GLOBAL PHASE PORTRAITS OF QUADRATIC SYSTEMS WITH AN ELLIPSE AND A STRAIGHT LINE AS INVARIANT ALGEBRAIC CURVES

JAUME LLIBRE AND JIANG YU

Abstract. In this paper we study a new class of integrable quadratic systems and classify all its phase portraits. More precisely, we characterize the class of all quadratic polynomial differential systems in the plane having an ellipse and a straight line as invariant algebraic curves. We show that this class is integrable and we provide all the different topological phase portraits that this class exhibits in the Poincaré disc.

1. Introduction and statement of the main results

A *planar polynomial differential system* is a differential system of the form

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where P and Q are real polynomials. We say that the polynomial differential system (1) has *degree* n, if n is the maximum of the degrees of the polynomials P and Q. Usually a *polynomial differential system of degree* 2 is denoted simply as a *quadratic system*. The dot in (1) denotes derivative with respect to the independent variable t.

Let U be a dense and open subset of $\mathbb{R}^2$. A non–locally constant function $H : U \to \mathbb{R}$ is a *first integral* of the differential system (1) if H is constant on the orbits of (1) contained in U, i.e.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}(x,y)P(x,y) + \frac{\partial H}{\partial y}(x,y)Q(x,y) = 0$$

in the points $(x,y) \in U$. We say that a quadratic system is *integrable* if it has a first integral $H : U \to \mathbb{R}$.

Quadratic systems have been studied intensively, and more than one thousand papers have been published about these polynomial differential equations of degree 2, see for instance the references quoted in the

2010 *Mathematics Subject Classification.* Primary 34C05.

Key words and phrases. quadratic system, first integral, global phase portraits, invariant ellipse, invariant straight line.

books of Ye [26, 27] and Reyn [22]. But the problem of classifying all the integrable quadratic system remains open.

For a quadratic system the notion of integrability reduces to the existence of a first integral, so the following natural question arises: *Given a quadratic system, how to recognize if it has a first integral*?, or *Given a class of quadratic systems depending on parameters, how to determine the values of the parameters for which the system has a first integral*? For the moment these questions do not have a good answers.

Many classes of integrable quadratic systems have been studied, and for them all the possible global topological phase portraits have been classified. One of the first of these classes studied was the classification of the quadratic centers and their first integrals which started with the works of Dulac [5], Kapteyn [9, 10], Bautin [3], Lunkevich and Sibirskii [15], Schlomiuk [23], Żołądek [29], Ye and Ye [28], Artés, Llibre and Vulpe [2], ... The class of the homogeneous quadratic systems, see Lyagina [16], Markus [17], Korol [11], Sibirskii and Vulpe [24], Newton [20], Date [4] and Vdovina [25],... Another class is the one formed by the Hamiltonian quadratic systems, see Artés and Llibre [1], Kalin and Vulpe [8] and Artés, Llibre and Vulpe [2].

In this paper we want to study a new class of integrable quadratic systems and classify all its phase portraits. More precisely we analyze the class of all quadratic polynomial differential systems having an ellipse and a straight line as invariant algebraic curves.

Our first result is to provide a normal form for all quadratic polynomial differential systems having an ellipse and a straight line as invariant algebraic curves.

Theorem 1. *A planar polynomial differential system of degree* 2 *having an ellipse and a straight line as invariant algebraic curves, after an affine change of coordinates, can be written as*

$$\begin{aligned} \dot{x} &= -cy(x-r), \\ \dot{y} &= C(x^2+y^2-1)+cx(x-r), \end{aligned} \tag{2}$$

where $c, C \in \mathbb{R}$.

Theorem 1 is proved in section 2.

In the next result we present the first integrals of the polynomial differential system of degree 2 having an ellipse and a straight line as invariant algebraic curves.

Theorem 2. *The quadratic polynomial differential systems* (2) *have the following first integrals:*

(a) $H = x^2 + y^2$ *if* $C = 0$ *and* $c \neq 0$*;*

(b) $H = x$ *if* $C \neq 0$ *and* $c = 0$*;*

(c) $H = (x - r)^{2C/c}(x^2 + y^2 - 1)$ *if* $Cc \neq 0$.

Moreover, the quadratic polynomial differential systems (2) *have no limit cycles.*

Theorem 2 is proved in section 2.

In the next theorem we present the topological classification of all the phase portraits of planar polynomial differential system of degree 2 having an ellipse and a straight line as invariant algebraic curves in the Poincaré disc. For a definition of the Poincaré compactification and Poincaré disc see section 3, and for a definition of a topological equivalent phase portraits of a polynomial differential system in the Poincaré disc see sections 3 and 4.

