LIMIT CYCLES FOR A CLASS OF THIRD–ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the limit cycles of the third–order differential equation $\dddot{x} - \mu\ddot{x} + \dot{x} - \mu x = \varepsilon F(x, \dot{x}, \ddot{x}, t)$ where $\mu \neq 0$, ε is small enough and $F \in \mathcal{C}^2$ is a 2π–periodic function of variable t.

1. Introduction and statement of the main results

One of the main problems in the theory of differential equations is the study of their periodic orbits, their existence, their number and their stability. As usual a limit cycle of a differential equation is a periodic orbit isolated in the set of all periodic orbits of the differential equation.

In this paper we study the limit cycles of the following class of third–order ordinary differential equations

$$\dddot{x} - \mu\ddot{x} + \dot{x} - \mu x = \varepsilon F(x, \dot{x}, \ddot{x}, t), \tag{1}$$

where $\mu \neq 0$ and the dot means derivative with respect to the variable t, ε is small enough and $F \in \mathcal{C}^2$ is a 2π–periodic function of variable t. Here the variables x and t, and the parameters μ and ε are real.

There are many papers studying the periodic orbits of third–order differential equations. Thus our class of equations is not far from the ones studied in [13] and [3]. But our main tool for studying the periodic orbits of equation (1) is completely different to the tools of the mentioned papers. We shall use the averaging theory, more precisely the Theorem 5 of the Appendix. Many of the papers dealing with the periodic orbits of third–order differential equations use Schauder's or Leray–Schauder's fixed point theorem, see for instance [5, 8, 9], or the nonlocal reduction method see [2], and others [6].

In order to state our main result we need some preliminaries. We define

$$f_1(X_0, Y_0) = \int_0^{2\pi} \sin t \; F\left(\frac{\mu X(t) - Y(t)}{1+\mu^2}, -\frac{X(t) + \mu Y(t)}{1+\mu^2}, \frac{-\mu X(t) + Y(t)}{1+\mu^2}, t\right) dt,$$
$$f_2(X_0, Y_0) = \int_0^{2\pi} \cos t \; F\left(\frac{\mu X(t) - Y(t)}{1+\mu^2}, -\frac{X(t) + \mu Y(t)}{1+\mu^2}, \frac{-\mu X(t) + Y(t)}{1+\mu^2}, t\right) dt,$$

where

$$X(t) = X_0 \cos t - Y_0 \sin t, \quad Y(t) = Y_0 \cos t + X_0 \sin t. \tag{2}$$

Our main result is the following.

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Theorem 1. *If there exists* $(X_0, Y_0) \in \mathbb{R}^2$ *such that* $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$ *and* $\det\big(\partial(f_1, f_2)/\partial(X_0, Y_0)\big) \neq 0$, *then for* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *with* $\varepsilon_0 > 0$ *sufficiently small there is a* 2π*–periodic solution* $x(t, \varepsilon)$ *of the third–order differential equation* (1) *such that*

$$\big(x(0,\varepsilon), \dot{x}(0,\varepsilon), \ddot{x}(0,\varepsilon)\big) \to \left(\frac{\mu X_0 - Y_0}{1+\mu^2}, \frac{-X_0 - \mu Y_0}{1+\mu^2}, \frac{-\mu X_0 + Y_0}{1+\mu^2}\right)$$

as $\varepsilon \to 0$. *Moreover, for* $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ *the* 2π*–periodic solution* $x(t,\varepsilon)$ *is a limit cycle.*

Theorem 1 will be proved in Section 2.

The linear differential equation of third–order $\dddot{x} - \mu\ddot{x} + \dot{x} - \mu x = 0$ provides a linear system in $\mathbb{R}^3$ having a 2-dimensional center. Theorem 1 reduces the study of the limit cycles of the differential equation of third–order (1) bifurcating from the periodic orbits of that center to find the nondegenerate zeros of the system of two equations and two unknowns given by $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$. The zeros are nondegenerate in the sense that the Jacobian of the system on them must be nonzero. In general the problem of finding the zeros of two nonlinear equations with two unknowns is not easy, but of course is easier than to look for the periodic orbits directly.

Using Theorem 1 we have studied the limit cycles of some third–order differential equations. Thus in the next result we present a third–order differential equation (1) having as many limit cycles as we want.

