tag{22}$$

We denote by S_i the set of solutions of equation (i). Then by a directly calculation we get that the set S_{21} of solutions of equation (21) is

$$\begin{array}{ll}
\big\{(x,y,z,w):\ \{(0,0)\}\times\mathbb{R}_+^2\cup\{(0,-\frac{a}{d})\}\times\mathbb{R}_+^2\big\} & \text{if } ad<0;\\
\big\{(x,y,z,w):\ \{(0,0)\}\times\mathbb{R}_+^2\ \big\} & \text{if } ad\geq 0 \text{ and } a^2+d^2\neq 0;\\
\big\{(x,y,z,w):\ \{(0,0)\}\times\mathbb{R}_+^3\ \big\} & \text{if } a=d=0;\\
\big\{(x,y,z,w):\ \{(-\frac{a}{c},0)\}\times\mathbb{R}_+^2\cup\{(-\frac{a}{3c},-\frac{a}{3d})\}\times\mathbb{R}_+^2\big\} & \text{if } ac<0 \text{ and } ad<0;\\
\big\{(x,y,z,w):\ \{(-\frac{a}{c},0)\}\times\mathbb{R}_+^2\ \big\} & \text{if } ac<0 \text{ and } ad>0;\\
\big\{(x,y,z,w):\ \mathbb{R}_+\times\{0\}\times\mathbb{R}_+^2\ \big\} & \text{if } a=c=0 \text{ and } d\neq 0;\\
\big\{(x,y,z,w):\ \{(0,0)\}\times\mathbb{R}_+^3\ \big\} & \text{if } a=d=0 \text{ and } c\neq 0,
\end{array}$$

and the set S_{22} of solutions of equation (22), which is equal to

$$\begin{array}{ll}
\Big\{(x,y,z,w):\ \mathbb{R}_+^2\times\{(0,0)\}\cup\mathbb{R}_+^2\times\{(0,-\frac{b}{f})\}\Big\} & \text{if } bf<0;\\
\big\{(x,y,z,w):\ \mathbb{R}_+^2\times\{(0,0)\}\ \big\} & \text{if } bf\geq 0 \text{ and } b^2+f^2\neq 0;\\
\big\{(x,y,z,w):\ \mathbb{R}_+^3\times\{0\}\ \big\} & \text{if } b=e=0;\\
\Big\{(x,y,z,w):\ \mathbb{R}_+^2\times\{(-\frac{b}{e},0)\}\cup{\mathbb{R}_+}^2\times\{(-\frac{b}{3e},-\frac{b}{3f})\}\Big\} & \text{if } be<0 \text{ and } bf<0;\\
\big\{(x,y,z,w):\ \mathbb{R}_+^2\times\{(-\frac{b}{e},0)\}\ \big\} & \text{if } be<0 \text{ and } bf>0.\\
\big\{(x,y,z,w):\ \mathbb{R}_+^2\times\{0\}\times\mathbb{R}_+\ \big\} & \text{if } b=f=0 \text{ and } e\neq 0;\\
\big\{(x,y,z,w):\ \mathbb{R}_+^3\times\{0\}\ \big\} & \text{if } b=e=0 \text{ and } f\neq 0.
\end{array}$$

Therefore, the set of critical points of the map (F_1,F_2) is the union of S_{21} and S_{22}.

From these critical points we can calculate the critical values of the maps F_1 and F_2, respectively. $F_1(x,y,z,w)$ has only two critical values: 0 and $a^3/(27cd)$. And $F_2(x,y,z,w)$ has only two critical values: 0 and $b^3/(27ef)$. Hence, I_{h_1} is a submanifold in $\mathbb{R}_+^4$ if $h_1\neq 0, a^3/(27cd)$. And I_{h_2} is a submanifold in $\mathbb{R}_+^4$ if $h_2\neq 0, b^3/(27ef)$. The set $I_{h_1h_2}$ is also a submanifold in $\mathbb{R}_+^4$ if $h_1\neq 0, a^3/(27cd)$ and $h_2\neq 0, b^3/(27ef)$.

Note that system (4) (or system (5)) has a family of periodic orbits if $ac<0$ and $ad<0$ (resp. $be<0$ and $bf<0$) (see (d) in figure 2). From Theorem 10, we have the following result.

Lemma 11. *Assume that $ac<0$, $ad<0$, $be<0$ and $bf<0$. If $h_1\in\big(a^3/(27cd),0\big)$ and $h_2\in\big(b^3/(27ef),0\big)$ (resp. $h_2\in\big(0,b^3/(27ef)\big)$) as $b<0$ (resp. $b>0$), then $I_{h_1h_2}$ is homeomorphic to a two–dimensional torus in the interior of $\mathbb{R}_+^4$, and the hypersurfaces S_h in $\mathbb{R}_+^4$ is foliated by the two-dimensional tori $I_{h_1h_2}$ with $h=h_1+h_2$ defined by*

$$\{(x,y,z,w)\in\mathbb{R}_+^4:\ axy+cx^2y+dxy^2=h_1,\ bwz+fw^2z+ewz^2=h_2\}.$$

Moreover, there exist some pairs of h_1 and h_2 such that $T_1(h_1)/T_2(h_2)$ is rational and every orbit of system (3) is periodic on the two–dimensional torus $I_{h_1h_2}$. And there also exist some pairs of h_1 and h_2 such that $T_1(h_1)/T_2(h_2)$ is irrational, system (3) has two periodic orbits and the other orbits are quasi–periodic on the two–dimensional tori $I_{h_1h_2}$.

