RATIONAL FIRST INTEGRALS IN THE DARBOUX THEORY OF INTEGRABILITY IN $\mathbb{C}^n$

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Abstract. In 1976 Jouanolou showed that if the number of invariant algebraic hypersurfaces of a polynomial vector field in $\mathbb{R}^n$ or $\mathbb{C}^n$ of degree d is at least $\begin{pmatrix} d+n-1 \\ n \end{pmatrix} + n$, then the vector field has a rational first integral. His proof used sophisticated tools of algebraic geometry. We provide an easy and elementary proof of Jouanolou's result using linear algebra.

1. Introduction

Nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system or a vector field defined in $\mathbb{R}^n$ or $\mathbb{C}^n$ the existence of a first integral reduces the study of its dynamics in one dimension; of course working with real or complex time, respectively. So a natural question is: *Given a vector field on $\mathbb{R}^n$ or $\mathbb{C}^n$, how to recognize if this vector field has a first integral?* This question has no a satisfactory answer up to now. Many different methods have been used for studying the existence of first integrals of vector fields. Some of these methods based on: Noether symmetries [4], the Darboux theory of integrability [7], the Lie symmetries [13], the Painlevé analysis [2], the use of Lax pairs [11], the direct method [8] and [9], the linear compatibility analysis method [14], the Carlemann embedding procedure [3] and [1], the quasimonomial formalism [2], etc.

In this paper we shall study the existence of rational first integrals of a polynomial vector field in $\mathbb{R}^n$ or $\mathbb{C}^n$. The best answer to this question was given by Jouanolou [10] in 1979 inside the Darboux theory of integrability. This theory of integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic hypersurfaces that they have.

Darboux [7] showed how can be constructed a first integral of polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$ possessing sufficient invariant algebraic curves. In particular he proved that if a planar polynomial vector field in $\mathbb{R}^2$ or $\mathbb{C}^2$ of degree d has at least $\begin{pmatrix} d+1 \\ 2 \end{pmatrix} + 1$ invariant algebraic curves, then it has a first integral, which can be computed using these invariant algebraic curves. Jouanolou [10] shows that if the number of invariant algebraic curves of a planar polynomial vector field in $\mathbb{R}^2$ or $\mathbb{C}^2$ of degree d is at least $\begin{pmatrix} d+1 \\ 2 \end{pmatrix} + 2$, then the vector field has a rational first integral, which also can be computed using the invariant algebraic curves.

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In fact the results of the previous paragraph for polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$ extend to polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$. Thus it is known (see for instance [12]) that if a polynomial vector field of degree d in $\mathbb{R}^n$ or $\mathbb{C}^n$ has at least $\binom{d+n-1}{n}+1$ invariant algebraic hypersurfaces, then it has a first integral, which can be computed using these invariant algebraic hypersurfaces. Jouanolou [10] shows that if the number of invariant algebraic hypersurfaces of a polynomial vector field in $\mathbb{R}^n$ or $\mathbb{C}^n$ of degree d is at least $\binom{d+n-1}{n}+n$, then the vector field has a rational first integral, which again can be computed using these invariant algebraic hypersurfaces.

The proof of Jouanolou uses sophisticated techniques of algebraic geometry. For polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$ an elementary proof of Jouanolou's result was given in [5, 6]. Up to now an easy proof of Jouanolou's result in $\mathbb{R}^n$ or $\mathbb{C}^n$ was not given. The goal of this paper is to provided such elementary proof. Our proof is shorter, self–contained and only uses linear algebra.

The paper is organized as follows. In Section 2 we provide the notation and definitions, and we state the Jouanolou's result. In Section 3 we work with the notion of functionally independence and first integrals. Finally in Section 4 we prove Jouanolou's result.

