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HOPF BIFURCATION IN HIGHER DIMENSIONAL DIFFERENTIAL SYSTEMS VIA THE AVERAGING METHOD

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Dedicated to the memory of Professor Ye Yanqian

We study the Hopf bifurcation of $\mathscr{C}^3$ differential systems in $\mathbb{R}^n$ showing that l limit cycles can bifurcate from one singularity with eigenvalues $\pm bi$ and $n-2$ zeros with $l \in \{0, 1, \dots, 2^{n-3}\}$. As far as we know this is the first time that it is proved that the number of limit cycles that can bifurcate in a Hopf bifurcation increases exponentially with the dimension of the space. To prove this result, we use first-order averaging theory. Further, in dimension 4 we characterize the shape and the kind of stability of the bifurcated limit cycles. We apply our results to certain fourth-order differential equations and then to a simplified Marchuk model that describes immune response.

1. Introduction and statement of the main results

In this work we study the Hopf bifurcation of $\mathscr{C}^3$ differential systems in $\mathbb{R}^n$ with $n \geq 3$ by using first-order averaging theory. We assume that these systems have a singularity at the origin, whose linear part has eigenvalues $\varepsilon a \pm bi$ and εc_k for $k = 3, \dots, n$, where ε is a small parameter. Such systems can be written in the form

$$
\begin{aligned}
\dot{x} &= \varepsilon a x - b y + \sum_{i_1+\cdots+i_n=2} a_{i_1\cdots i_n} x^{i_1} y^{i_2} z_3^{i_3} \cdots z_n^{i_n} + \mathscr{A}, \\
\dot{y} &= b x + \varepsilon a y + \sum_{i_1+\cdots+i_n=2} b_{i_1\cdots i_n} x^{i_1} y^{i_2} z_3^{i_3} \cdots z_n^{i_n} + \mathscr{B}, \\
\dot{z}_k &= \varepsilon c_k z_k + \sum_{i_1+\cdots+i_n=2} c^{(k)}_{i_1\cdots i_n} x^{i_1} y^{i_2} z_3^{i_3} \cdots z_n^{i_n} + \mathscr{C}_k, \quad k = 3, \dots, n,
\end{aligned} \tag{1}
$$

where $a_{i_1\cdots i_n}$, $b_{i_1\cdots i_n}$, $c^{(k)}_{i_1\cdots i_n}$, a, b and c_k are real parameters, $ab \neq 0$, and $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}_k$ are the Lagrange expressions of the error function of third order in the expansion of the functions of the system in Taylor series.

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