Proposition 2. *We consider the third–order differential equation*

$$\dddot{x} - \ddot{x} + \dot{x} - x = \varepsilon\cos(x + t). \tag{3}$$

Then for all positive integer m *there is* $\varepsilon_m > 0$ *such that if* $\varepsilon \in [-\varepsilon_m, \varepsilon_m] \setminus \{0\}$ *equation* (3) *has at least* m *limit cycles.*

Proposition 2 will be proved in Section 3.

The following third–order differential equation (1) only has finitely many limit cycles obtained using Theorem 1. As usual $[\cdot]$ denotes the integer part function.

Proposition 3. *We consider the third–order differential equation*

$$\dddot{x} - \ddot{x} + \dot{x} - x = \varepsilon\Big(\sum_{\substack{i+j+k=0\\ i,j,k\geq 0}}^{n} a_{ijk}x^i\dot{x}^j\ddot{x}^k + \cos t\Big). \tag{4}$$

Then for $\varepsilon \neq 0$ *sufficiently small equation* (4) *has at least* $m \in \{1, 2, \ldots, 2[(n-1)/2] + 1\}$ *limit cycles choosing conveniently the coefficients* a_{ijk}.

Proposition 3 will be proved in Section 4.

The third–order differential equation studied in the next proposition when $\mu = \varepsilon$ is close to the equation studied in the example 1 of [3]. Moreover that equation without the term $(\cos t)/2$ is the equation of the Ezeilo problem mentioned in [2].

Proposition 4. *We consider the third–order differential equation*

$$\dddot{x} - \mu\ddot{x} + \dot{x} - \mu x = \varepsilon\left(\sin x - x + \frac{1}{2}\cos t\right). \tag{5}$$

Then for $\varepsilon \neq 0$ *sufficiently small equation* (5) *has at least two limit cycles.*

Proposition 4 will be proved in Section 5. As we shall see in its proof Theorem 1 applied to equation (5) provides at most one limit cycle bifurcating from the linear center of system (5) with $\varepsilon = 0$. But eventually equation (5) can have for $\varepsilon \neq 0$ other limit cycles which do not bifurcate from the mentioned linear center.

2. Proof of Theorem 1

If $y = \dot{x}$ and $z = \ddot{x}$, then system (1) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= \mu x - y + \mu z + \varepsilon F(x, y, z, t). \end{aligned} \tag{6}$$

The origin $(0, 0, 0)$ is the unique singular point of system (6) when $\varepsilon = 0$. The eigenvalues of the linearized system at this singular point are $\pm i$ and μ. By the linear invertible transformation $(X, Y, Z)^T = C(x, y, z)^T$, where

$$C = \begin{pmatrix} \mu & -1 & 0 \\ 0 & -\mu & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

we transform the system in another such that its linear part is the real Jordan normal form of the linear part of system (6) with $\varepsilon = 0$, i.e.

$$\begin{aligned} \dot{X} &= -Y, \\ \dot{Y} &= X + \varepsilon\tilde{F}(X, Y, Z, t), \\ \dot{Z} &= \mu Z + \varepsilon\tilde{F}(X, Y, Z, t), \end{aligned} \tag{7}$$

where

$$\tilde{F}(X, Y, Z, t) = F\left(\frac{\mu X - Y + Z}{1 + \mu^2}, \frac{-X - \mu Y + \mu Z}{1 + \mu^2}, \frac{-\mu X + Y + \mu^2 Z}{1 + \mu^2}, t\right).$$

Using the notation introduced in the Appendix we have that $\mathbf{x} = (X, Y, Z)$, $F_0(\mathbf{x}, t) = (-Y, X, \mu Z)$, $F_1(\mathbf{x}, t) = (0, \tilde{F}, \tilde{F})$ and $F_2(\mathbf{x}, t) = 0$. Let $\mathbf{x}(t; X_0, Y_0, Z_0, \varepsilon)$ be the solution of system (7) such that $\mathbf{x}(0; X_0, Y_0, Z_0, \varepsilon) = (X_0, Y_0, Z_0)$. Clearly the unperturbed system (7) with $\varepsilon = 0$ has a linear center at the origin in the (X, Y)–plane, which is an invariant plane under the flow of the unperturbed system, and the periodic solution $\mathbf{x}(t; X_0, Y_0, 0, 0) = (X(t), Y(t), Z(t))$ is

$$X(t) = X_0 \cos t - Y_0 \sin t, \quad Y(t) = Y_0 \cos t + X_0 \sin t, \quad Z(t) = 0. \tag{8}$$

Note that all these periodic orbits have period 2π.