Theorem 3. *Given a planar polynomial differential system of degree* 2 *having an ellipse and a straight line as invariant algebraic curves its phase portrait is topological equivalent to one of the* 18 *phase portraits of Figure 1.*

Theorem 3 is proved in section 5.

2. Proofs of Theorems 1 and 2

Suppose that a polynomial differential system in the plane has an invariant ellipse and an invariant straight line. Then, first we do an affine transformation that change the ellipse to a circle and of course the straight line to another straight line, second we translate the center of the circle to the origin of coordinates, third we rescale the coordinates in order that the circle has radius one, and finally we rotate the coordinates around the origin until the straight line takes the form $x - r = 0$ with $r \geq 0$. Hence, we can assume that the systems having an ellipse and a straight line as invariant algebraic curves, without loss of generality, these curves are

$$f_1(x, y) = x^2 + y^2 - 1 = 0, \quad \text{and} \quad f_2(x, y) = x - r = 0, \;\; r \geq 0.$$

We shall need the following result which is a consequence of Corollary 6 of [14], which characterizes all rational differential systems having two curves $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves. Since this result plays a main role in this work and its proof given in Theorem 2.1 of [13] is shorter, for completeness we present it here.

Theorem 4. *Let* f_1 *and* f_2 *be polynomials in* $\mathbb{R}[x, y]$ *such that the Jacobian* $\{f_1, f_2\} \not\equiv 0$*. Then any planar polynomial differential system*

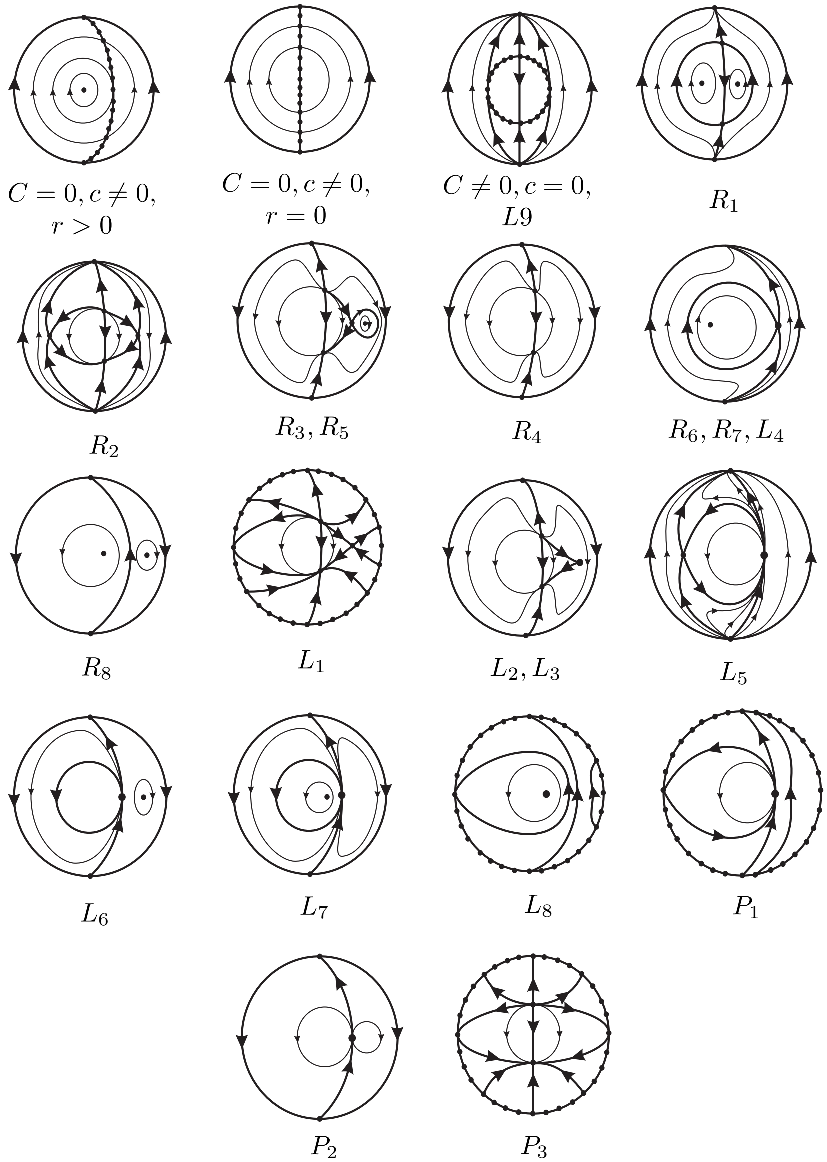

FIGURE 1. Phase portraits of systems (2).

