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Darboux theory of integrability in $\mathbb{C}^n$ taking into account the multiplicity

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ABSTRACT

Darboux in 1878 provided a theory on the existence of first integrals of polynomial systems based on the existence of sufficient invariant algebraic hypersurfaces, called now the Darboux theory of integrability. In 1979 Jouanolou successfully improved the Darboux theory of integrability characterizing the existence of rational first integrals, for this he used sophisticated tools of algebraic geometry. The aim of this paper is to improve the classical result of Darboux and the new one of Jouanolou taking into account the multiplicity of the invariant algebraic hypersurfaces. Additionally our proof of the improved result of Jouanolou is much simpler and elementary than the original one. Some examples show that the improved Darboux theory of integrability with multiplicity is much useful than the classical one.

1. Introduction

In many branches of applied mathematics, physics and, in general, in applied sciences appear nonlinear ordinary differential equations. If a differential equation or vector field defined in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a first integral, then its study can be reduced in one dimension; of course working with real or complex time, respectively. Therefore a natural question is: *Given a vector field in $\mathbb{R}^n$ or $\mathbb{C}^n$, how to recognize if this vector field has a first integral*? This question has no a good answer up to now.

In this paper we shall study the existence of Darboux first integrals of polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$, and in particular of rational first integrals. The best answer to this question using Darboux first integral was given by Darboux [5,6] in 1878, and for the rational first integral the best

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