For our system the V and the α of Theorem 5 of the Appendix are $V = \{(X, Y, 0) : 0 < X^2 + Y^2 < \rho\}$ for some arbitrary $\rho > 0$ and $\alpha = (X_0, Y_0) \in V$.

The fundamental matrix solution $M(t)$ of the variational equation of the unperturbed system $(7)_{\varepsilon=0}$ with respect to the periodic orbits (8) satisfying that $M(0)$ is the identity matrix is

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{\mu t} \end{pmatrix}.$$

We remark that it is independent of the initial condition $(X_0, Y_0, 0)$. Moreover an easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi\mu} \end{pmatrix}.$$

In short we have shown that all the assumptions of Theorem 5 of the Appendix hold. Hence we shall study the zeros $\alpha = (X_0, Y_0) \in V$ of the two components of the function $\mathcal{F}(\alpha)$ given in (22). More precisely we have $\mathcal{F}(\alpha) = (\mathcal{F}_1(\alpha), \mathcal{F}_2(\alpha))$ where

$$\begin{aligned} \mathcal{F}_1(\alpha) &= \int_0^{2\pi} \sin t \tilde{F}(\mathbf{x}(t; X_0, Y_0, 0, 0), t) dt \\ &= \int_0^{2\pi} \sin t F\left(\frac{\mu X(t) - Y(t)}{1+\mu^2}, -\frac{X(t) + \mu Y(t)}{1+\mu^2}, \frac{-\mu X(t) + Y(t)}{1+\mu^2}, t \right) dt, \\ \mathcal{F}_2(\alpha) &= \int_0^{2\pi} \cos t \tilde{F}(\mathbf{x}(t; X_0, Y_0, 0, 0), t) dt \\ &= \int_0^{2\pi} \cos t F\left(\frac{\mu X(t) - Y(t)}{1+\mu^2}, -\frac{X(t) + \mu Y(t)}{1+\mu^2}, \frac{-\mu X(t) + Y(t)}{1+\mu^2}, t \right) dt, \end{aligned}$$

where $X(t), Y(t)$ are given by (8). Now the rest of the proof of Theorem 1 follows directly from the statement of Theorem 5 in Appendix.

3. Proof of Proposition 2

First we consider the third–order differential equation (3). For this equation we have that

$$\begin{aligned} f_1(X_0, Y_0) &= \int_0^{2\pi} \sin t \cos\left(t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t)}{2} \right) dt, \\ f_2(X_0, Y_0) &= \int_0^{2\pi} \cos t \cos\left(t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t)}{2} \right) dt. \end{aligned}$$

To simplify the computation of these two previous integrals we do the change of variables $(X_0, Y_0) \longmapsto (r, s)$ given by

$$X_0 - Y_0 = 2r\cos s, \quad X_0 + Y_0 = -2r \sin s, \tag{9}$$

where $r > 0$ and $s \in [0, 2\pi)$. From now on and until the end of the paper we write $f_1(r, s)$ instead of

$$f_1(X_0, Y_0) = f_1\big(r(\cos s - \sin s), -r(\cos s + \sin s)\big).$$

Similarly for $f_2(r, s)$.

We compute the two previous integrals and we get

$$\begin{aligned} f_1(r, s) &= -\pi J_2(r) \sin 2s, \\ f_2(r, s) &= 2\pi \left(\frac{1}{r} J_1(r) - J_2(r) \cos^2 s \right), \end{aligned} \tag{10}$$

where J_1 and J_2 are the first and second Bessel functions of first kind. These computations become easier with the help of an algebraic manipulation as Mathematica or Maple.

Using the asymptotic expressions of the Bessel functions of first kind it follows that Bessel functions $J_1(r)$ and $J_2(r)$ have different zeros. Hence $f_i(r,s) = 0$ for $i = 1,2$ imply that either $s \in \{0, \pi/2, \pi, 3\pi/2\}$. Therefore we have to study the zeros of

$$f_2(r,0) = f_2(r,\pi) = 2\pi\left(\frac{1}{r}J_1(r) - J_2(r)\right), \tag{11}$$

$$f_2(r,\pi/2) = f_2(r,3\pi/2) = \frac{2\pi}{r}J_1(r). \tag{12}$$

We claim that function (11) has also infinite zeros for $r \in (0,\infty)$. Note that if ρ is sufficiently large, and we choose $r < \rho$ also sufficiently large, then

$$J_n(r) \approx \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for} \quad n = 1,2,$$

are asymptotic estimations, see [1]. Considering (11) for r sufficiently large we obtain that

$$\begin{aligned} f_2(r,0) &\approx \frac{2}{r}\sqrt{\frac{2\pi}{r}}\left(\cos\left(r - \frac{3\pi}{4}\right) + r\cos\left(r - \frac{\pi}{4}\right)\right) \\ &= \frac{2}{r}\sqrt{\frac{\pi}{r}}((r-1)\cos r + (r+1)\sin r). \end{aligned}$$

The above function has infinite zeros because the equation

$$\tan r = \frac{1-r}{r+1}$$

has infinitely many solutions.