which admits $f_1 = 0$ *and* $f_2 = 0$ *as invariant algebraic curves can be written as*

$$\dot{x} = \varphi_1\{x, f_2\} + \varphi_2\{f_1, x\}, \quad \dot{y} = \varphi_1\{y, f_2\} + \varphi_2\{f_1, y\}, \tag{3}$$

where $\varphi_1 = \lambda_1 f_1$ *and* $\varphi_2 = \lambda_2 f_2$*, with* λ_1 *and* λ_2 *being arbitrary polynomial functions.*

Proof. Consider the following vector fields

$$\{*, f_2\} = \det \begin{pmatrix} \dfrac{\partial *}{\partial x} & \dfrac{\partial *}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} \quad \text{and} \quad \{f_1, *\} = \det \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial *}{\partial x} & \dfrac{\partial *}{\partial y} \end{pmatrix}.$$

Using this notation and denoting by X the vector field associated to system (3) we have

$$X(*) = \varphi_1\{*, f_2\} + \varphi_2\{f_1, *\}. \tag{4}$$

In this way

$$X(f_1) = \varphi_1\{f_1, f_2\} + \varphi_2\{f_1, f_1\} = \lambda_1 f_1\{f_1, f_2\} = K f_1.$$

Hence $f_1 = 0$ is an invariant algebraic curve of the polynomial vector field X associated to system (3) with cofactor $K = \lambda_1\{f_1, f_2\}$. Analogously we can show that $f_2 = 0$ is also an invariant algebraic curve of X.

Now we prove that the vector field X is the most general polynomial vector field which admits $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves. Indeed let $Y = (Y_1(x, y), Y_2(x, y))$ be an arbitrary polynomial vector field having $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves. Then taking

$$\varphi_1 = \frac{Y(f_1)}{\{f_1, f_2\}} \quad \text{and} \quad \varphi_2 = \frac{Y(f_2)}{\{f_1, f_2\}}$$

and substituting the expressions of φ_1 and φ_2 in the expression (4) of the vector field X we obtain for an arbitrary polynomial F that

$$X(F) = Y(f_1)\frac{\{F, f_2\}}{\{f_1, f_2\}} + Y(f_2)\frac{\{f_1, F\}}{\{f_1, f_2\}}.$$

Substituting

$$Y(f_1) = Y_1\frac{\partial f_1}{\partial x} + Y_2\frac{\partial f_1}{\partial y} \quad \text{and} \quad Y(f_2) = Y_1\frac{\partial f_2}{\partial x} + Y_2\frac{\partial f_2}{\partial y}$$

in $X(F)$ we have that $X(F) = Y(F)$. Therefore the theorem is proved, due to the arbitrariness of the function F. $\square$

Using this theorem we prove Theorem 1.

Proof of Theorem 1. Noting that

$$\{x, f_2\} = 0, \ \{y, f_2\} = -1, \ \{f_1, x\} = -2y, \ \{f_1, y\} = 2x,$$

and applying Theorem 4 we can write systems (1) of degree ≤ 3 having an ellipse and a straight line as invariant algebraic curves as

$$\begin{aligned} \dot{x} = & \; -2\lambda_2 y(x-r), \\ \dot{y} = & \; -\lambda_1(x^2+y^2-1) + 2\lambda_2 x(x-r), \end{aligned}$$

where λ_1, λ_2 are arbitrary constants. Then we have system (2). $\square$

Proof of Theorem 2. Statements (a) and (b) follow easily.

It is immediate that the function H given in statement (c) on the orbits of system (7) satisfies

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}(-cy(x-r)) + \frac{\partial H}{\partial y}(C(x^2+y^2-1) + cx(x-r)) = 0.$$

So H is a first integral of system (7), and this proves statement (c).

Since both first integrals are defined in the whole plane except perhaps on the invariant straight line $x = r$, the system has no limit cycles. This completes the proof of the theorem. $\square$

3. Poincaré compactification

Let

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be the planar polynomial vector field of degree n associated to the polynomial differential system (1) of degree n. The *Poincaré compactified vector field* $p(\mathcal{X})$ *associated to* $\mathcal{X}$ is an analytic vector field on $\mathbb{S}^2$ constructed as follows (see, for instance [7], or Chapter 5 of [6]).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at point y. We identify the plane $\mathbb{R}^2$ where we have our polynomial vector field $\mathcal{X}$ with the tangent plane $T_{(0,0,1)}\mathbb{S}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly the equator $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar polynomial vector field of degree n then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \backslash \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and knowing the behaviour of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behaviour of $\mathcal{X}$ at infinity.

The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc,* and it is denoted by $\mathbb{D}^2$. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

We say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$, preserving or reversing simultaneously the sense of all orbits.