2. Definitions and statement of the main result

Since any polynomial differential system in $\mathbb{R}^n$ can be thought as a polynomial differential inside $\mathbb{C}^n$ we shall work only in $\mathbb{C}^n$. If our initial differential system is in $\mathbb{R}^n$, once we get a complex first integral of this system thought inside $\mathbb{C}^n$ taking the square of the modulus of this complex integral we have a real first integral. Moreover if that complex first integral is rational, the real one defined as before also is rational. In short in the rest of the paper we work all the time in $\mathbb{C}^n$.

As usual $\mathbb{C}[x]=\mathbb{C}[x_1,\ldots,x_n]$ denotes the ring of all complex polynomials in the variables $x_1,\ldots,x_n$. We consider the *polynomial vector field* in $\mathbb{C}^n$

$$\mathcal{X}=\sum_{i=1}^{n} P_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}, \qquad (x_1,\ldots,x_n)\in\mathbb{C}^n, \tag{1}$$

where $P_i=P_i(x_1,\ldots,x_n)\in\mathbb{C}[x]$ for $i=1,\ldots,n$. The integer $d=\max\{\deg P_1,\ldots,\deg P_n\}$ is the *degree* of the vector field $\mathcal{X}$. Usually for simplicity the vector field $\mathcal{X}$ will be represented by $(P_1,\ldots,P_n)$.

Let $f=f(x)\in\mathbb{C}[x]$. We say that $\{f=0\}\subset\mathbb{C}^n$ is an *invariant algebraic hypersurface* of the vector field $\mathcal{X}$ if there exists a polynomial $k\in\mathbb{C}[x]$ such that

$$\mathcal{X}f=\sum_{i=1}^{n} P_i\frac{\partial f}{\partial x_i}=kf.$$

The polynomial k is called the *cofactor* of $f=0$. Note that from this definition the degree of k is at most $d-1$, and also that if an orbit $x(t)$ of the vector field $\mathcal{X}$ has a point on $\{f=0\}$, then the whole orbit is contained in $\{f=0\}$. This justifies the name of invariant algebraic hypersurface, it is invariant by the flow of the vector field $\mathcal{X}$.

Let $\mathcal{D}$ be an open subset of $\mathbb{C}^n$ having full Lebesgue measure in $\mathbb{C}^n$. A non–constant holomorphic function $H : \mathcal{D} \to \mathbb{C}$ is a *first integral* of the polynomial vector field $\mathcal{X}$ on $\mathcal{D}$ if it is constant on all orbits $x(t)$ of $\mathcal{X}$ contained in $\mathcal{D}$; i.e. $H(x(t)) =$ constant for all values of t for which the solution $x(t)$ is defined and contained in $\mathcal{D}$. Clearly H is a first integral of $\mathcal{X}$ on $\mathcal{D}$ if and only if $\mathcal{X}H = 0$ on $\mathcal{D}$. Of course a *rational first integral* is a first integral given by a rational function.

The Jouanolou's result mentioned in the introduction can be stated as follows.

Theorem 1. *Let $\mathcal{X}$ be a polynomial vector field defined in $\mathbb{C}^n$ of degree $d > 0$. Then $\mathcal{X}$ admits $\binom{d+n-1}{n} + n$ irreducible invariant algebraic hypersurfaces if and only if $\mathcal{X}$ has a rational first integral.*

Under the assumptions of Theorem 1 all the orbits of the vector field $\mathcal{X}$ are contained in invariant algebraic hypersurfaces.

3. Preliminary result

Assume that $H_j(x)$ for $j = 1, \ldots, m$ are holomorphic first integrals of system (1) defined in a full Lebesgue measurable subset $\mathcal{D}_1$ of $\mathbb{C}^n$. For each $x \in \mathcal{D}_1$ let $r(x)$ be the rank of the m vectors $\nabla H_1(x), \ldots, \nabla H_m(x)$ in $\mathbb{C}^n$, where $\nabla H_k(x)$ denotes the gradient of the function $H_k(x)$ with respect to x.

We say that $H_1, \ldots, H_m$ are *functionally independent* in $\mathcal{D}_1$ if $r(x) = m$ for all $x \in \mathcal{D}_1$ except possibly a subset of Lebesgue measure zero.