For every zero $r_0 > 0$ of the function (11) we have two zeros of system (10), namely $(r,s) = (r_0,0)$ and $(r,s) = (r_0,\pi)$.

We have from (10) that

$$\begin{aligned} \left|\frac{\partial(f_1,f_2)}{\partial(r,s)}\right|_{(r,s)=(r_0,0)} &= \frac{4\pi^2(J_0(r_0)r_0 - 2J_1(r_0))(J_0(r_0)r_0 + (r_0^2-2)J_1(r_0))}{r_0^3} \\ &= \frac{4\pi^2}{r_0}J_2(r_0)(J_1(r_0)r_0 - J_2(r_0)), \end{aligned} \tag{13}$$

where we have used several relation between the Bessel functions of first kind, see [1]. Clearly it is impossible that (11) and (13) are equal to zero at the same time. Therefore by Theorem 1 there is a periodic orbit of system (3) for each $(r_0,0)$, that is for each value of $(X_0,Y_0) = (r_0,-r_0)$.

In an analogous way there is a periodic orbit of system (3) for each (r_0,π), that is for each value of $(X_0,Y_0) = (-r_0,r_0)$. In fact, the periodic orbit with this initial conditions and the previous one with initial conditions $(X_0,Y_0) = (r_0,-r_0)$ are the same.

Similarly since $J_1(r)$ has infinitely many zeros (see [1]), the function (12) has infinitely many positive zeros r_1. Every one of these zeros provides two solutions of system (10), namely $(r,s) = (r_1,\pi/2)$ and $(r,s) = (r_1,3\pi/2)$.

Moreover we have from (10) that

$$\left|\frac{\partial(f_1,f_2)}{\partial(r,s)}\right|_{(r,s)=(r_1,\pi/2)} = \frac{4\pi^2}{r_1}J_2^2(r_1) \neq 0. \tag{14}$$

Therefore by Theorem 1 there is a periodic orbit of system (3) for each $(r_1, \pi/2)$, that is for each value of $(X_0, Y_0) = (-r_1, -r_1)$.

In an analogous way there is a periodic orbit of system (3) for each $(r_1, 3\pi/2)$, that is for each value of $(X_0, Y_0) = (r_1, r_1)$. In fact, the periodic orbit with this initial conditions and the previous one with initial conditions $(X_0, Y_0) = (-r_1, -r_1)$ are the same.

Taking the radius ρ of the disc $V = \{(X_0, Y_0, 0) : 0 < X^2 + Y^2 < \rho\}$ in the proof of Theorem 1 conveniently large we include in it as many zeros of the system $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$ as we want, so from Theorem 1, Proposition 2 follows.

4. Proof of Proposition 3

Now we consider the third–order differential equation (4). In order to estimate the number of the periodic solutions of equation (4), according with Theorem 1 we study the solutions of $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$.

For our equation the function F which appears in the integrals of the definitions of the functions f_1 and f_2 is

$$F\left(\frac{X(t) - Y(t)}{2}, -\frac{X(t) + Y(t)}{2}, \frac{-X(t) + Y(t)}{2}, t\right).$$

Therefore from the change (9) and the expressions (2), a monomial $x^i y^j z^k$ which appears in $F(x, y, z, t)$ becomes $(-1)^k r^{i+j+k} \cos^{i+k}(s-t) \sin^j(s-t)$. Hence we obtain the following expressions

$$\begin{aligned} f_1(r, s) &= \int_0^{2\pi} \sin t \sum_{i+j+k=0}^{n} (-1)^k a_{ijk}\, r^{i+j+k} \cos^{i+k}(s-t) \sin^j(s-t) dt, \\ f_2(r, s) &= \int_0^{2\pi} \cos t \sum_{i+j+k=0}^{n} (-1)^k a_{ijk}\, r^{i+j+k} \cos^{i+k}(s-t) \sin^j(s-t) dt + \pi. \end{aligned}$$