As $\mathbb{S}^2$ is a differentiable manifold, for computing the expression for $p(\mathcal{X})$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$ respectively. If we denote by (u, v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so (u, v) represents different things according to the local charts under consideration), then some easy computations give for $p(\mathcal{X})$ the following expressions:

$$(5)\; v^n\Delta(u,v)\left(Q\left(\frac{1}{v},\frac{u}{v}\right) - uP\left(\frac{1}{v},\frac{u}{v}\right), -vP\left(\frac{1}{v},\frac{u}{v}\right)\right) \quad \text{in } U_1,$$

$$(6)\; v^n\Delta(u,v)\left(P\left(\frac{u}{v},\frac{1}{v}\right) - uQ\left(\frac{u}{v},\frac{1}{v}\right), -vQ\left(\frac{u}{v},\frac{1}{v}\right)\right) \quad \text{in } U_2,$$

$$\Delta(u,v)\left(P(u,v), Q(u,v)\right) \quad \text{in } U_3,$$

where $\Delta(u, v) = (u^2 + v^2 + 1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $v = 0$ always denotes the points of $\mathbb{S}^1$. *In what follows we omit the factor $\Delta(u, v)$ by rescaling the vector field $p(\mathcal{X})$.* Thus we obtain a polynomial vector field in each local chart.

4. Separatrices and canonical regions

Let $p(\mathcal{X})$ be the Poincaré compactification in the Poincaré disc $\mathbb{D}$ of the polynomial differential system (1) defined in $\mathbb{R}^2$, and let Φ be its analytic flow. Following Markus [18] and Neumann [19] we denote by (U, Φ) the flow of a differential system on an invariant set $U \subset \mathbb{D}$ under the flow Φ. Two flows (U, Φ) and (V, Ψ) are *topologically equivalent* if and only if there exists a homeomorphism $h : U \to V$ which sends orbits of the flow Φ into orbits of the flow Ψ either preserving or reversing the orientation of all the orbits.

The flow (U, Φ) is said to be *parallel* if it is topologically equivalent to one of the following flows:

(i) The flow defined in $\mathbb{R}^2$ by the differential system $\dot{x} = 1$, $\dot{y} = 0$, called *strip flow*.
(ii) The flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the differential system in polar coordinates $\dot{r} = 0$, $\dot{\theta} = 1$, called *annular flow*.
(iii) The flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the differential system in polar coordinates $\dot{r} = r$, $\dot{\theta} = 0$, called *spiral or radial flow*.

It is known that the separatrices of the vector field $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$ are:

(I) all the orbits of $p(\mathcal{X})$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disc (i.e. at the infinity of $\mathbb{R}^2$),
(II) all the finite singular points of $p(\mathcal{X})$,
(III) all the limit cycles of $p(\mathcal{X})$, and
(IV) all the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$.

Moreover such vector fields $p(\mathcal{X})$, coming from polynomial vector fields (1) of $\mathbb{R}^2$ having finitely many singular points finite and infinite, have finitely many separatrices. For more details see for instance [12].

Let $\mathcal{S}$ be the union of the separatrices of the flow $(\mathbb{D}, \Phi)$ defined by $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$. It is easy to check that $\mathcal{S}$ is an invariant closed set. If N is a connected component of $\mathbb{D} \setminus \mathcal{S}$, then N is also an invariant set under the flow Φ of $p(\mathcal{X})$, and the flow $(N, \Phi|_N)$ is called a *canonical region* of the flow $(\mathbb{D}, \Phi)$.

Proposition 5. *If the number of separatrices of the flow* $(\mathbb{D}, \Phi)$ *is finite, then every canonical region of the flow* $(\mathbb{D}, \Phi)$ *is parallel.*

For a proof of this proposition see [19] or [12].

The *separatrix configuration* $\mathcal{S}_c$ of a flow $(\mathbb{D}, \Phi)$ is the union of all the separatrices $\mathcal{S}$ of the flow together with an orbit belonging to each canonical region. The separatrix configuration $\mathcal{S}_c$ of the flow $(\mathbb{D}, \Phi)$ is said to be topologically equivalent to the separatrix configuration $\mathcal{S}_c^*$ of the flow $(\mathbb{D}, \Phi^*)$ if there exists an orientation preserving homeomorphism from $\mathbb{D}$ to $\mathbb{D}$ which transforms orbits of $\mathcal{S}_c$ into orbits of $\mathcal{S}_c^*$, and orbits of $\mathcal{S}$ into orbits of $\mathcal{S}^*$.

Theorem 6 (Markus–Neumann–Peixoto)**.** *Let* $(\mathbb{D}, \Phi)$ *and* $(\mathbb{D}, \Phi^*)$ *be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields* (1)*. Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

For a proof of this result we refer the reader to [18, 19, 21].

