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POLYNOMIAL FIRST INTEGRALS OF QUADRATIC SYSTEMS

JAUME LLIBRE AND XIANG ZHANG

ABSTRACT. The main purpose of this paper is to give the classification and the topological phase portraits of all quadratic systems having minimal polynomial first integrals of degree less than 5, and to prove the existence of minimal polynomial first integrals of any degree for quadratic systems. Moreover, we prove that quadratic systems with minimal polynomial first integrals of degree larger than 1 have at most three invariant straight lines, and under convenient assumptions we give the greatest degree of the irreducible polynomial first integrals.

1. Introduction. By definition, a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y and the independent variable (the *time*) t are all real, and $P, Q \in \mathbf{R}[x, y]$, as usual $\mathbf{R}[x, y]$ denotes the ring of polynomials in the variables x and y with real coefficients. In what follows, all mentioned functions are in $\mathbf{R}[x, y]$ and all constants are real. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. The polynomial systems of degree 2 will be called *quadratic systems*.

Quadratic systems have been investigated intensively, and nearly 1000 papers have been published about these systems (see, for instance, [**23**], [**26**] and [**27**]). But it is an open problem to know what are the integrable quadratic systems. We must define what it means that a polynomial system is integrable because this notation changes with the authors, see, for instance, [**8**].

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