Remark: When $T_1(h_1)/T_2(h_2)$ is rational, the periodic solution $\phi(t, x_0, y_0, z_0, w_0)$ of system (20) with $(x_0, y_0, z_0, w_0) \in \text{int}\mathbb{R}^2_+$ on the two–dimensional torus $I_{h_1h_2}$ is a knot (see [17]), where $\text{int}\mathbb{R}^2_+$ is the interior of $\mathbb{R}^2_+$.

On the other hand, based on the global dynamics of system (4) and system (5) (see Figure 1 – 3) and results in [3], we have

Lemma 12. *Assume that at least one of the inequalities $ac < 0$, $ad < 0$, $be < 0$ and $bf < 0$ does not hold. If $h_1 \neq 0, a^3/(27cd)$ and $h_2 \neq 0, b^3/(27ef)$, then $I_{h_1h_2}$ is a two–dimensional noncompact submanifold in $\mathbb{R}^4_+$, which is topological homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$, and the hypersurfaces S_h in $\mathbb{R}^4_+$ is foliated by the two–dimensional noncompact submanifold $I_{h_1h_2}$ with $h = h_1 + h_2$ defined by*

$$\{(x, y, z, w) \in \mathbb{R}^4_+ : \ axy + cx^2y + dxy^2 = h_1, \ bwz + fw^2z + ewz^2 = h_2\}.$$

We are now in the position to prove Theorem 6.

Proof of Theorem 6. From Lemmas 11 and 12, we know that the hypersurface S_h is a compact orientable 3–manifold if and only if the conditions (i) or (ii) in the theorem holds. We only need to prove the compact submanifold S_h is topological homeomorphic to $\mathbb{S}^3$ in two cases: (i) $b > 0$ and (ii) $b < 0$.

If $b > 0$, then for any given $h \in \big(a^3/(7cd), b^3/(27fe)\big)$, we can find an $h_{10} \in \big(a^3/(7cd), 0\big)$ such that $h - h_{10} \in \big(0, b^3/(27fe)\big)$. Thus, the interval $\big(a^3/(7cd), 0\big)$ is divided into two subintervals $\big(a^3/(7cd), h_{10}\big]$ and $[h_{10}, 0)$, and also the interval $\big(0, b^3/(27fe)\big)$ is divided into two subintervals $\big[h - h_{10}, h - a^3/(7cd)\big) \cap \big(0, b^3/(27fe)\big)$ and $(h, h - h_{10}] \cap \big(0, b^3/(27fe)\big)$.

From Lemma 11, for any $h_1 \in \big(a^3/(7cd), h_{10}\big]$ and corresponding $h_2 \in [h - h_{10}, h - a^3/(7cd)) \cap \big(0, b^3/(27fe)\big)$ with $h_1 + h_2 = h$, the invariant set $I_{h_1h_2}$ of system (3) is homeomorphic to a two–dimensional torus in the interior of $\mathbb{R}^4_+$, which foliates the hypersurface S_h. As h_1 varies continuously in the interval $\big(a^3/(7cd), h_{10}\big]$, it becomes a solid torus. Similarly, as h_1 varies continuously in the interval $[h_{10}, 0)$ and h_2 varies corresponding in the interval $(h, h - h_{10}] \cap \big(0, b^3/(27fe)\big)$, the set $\cup_{h_1 \in [h_{10}, 0)} I_{h_1h_2}$ is another solid torus. Hence, the hypersurface S_h can be splitted into two solid tori by a Heegaard splitting [17], which implies that hypersurface S_h is topological homeomorphic to $\mathbb{S}^3$.

If $b < 0$, then for any given $h \in \big(a^3/(7cd) + b^3/(27fe), 0\big)$, we can find an $h_{10} \in \big(a^3/(7cd), 0\big)$ such that $h - h_{10} \in \big(b^3/(27fe), 0\big)$. Thus, the interval $\big(a^3/(7cd), 0\big)$ is divided into two subintervals $\big(a^3/(7cd), h_{10}\big]$ and $[h_{10}, 0)$, and also the interval $\big(b^3/(27fe), 0\big)$ is divided into two subintervals $\Big[h - h_{10}, h - \frac{a^3}{27cd}\Big) \cap \big(b^3/(27fe), 0\big)$ and $(h, h - h_{10}] \cap \big(b^3/(27fe), 0\big)$.

Similar to the arguments in the case $b > 0$, the hypersurface S_h can be splitted into two solid tori by a Heegaard splitting. This leads that hypersurface S_h is topological homeomorphic to $\mathbb{S}^3$. This completes the proof. □

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