Taking $u = s - t$ the functions f_1 and f_2 can be written as

$$\begin{aligned} f_1(r, s) &= I_1(r) \sin s - I_2(r) \cos s, \\ f_2(r, s) &= I_1(r) \cos s + I_2(r) \sin s + \pi, \end{aligned} \tag{15}$$

where

$$\begin{aligned} I_1(r) &= -\int_0^{2\pi} \cos u \sum_{i+j+k=0}^{n} (-1)^k a_{ijk}\, r^{i+j+k} \cos^{i+k} u \, \sin^j u \, du \\ &= \sum_{i+j+k=0}^{n} (-1)^{k+1} a_{ijk}\, r^{i+j+k} \int_0^{2\pi} \cos^{i+k+1} u \, \sin^j u \, du, \end{aligned}$$

$$\begin{aligned} I_2(r) &= -\int_0^{2\pi} \sin u \sum_{i+j+k=0}^{n} (-1)^k a_{ijk}\, r^{i+j+k} \cos^{i+k} u \, \sin^j u \, du \\ &= \sum_{i+j+k=0}^{n} (-1)^{k+1} a_{ijk}\, r^{i+j+k} \int_0^{2\pi} \cos^{i+k} u \, \sin^{j+1} u \, du. \end{aligned}$$

Using symmetries the integral $\int_0^{2\pi} \cos^p u \, \sin^q u \, du$ is not zero if and only if p and q are even. Hence $I_1(r)$ and $I_2(r)$ are polynomials in r having all their monomials of odd degree. Moreover if n is even the degree in the variable r of the polynomials

$I_1(r)$ and $I_2(r)$ is $n-1$, and if n is odd that degree is n. So their degree always is odd and equal to $2[(n-1)/2]+1$. Of course we are playing with the fact that the coefficients of those polynomials can be chosen arbitrarily.

It is clear that the system $f_1 = f_2 = 0$ given by (15) is equivalent to the system

$$\begin{pmatrix} I_1(r) \\ I_2(r) \end{pmatrix} = \begin{pmatrix} \sin s & -\cos s \\ \cos s & \sin s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \pi \end{pmatrix} = \pi \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}. \tag{16}$$

We claim that (16) has at most $2[(n-1)/2]+1$ solutions providing different limit cycles of the third–order differential equation (4), and that this number is reached.

For proving the claim first we observe that system (16) is equivalent to the system

$$I_1^2(r) + I_2^2(r) = \pi^2, \quad \frac{I_2(r)}{I_1(r)} = \tan s. \tag{17}$$

Since the first equation of system (17) is a polynomial equation in the variable r^2 of degree $2[(n-1)/2]+1$ playing with the fact that the coefficients of the polynomials $I_1(r)$ and $I_1(r)$ are arbitrary, it follows that it has at most $2[(n-1)/2]+1$ zeros in $(0,\infty)$, and we can choose the coefficients a_{ijk} such that it has exactly m simple zeros $r_i > 0$ with $m \in \{1, 2, \ldots, 2[(n-1)/2]+1\}$.

There are two solutions s_i and $s_i + \pi$ in $[0, 2\pi)$ of the second equation for each zero $r_i > 0$ of the first equation of (17). But as in the proof of Proposition 2 these two solutions only provide two different initial conditions of the same periodic orbit. In short applying Theorem 1 we would get at most $2[(n-1)/2]+1$ limit cycles for the third–order differential equation (4) if the Jacobian $\det\big(\partial(f_1, f_2)/\partial(r, s)\big) \neq 0$ at $(r, s) = (r_i, s_i)$.

Playing with the coefficients a_{ijk} we get

$$I_1(r_i)I_1'(r_i) + I_2(r_i)I_2'(r_i) \neq 0, \tag{18}$$

for every solution (r_i, s_i) of system (17). Then we compute the Jacobian of $g_1(r, s) = I_1(r) - \pi\cos s$ and $g_2(r, s) = I_2(r) - \pi \sin s$, i. e.

$$\begin{aligned} \left|\frac{\partial(g_1(r,s), g_2(r,s))}{\partial(r,s)}\right|_{(r,s)=(r_i,s_i)} &= -\pi\big(I_1'(r_i)\cos s_i + I_2'(r_i)\sin s_i\big) \\ &= I_1(r_i)I_1'(r_i) + I_2(r_i)I_2'(r_i) \neq 0. \end{aligned}$$

Hence it is easy to check that

$$\left|\frac{\partial(f_1(r,s), f_2(r,s))}{\partial(r,s)}\right|_{(r,s)=(r_i,s_i)} \neq 0.$$

In short the claim is proved and consequently the Proposition 3.