It follows from the previous theorem that in order to classify the phase portraits in the Poincaré disc of a planar polynomial differential system having finitely many separatrices finite and infinite, it is enough to describe their separatrix configuration. This is what we have done in Figure 1, where we also have added the invariant straight line $x = r$ with $r \geq 0$ and the invariant circle $x^2 + y^2 = 1$.

5. Phase portraits

It is clear that the phase portrait of the quadratic polynomial differential system (2) with $C = 0$, if formed by all the invariant circles centered at the origin of coordinates, intersected with the invariant straight line $x = r$ filled of equilibria, providing the two first phase portraits of Figure 1.

In what follows we shall study the phase portraits of system (2) with $C \neq 0$.

Doing the rescaling of the time $\tau = Ct$, and renaming c/C again by c, we have the quadratic system

$$
\begin{aligned}
\dot{x} &= -cy(x-r), \\
\dot{y} &= x^2 + y^2 - 1 + cx(x-r),
\end{aligned}
\tag{7}
$$

with $c \in \mathbb{R}$ and $r \geq 0$.

Remark 7. *System* (7) *is reversible because it does not change under the transformation* $(x, y, t) \to (x, -y, -t)$. *Hence we know that the phase portrait of system* (7) *is symmetric with respect to the* $x-$*axis.*

The way for studying the phase portraits of systems (7) is the following. First we shall characterize all the finite equilibria of those systems together with their local phase portraits. After we do the same for the infinite equilibria, and finally using this information on the equilibria and the existence of the invariant straight line $x = r$ with $r \geq 0$, and of the invariant ellipse $x^2 + y^2 = 1$ we shall provide the classification of all the phase portraits of systems (7).

5.1. The finite singular points.

The finite singular points of system (7) are characterized in the next result.

Proposition 8. *System* (7) *has the following finite singular points:*

(a) *if* $c = 0$ *all the points of circle* $x^2 + y^2 = 1$;

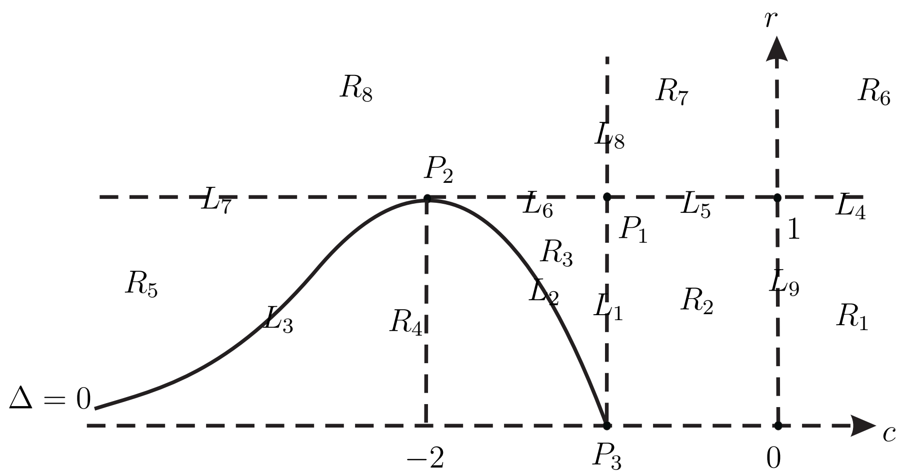

FIGURE 2. The bifurcation diagram.

(b) *if* $c \notin \{-1, 0\}$ *the singular points are*

$$
\begin{aligned}
&M_{\pm} &&= (r, \pm\sqrt{1-r^2}) && \textit{if } 0 \le r < 1,\\
&M &&= (1, 0) && \textit{if } r = 1,\\
&N_{\pm} &&= (x^*_{\pm}, 0) = \left(\frac{cr \pm \sqrt{\Delta}}{2(c+1)}, 0\right) && \textit{if } \Delta > 0,\\
&N &&= (x^*, 0) = \left(\frac{cr}{2(c+1)}, 0\right) && \textit{if } \Delta = 0,
\end{aligned}
\tag{8}
$$

where $\Delta = c^2r^2 + 4(c+1)$*;*

(d) *if* $c = -1$ *the singular points are*

$$
\begin{aligned}
&(0, \pm 1) && \textit{if } r = 0,\\
&\left(\frac{1}{r}, 0\right) \quad \textit{and} \quad (r, \pm\sqrt{1-r^2}) && \textit{if } 0 < r < 1,\\
&\left(\frac{1}{r}, 0\right) && \textit{if } r \ge 1.
\end{aligned}
\tag{9}
$$

Proof. The proof follows easily studying the real solutions of the system $cy(x-r) = 0$, $x^2 + y^2 - 1 + cx(x-r) = 0$. □

We write the curve $\Delta = 0$ in the strip $\{(c, r) \ : \ c \in \mathbb{R}, \ 0 \le r \le 1\}$ as

$$
c_{\pm}(r) = \frac{-2 \pm 2\sqrt{1-r^2}}{r^2}. \tag{10}
$$

Obviously $c_-(r) \le -2 \le c_+(r)$.