We say that $H_1, \ldots, H_m$ are *k–functionally independent* in $\mathcal{D}_1$ if there exist k of these $H_1, \ldots, H_m$ which are functionally independent in $\mathcal{D}_1$, and any $k + 1$ elements of $\{H_1, \ldots, H_m\}$ are not functionally independent in any positive Lebesgue measurable subset of $\mathcal{D}_1$.

It is easy to check that if m first integrals $H_1, \ldots, H_m$ of a polynomial vector field in $\mathbb{C}^n$ are k–functionally independent then $k \leq n - 1$.

Theorem 2. *For $k < m$ we assume that $H_1, \ldots, H_m$ are k–functionally independent first integrals of the polynomial vector field $\mathcal{X}$ given by* (1)*. Without loss of generality we can assume that $H_1, \ldots, H_k$ are functionally independent.*

(a) *For each $s \in \{k + 1, \ldots, m\}$ there exist holomorphic functions $C_{s1}(x), \ldots,$ $C_{sk}(x)$ defined on a full Lebesgue measurable subset of $\mathbb{C}^n$ such that*

$$\nabla H_s(x) = C_{s1}(x)\nabla H_1(x) + \ldots + C_{sk}(x)\nabla H_k(x). \tag{2}$$

(b) *For every $s \in \{k+1, \ldots, m\}$ and $j \in \{1, \ldots, k\}$ the function $C_{sj}(x)$* (*if not a constant*) *is a first integral of system* (1)*.*

Proof. Let $\mathcal{D}_1$ be the full Lebesgue measurable subset of $\mathbb{C}^n$ where the first integrals $H_1, \ldots, H_m$ are k–functionally independent.

From the assumptions there exists a full measurable subset $\mathcal{D}_2 \subset \mathcal{D}_1$ such that for each $x \in \mathcal{D}_2$, $\nabla H_1(x), \ldots, \nabla H_k(x)$ are linearly independent in $\mathbb{C}^n$, and such that for each $x \in \mathcal{D}_2$, $s \in \{k + 1, \ldots, m\}$, the vector $\nabla H_s(x)$ is linearly dependent on $\nabla H_1(x), \ldots, \nabla H_k(x)$ in $\mathbb{C}^n$. So there exist functions $C_{s1}(x), \ldots, C_{sk}(x)$ such that the equality (2) holds for every $x \in \mathcal{D}_2$. These functions $C_{s1}(x), \ldots, C_{sk}(x)$ defined on $\mathcal{D}_2$ can be expressed in function of the ∇H_j's for $j = 1, \ldots, k, s$ using the Cramer's rule. So they are holomorphic in $\mathcal{D}_2$ because the functions $H_1, \ldots, H_k$

and H_s are holomorphic and the gradient vectors of the functions $H_1, \ldots, H_k$ has rank k. This proves statement (a).

The points x which appear in the following expressions are points of $\mathcal{D}_2$. For any $i, j \in \{1, \ldots, n\}$ from (2) we have

$$\frac{\partial H_s}{\partial x_i} = C_{s1}(x)\frac{\partial H_1}{\partial x_i} + \ldots + C_{sk}(x)\frac{\partial H_k}{\partial x_i} \text{ and } \frac{\partial H_s}{\partial x_j} = C_{s1}(x)\frac{\partial H_1}{\partial x_j} + \ldots + C_{sk}(x)\frac{\partial H_k}{\partial x_j}.$$

Derivating these two equations with respect to x_j and x_i respectively, and subtracting the two resulting equations we get

$$\frac{\partial C_{s1}}{\partial x_i}\frac{\partial H_1}{\partial x_j} - \frac{\partial C_{s1}}{\partial x_j}\frac{\partial H_1}{\partial x_i} + \ldots + \frac{\partial C_{sk}}{\partial x_i}\frac{\partial H_k}{\partial x_j} - \frac{\partial C_{sk}}{\partial x_j}\frac{\partial H_k}{\partial x_i} = 0. \tag{3}$$