5. Proof of Proposition 4

In this section we consider equation (5). We remark here that $F(x, y, z, t) = \sin x - x + \cos t/2$. Doing the change

$$\mu X_0 - Y_0 = (1+\mu^2) r \cos s, \quad X_0 + \mu Y_0 = -(1+\mu^2) r \sin s,$$

where $r > 0$ and $s \in [0, 2\pi)$. We compute the two integrals of f_1 and f_2 and we get

$$f_1(r,s) = \int_0^{2\pi} \sin t(\sin(r\cos(t-s)) - r\cos(t-s))dt,$$
$$f_2(r,s) = \int_0^{2\pi} \cos t(\sin(r\cos(t-s)) - r\cos(t-s))dt + \frac{\pi}{2}.$$

Taking $u = t - s$, then we have

$$f_1(r,s) = \sin s\left(\int_0^{2\pi} \cos u \sin(r\cos u)\ du - \pi r\right) = \pi(2J_1(r) - r)\sin s,$$
$$f_2(r,s) = \cos s\left(\int_0^{2\pi} \cos u \sin(r\cos u)\ du - \pi r\right) + \frac{\pi}{2} = \pi(2J_1(r) - r)\cos s + \frac{\pi}{2}.$$

It is clear that if $f_1(r,s) = 0$ and $f_2(r,s)$=0, then $\sin s = 0$. Consequently we need to estimate the zeros of the following function for $s = 0$ or π

$$g_\pm(r) = \pm\left(\int_0^{2\pi} \cos u \sin(r\cos u)\ du - \pi r\right) + \frac{\pi}{2} = \pm\pi(2J_1(r) - r) + \frac{\pi}{2}.$$

We claim that there is a unique zero of the function $g_\pm(r)$ for $r > 0$. For example we note that for $s = 0$ and $r > 0$ we have that

$$g'_+(r) = \frac{dg_+(r)}{dr} = \int_0^{2\pi} \cos^2 u \cos(r\cos u)\ du - \pi < \int_0^{2\pi} \cos^2 u\ du - \pi = 0.$$

Hence our claim is true from the fact that $g_+(0) = \pi/2 > 0$, $g_+(r) < 0$ for r sufficiently large and $g_+(r)$ is strictly decreasing. We denote by $r_\pm$ the unique zero of the function $g_\pm(r)$ for $r > 0$.

Writing the zeros of f_1 and f_2 as $(r_+, 0)$ and (r_-, π), we have that

$$\left|\frac{\partial(f_1(r,s), f_2(r,s))}{\partial(r,s)}\right|_{(r,s)=(r_0,0)} = \frac{\pi}{2}g'_+(r_0) < 0.$$

Similarly it can be shown that this Jacobian at the point (r_0, π) is also different from zero. This implies that the system $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$ has two solutions $(X_0, Y_0) = r_+(\mu, -1)$ and $(X_0, Y_0) = r_-(-\mu, 1)$ corresponding to $(r_+, 0)$ and (r_-, π) with Jacobian different from zero. Since $r_+ \neq r_-$ these two solutions provide distinct periodic orbit of the linear center from which it bifurcates one limit cycle of equation (5). So Proposition 4 is proved.

6. Appendix

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\mathbf{x}'(t) = F_0(\mathbf{x}, t) + \varepsilon F_1(\mathbf{x}, t) + \varepsilon^2 F_2(\mathbf{x}, t, \varepsilon), \tag{19}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \Omega \times \mathbb{R} \to \mathbb{R}^n$ and $F_2 : \Omega \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the variable t, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\mathbf{x}'(t) = F_0(\mathbf{x}, t), \tag{20}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst [11], and of Verhulst [12].

Let $\mathbf{x}(t,\mathbf{z})$ be the solution of the unperturbed system (20) such that $\mathbf{x}(0,\mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t,\mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(\mathbf{x}(t,\mathbf{z}),t)\mathbf{y}. \tag{21}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (21), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_k)$.

Theorem 5. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha))\,,\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha)$ of (20) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (21) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the right up corner the $k \times (n-k)$ zero matrix, and in the right down corner a $(n-k)\times(n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(\mathbf{x}(t,\mathbf{z}_\alpha),t)dt\right). \tag{22}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\alpha\right)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t,\varepsilon)$ of system (19) such that $\varphi(0,\varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 5 goes back to Malkin [7] and Roseau [10], for a shorter proof see [4].

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