Now we define the regions

$$
\begin{aligned}
R_1 = &\ \{(c,r)\ :\ 0 \le r < 1,\ c > 0\},\\
R_2 = &\ \{(c,r)\ :\ 0 \le r < 1,\ -1 < c < 0\},\\
R_3 = &\ \{(c,r)\ :\ 0 < r < 1,\ c_+(r) < c < -1\},\\
R_4 = &\ \{(c,r)\ :\ 0 \le r < 1,\ c_-(r) < c < c_+(r)\},\\
R_5 = &\ \{(c,r)\ :\ 0 < r < 1,\ c < c_-(r)\},\\
R_6 = &\ \{(c,r)\ :\ 1 < r,\ c > 0\},\\
R_7 = &\ \{(c,r)\ :\ 1 < r,\ -1 < c < 0\},\\
R_8 = &\ \{(c,r)\ :\ 1 < r,\ c < -1\},
\end{aligned}
$$

the curves

$$
\begin{aligned}
L_1 = &\ \{(c,r)\ :\ 0 < r < 1,\ c = -1\},\\
L_2 = &\ \{(c,r)\ :\ 0 < r < 1,\ -2 < c = c_+(r)\},\\
L_3 = &\ \{(c,r)\ :\ 0 < r < 1,\ c = c_-(r) < -2\},\\
L_4 = &\ \{(c,r)\ :\ r = 1,\ 0 < c\},\\
L_5 = &\ \{(c,r)\ :\ r = 1,\ -1 < c < 0\},\\
L_6 = &\ \{(c,r)\ :\ r = 1,\ -2 < c < -1\},\\
L_7 = &\ \{(c,r)\ :\ r = 1,\ c < -2\},\\
L_8 = &\ \{(c,r)\ :\ r > 1,\ c = -1\},\\
L_9 = &\ \{(c,r)\ :\ r \ge 0,\ c = 0\}.
\end{aligned}
$$

and the points $P_1(c,r) = (-1,1)$, $P_2(c,r) = (-2,1)$ and $P_3(c,r) = (-1,0)$. See Figure 2.

For definitions of elliptic and hyperbolic sectors, cusp, and hyperbolic, semi–hyperbolic and nilpotent singular points see [6].

Proposition 9. *System* (7) *has the following finite singular points if its parameters* (c,r) *are in*

- (R_1) *two hyperbolic saddles* $M_\pm$ *and two centers* $N_\pm$.
- (R_2) *four hyperbolic singular points:* M_+ *is an unstable node,* M_- *is a stable node, and* $N_\pm$ *are saddles.*
- (R_3) *three hyperbolic singular points:* M_+ *is an unstable node,* M_- *is a stable node, and* N_+ *is saddle; and a center* N_-.
- (R_4) *two hyperbolic singular points:* M_+ *is an unstable node, and* M_- *is a stable node.*
- (R_5) *three hyperbolic singular points:* M_+ *is an unstable node,* M_- *is a stable node, and* N_- *is saddle; and a center* N_+.
- $(R_{6,7})$ *one hyperbolic saddle* N_+ *and a center* N_-.
- (R_8) *two centers* $N_\pm$.
- (L_1) *three hyperbolic singular points:* M_+ *is an unstable node,* M_- *is a stable node, and* N *is a saddle.*

($L_{2,3}$) *two hyperbolic singular points: M_+ is an unstable node and M_- is a stable node, and a nilpotent cusp N.*

(L_4) $M = (1, 0)$ *is a nilpotent saddle and* $N = (-1/(c + 1), 0)$ *is a center.*

(L_5) $M = (1, 0)$ *is a nilpotent singular point formed by one elliptic sector and one hyperbolic sector, and* $N = (-1/(c + 1), 0)$ *is a hyperbolic saddle.*

($L_{6,7}$) $M = (1, 0)$ *is a nilpotent singular point formed by one elliptic sector and one hyperbolic sector, and* $N = (-1/(c + 1), 0)$ *is a center.*

(L_8) $N(1/r, 0)$ *is a center.*

(L_9) *all the points of circle* $x^2 + y^2 = 1$ *are singular points.*

(P_1) $M = (1, 0)$ *is a nilpotent singular point formed by one elliptic sector and one hyperbolic sector.*

(P_2) $M = (1, 0)$ *is a degenerated singular point formed by the union of two elliptic sectors.*

(P_3) *two hyperbolic singular points: M_+ is an unstable node and M_- is a stable node.*

Proof. On the curve L_9 we have that c is zero, so the straight lines $x =$ constant are invariant by system (7), see the phase portrait L_9 in Figure 1.

