GLOBAL PHASE PORTRAITS OF QUADRATIC SYSTEMS WITH AN ELLIPSE AND A STRAIGHT LINE AS INVARIANT ALGEBRAIC CURVES

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Abstract. In this paper we study a new class of integrable quadratic systems and classify all its phase portraits. More precisely, we characterize the class of all quadratic polynomial differential systems in the plane having an ellipse and a straight line as invariant algebraic curves. We show that this class is integrable and we provide all the different topological phase portraits that this class exhibits in the Poincaré disc.

1. Introduction and statement of the main results

A *planar polynomial differential system* is a differential system of the form

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where P and Q are real polynomials. We say that the polynomial differential system (1) has *degree* n, if n is the maximum of the degrees of the polynomials P and Q. Usually a *polynomial differential system of degree* 2 is denoted simply as a *quadratic system*. The dot in (1) denotes derivative with respect to the independent variable t.

Let U be a dense and open subset of $\mathbb{R}^2$. A non–locally constant function $H : U \to \mathbb{R}$ is a *first integral* of the differential system (1) if H is constant on the orbits of (1) contained in U, i.e.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}(x,y)P(x,y) + \frac{\partial H}{\partial y}(x,y)Q(x,y) = 0$$

in the points $(x,y) \in U$. We say that a quadratic system is *integrable* if it has a first integral $H : U \to \mathbb{R}$.

Quadratic systems have been studied intensively, and more than one thousand papers have been published about these polynomial differential equations of degree 2, see for instance the references quoted in the

2010 *Mathematics Subject Classification.* Primary 34C05.
Key words and phrases. quadratic system, first integral, global phase portraits, invariant ellipse, invariant straight line.