Since $k \le n-1$. We consider two cases. First we assume that $k = n-1$. From (3) we get

$$\sum_{1 \le i < j \le n} \left(\left(\frac{\partial C_{s1}}{\partial x_i}\frac{\partial H_1}{\partial x_j} - \frac{\partial C_{s1}}{\partial x_j}\frac{\partial H_1}{\partial x_i} + \ldots + \frac{\partial C_{sk}}{\partial x_i}\frac{\partial H_k}{\partial x_j} - \frac{\partial C_{sk}}{\partial x_j}\frac{\partial H_k}{\partial x_i} \right) \cdot \right.$$
$$\left. \sum_{\sigma(k_1, k_2 \ldots, k_{n-2})} (-1)^{\tau(ijk_1k_2\ldots,k_{n-2})} \frac{\partial H_2}{\partial x_{k_1}}\frac{\partial H_3}{\partial x_{k_2}} \cdots \frac{\partial H_{n-1}}{\partial x_{k_{n-2}}} \right) = 0,$$

where σ is a permutation of $\{1, \ldots, n\} \setminus \{i, j\}$ and the second summation is taken over all these possible permutations; τ evaluated on a permutation of $\{1, \ldots, n\}$ is the minimum number of transpositions for passing the permutation to the identity. In fact this last equation can be written as

$$\begin{vmatrix} \dfrac{\partial C_{s1}}{\partial x_1} & \dfrac{\partial C_{s1}}{\partial x_2} & \cdots & \dfrac{\partial C_{s1}}{\partial x_n} \\ \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \cdots & \dfrac{\partial H_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial H_{n-1}}{\partial x_1} & \dfrac{\partial H_{n-1}}{\partial x_2} & \cdots & \dfrac{\partial H_{n-1}}{\partial x_n} \end{vmatrix} = 0. \tag{4}$$

This equality follows from the following two facts

$$\sum_{1 \le i < j \le n} \left(\frac{\partial C_{s1}}{\partial x_i}\frac{\partial H_1}{\partial x_j} - \frac{\partial C_{s1}}{\partial x_j}\frac{\partial H_1}{\partial x_i} \right) \sum_{\sigma(k_1, k_2 \ldots, k_{n-2})} (-1)^{\tau(ijk_1k_2\ldots,k_{n-2})} \frac{\partial H_2}{\partial x_{k_1}}\frac{\partial H_3}{\partial x_{k_2}} \cdots \frac{\partial H_{n-1}}{\partial x_{k_{n-2}}}$$
$$= \begin{vmatrix} \dfrac{\partial C_{s1}}{\partial x_1} & \dfrac{\partial C_{s1}}{\partial x_2} & \cdots & \dfrac{\partial C_{s1}}{\partial x_n} \\ \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \cdots & \dfrac{\partial H_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial H_{n-1}}{\partial x_1} & \dfrac{\partial H_{n-1}}{\partial x_2} & \cdots & \dfrac{\partial H_{n-1}}{\partial x_n} \end{vmatrix},$$

and for $l = 2, \ldots, k$

$$\sum_{1\le i<j\le n}\left(\frac{\partial C_{sl}}{\partial x_i}\frac{\partial H_l}{\partial x_j}-\frac{\partial C_{sl}}{\partial x_j}\frac{\partial H_l}{\partial x_i}\right)\sum_{\sigma(k_1,k_2\ldots,k_{n-2})}(-1)^{\tau(ijk_1k_2\ldots,k_{n-2})}\frac{\partial H_2}{\partial x_{k_1}}\frac{\partial H_3}{\partial x_{k_2}}\cdots\frac{\partial H_{n-1}}{\partial x_{k_{n-2}}}$$