In the following we always assume $c \neq 0$. We distinguish two cases in the study of the finite singular points of system (7).

Case 1: Singular points on the invariant straight line $x = r$. Clearly system (7) has no singular point on $x = r$ when $r > 1$, a unique singular point M when $r = 1$, and the singular points $M_\pm$ for $0 \leq r < 1$, see (8).

Subcase 1.1: $0 \leq r < 1$. The eigenvalues of the Jacobian matrix of system (7) at M_+ are $-c\sqrt{1 - r^2}$ and $2\sqrt{1 - r^2}$, and at M_- are $c\sqrt{1 - r^2}$ and $-2\sqrt{1 - r^2}$. Therefore, $M_\pm$ are hyperbolic saddles if $c > 0$; and M_+ is an unstable hyperbolic node and M_- is a stable hyperbolic node if $c < 0$, see for more details Theorem 2.15 of [6] where are described the local phase portraits of the hyperbolic singular points.

Subcase 1.2: $r = 1$. The Jacobian matrix of system (7) at $M(1, 0)$ is

$$J_M = \begin{pmatrix} 0 & 0 \\ c+2 & 0 \end{pmatrix}.$$

Subcase 1.2.1: $c \neq -2$. So M is a nilpotent singular point. Using Theorem 3.5 of [6] for studying the local phase portraits of the nilpotent singular points we get that M a nilpotent saddle if $c > 0$, and if

$c<0$ and different from -2 is reunion of one elliptic sector with one hyperbolic sector.

Subcase 1.2.2: $c=-2$. Using the polar blowing–up centered at M, i.e. $x=\rho\cos\theta+1$ and $y=\rho\sin\theta$, system (7) becomes

$$\begin{aligned}\dot\rho &= \rho^2\sin\theta,\\ \dot\theta &= -\rho\cos\theta.\end{aligned} \tag{11}$$

The singular points of (11) on $\{\rho=0\}$ are located at $\theta=\pm\pi/2$. Then $(0,\pi/2)$ is an unstable hyperbolic node and $(0,-\pi/2)$ is a stable hyperbolic node. Doing a blowing down we obtain that M is formed by the union of two elliptic sectors. See picture P_2 in Figure 1.

Case 2: Singular points on the straight line $y=0$.

Subcase 2.1: $\Delta>0$ and $c\neq-1$. Then system (7) has two singular points $N_\pm=(x^*_\pm,0)$, see (8). The Jacobian matrix of system (7) at the points $N_\pm$ is

$$J=\begin{pmatrix}0 & -c(x^*_\pm-r)\\ \pm\sqrt{\Delta} & 0\end{pmatrix}. \tag{12}$$

It is easy to check that

$$(x^*_+-r)(x^*_--r)=\frac{r^2-1}{c+1},\quad (x^*_+-r)+(x^*_--r)=-\frac{(c+2)r}{c+1}.$$

Subcase 2.1.1: $0\le r<1$. Then we have $x^*_-<r<x^*_+$ when $c>-1$, and $x^*_\pm>r$ when $-2\le c<-1$, and $x^*_\pm<r$ when $c\le-2$. Using the fact that system (7) is reversible with respective to x–axis, and the eigenvalues of the Jacobian matrix (12), we obtain that $N_\pm$ are centers when $c>0$, and saddles when $-1<c<0$. Moreover, N_+ is saddle and N_- a center when $-2<c<-1$; N_+ is center and N_- is saddle when $c<-2$.

Subcase 2.1.2: $r>1$. Then $x^*_\pm<r$ when $c>-1$; $x^*_+<r<x^*_-$when $c<-1$. Since system (7) is reversible with respective to x–axis, using the eigenvalues of the Jacobian matrix (12) we get that N_+ is a saddle and N_- a center when $c>0$; N_- is a saddle and N_+ a center when $-1<c<0$; and $N_\pm$ are centers when $c<-1$.

Subcase 2.1.3: $r=1$. This case has been studied inside the Case 1.

Subcase 2.1.3.1: $c\neq-2$. Then N_+ meets with $M_\pm$, i.e., system (7) has the singular points $N_+=M_\pm=M(1,0)$ and $N_-=N(-1/(c+1),0)$. Here we only need to study the local phase portrait of the singular point N, because the local phase portrait of M has been study in Case 1. The eigenvalues of the Jacobian matrix of system (7) at N are $\pm(c+2)\sqrt{-c/(c+1)}$. Therefore N is a saddle if $-1<c<0$, and N

is a linear center for $c > 0$ or $c < -1$ and $c \neq -2$, but N is a center of system (7) because this system is reversible with respect to x–axis.

Subcase 2.1.3.2: $c = -2$.

Subcase 2.2: $\Delta = 0$ and $c \neq -1$, we have from (10) that $c = c_{\pm}(r)$, and from (12) the singular point $N = (x^*, 0)$ is nilpotent. Taking $(x, y) = (X + x^*, Y)$, after $(X, Y) = (x, y)$, and rescaling the independent variable t by $\tau = rc(c+2)t/(2(c+1))$, we obtain

$$\begin{aligned} \dot{x} = & \quad y - \frac{2(c+1)}{r(c+2)} xy, \\ \dot{y} = & \quad \frac{2(c+1)}{rc(c+2)} \left((c+1)x^2 + y^2 \right). \end{aligned}$$

By Theorem 3.5 of [6] the origin of the previous system is a cusp.

Subcase 2.3: $c = -1$. On $y = 0$ there is the unique singular point $N(1/r, 0)$ when $r > 0$.

Subcase 2.3.1: $r \notin \{0, 1\}$. The eigenvalues of the Jacobian matrix of system (7) at N are $\pm\sqrt{1 - r^2}$, which implies that N is obviously a linear center when $r > 1$ (and consequently a center due to the reversibility of the system), and N is a saddle when $0 < r < 1$.

Subcase 2.3.2: $r = 1$. The unique singular point of the system is $(1, 0)$, the Jacobian matrix on it is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So it is a nilpotent singular point. By Theorem 3.5 of [6] its local phase portrait is formed by one hyperbolic and one elliptic sector (see picture P_1 in Figure 1).

Subcase 2.3.3: $r = 0$. No singular points on $y = 0$.

Finally, taking into account all this information on the finite singular points, we can organize it, as it appears in the statements of the proposition. □

5.2. **The infinite singular points.**

Proposition 10. *The following two statements hold.*

(a) *If* $c \neq -1$ *system* (7) *has a pair of infinite singular points, which are saddles if* $c < -1$*, and nodes if* $c > -1$*.*

(b) *If* $c = -1$ *the infinity of system* (7) *is filled of singular points.*

Proof. Considering the infinite singular of (7), we take

$$x = \frac{1}{v}, \qquad y = \frac{u}{v}. \tag{13}$$

and the time rescaling $t = v\tau$. Then system (7) in the local chart (13) is

$$\begin{aligned} \dot{u} &= (u^2+1)(1+c) - v^2 - cr(u^2+1)v, \\ \dot{v} &= cu(1-rv)v. \end{aligned} \tag{14}$$

If $c \neq -1$, there is no singular point of system (14) on $v = 0$. Taking

$$x = \frac{u}{v}, \qquad y = \frac{1}{v}. \tag{15}$$

and the time rescaling $t = v\tau$. Then system (7) in the local chart (15) is

$$\begin{aligned} \dot{u} &= uv^2 + cr(1+u^2)v - u(1+u^2)(1+c), \\ \dot{v} &= v(v^2 - 1 + cruv - (c+1)u^2). \end{aligned} \tag{16}$$

If $c \neq -1$, the origin is a singular point of (16). It is easy to get that the eigenvalues of the Jacobian matrix at the origin are -1 and $-(c+1)$, which implies that system (7) has a pair of infinite saddles if $c < -1$, and a pair of node if $c > -1$.

If $c = -1$, it is obtained from (14) and (16) that the infinity $v = 0$ of the Poincaré disc is filled with singular points. Furthermore, we reduce (14) into

$$\begin{aligned} \dot{u} &= -z + r(u^2+1), \\ \dot{z} &= -u(1-rz). \end{aligned} \tag{17}$$

If $r = 0$, the origin of system (17) is a saddle, that is, there is a pair of infinite singular point of system (17). $\square$

According to Theorem 2, Proposition 9, Proposition 10 and using the invariant straight line $x = 0$ with $r \geq 0$ and the invariant circle $x^2 + y^2 = 1$, we obtain the global phase portraits of system (7) in Poincaré disc described in Figure 1.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014 SGR568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, the grant UNAB13-4E-1604, and from the recruitment program of high–end foreign experts of China.

The second author is partially supported by the state key program of NNSF of China grant number 11431008, NSF of Shanghai grant number 15ZR1423700 and NSF of Jiangsu, BK 20131285.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, P.R. China

E-mail address: jiangyu@sjtu.edu.cn