$$=\begin{vmatrix} \dfrac{\partial C_{sl}}{\partial x_1} & \dfrac{\partial C_{sl}}{\partial x_2} & \cdots & \dfrac{\partial C_{sl}}{\partial x_n} \\ \dfrac{\partial H_l}{\partial x_1} & \dfrac{\partial H_l}{\partial x_2} & \cdots & \dfrac{\partial H_l}{\partial x_n} \\ \dfrac{\partial H_2}{\partial x_1} & \dfrac{\partial H_2}{\partial x_2} & \cdots & \dfrac{\partial H_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial H_{n-1}}{\partial x_1} & \dfrac{\partial H_{n-1}}{\partial x_2} & \cdots & \dfrac{\partial H_{n-1}}{\partial x_n} \end{vmatrix}=0.$$

From (4) we have that for each $x \in \mathcal{D}_2$ the vector $\nabla C_{s1}(x)$ belongs to the $n-1$ dimensional vectorial space generated by $\{\nabla H_1(x), \ldots, \nabla H_{n-1}(x)\}$, denoted by $\mathcal{P}_{n-1}(x)$. By the definition of first integral we have that for all $x \in \mathcal{D}_2$

$$\frac{\partial H_j(x)}{\partial x_1}P_1(x)+\ldots+\frac{\partial H_j(x)}{\partial x_n}P_n(x)=0, \qquad \text{for } j=1,\ldots,n-1.$$

So for each $x \in \mathcal{D}_2$ the vector $\mathcal{X}(x) = (P_1(x), \ldots, P_n(x))$ is orthogonal to the $n-1$ dimensional vectorial space $\mathcal{P}_{n-1}(x)$. Hence we have

$$\frac{\partial C_{s1}(x)}{\partial x_1}P_1(x)+\ldots+\frac{\partial C_{s1}(x)}{\partial x_n}P_n(x)=0, \qquad \text{for all } x\in\mathcal{D}_2.$$

This proves that the function C_{s1} (if not a constant) is a first integral of the vector field $\mathcal{X}$ defined on $\mathcal{D}_2$.

Similar arguments can verify that the functions C_{sj} (if not constants), $j = 2, \ldots, k$, are also first integrals of $\mathcal{X}$. Hence statement (b) is proved if $k = n-1$.

Now we suppose that $k < n-1$. Working in a similar way to the proof of the case $k = n-1$ and taking into account that the functions $H_1, \ldots, H_m$ are k–functionally independent in $\mathcal{D}_2$, for any $i_1, \ldots, i_{k+1}$ such that $1 \le i_1 < i_2 < \ldots < i_{k+1} \le n$ and for each $x \in \mathcal{D}_2$ we have that

$$\begin{vmatrix} \dfrac{\partial C_{s1}}{\partial x_{i_1}} & \dfrac{\partial C_{s1}}{\partial x_{i_2}} & \cdots & \dfrac{\partial C_{s1}}{\partial x_{i_{k+1}}} \\ \dfrac{\partial H_1}{\partial x_{i_1}} & \dfrac{\partial H_1}{\partial x_{i_2}} & \cdots & \dfrac{\partial H_1}{\partial x_{i_{k+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial H_k}{\partial x_{i_1}} & \dfrac{\partial H_k}{\partial x_{i_2}} & \cdots & \dfrac{\partial H_k}{\partial x_{i_{k+1}}}. \end{vmatrix}=0. \tag{5}$$

This implies for all $x \in \mathcal{D}_2$ that $\nabla C_{s1}(x)$ belongs to the k-dimensional vectorial space generated by $\{\nabla H_1(x), \ldots, \nabla H_k(x)\}$, denoted by $\mathcal{P}_k(x)$.

On the other hand since the functions $H_j(x)$ for $j = 1, \ldots, k$ are first integrals of the vector field $\mathcal{X}$, for each $x \in \mathcal{D}_2$ the vector $\mathcal{X}(x)$ is orthogonal to the vectorial space $\mathcal{P}_k(x)$, and so $\mathcal{X}(x)$ is orthogonal to $\nabla C_{s1}(x)$. This means that $C_{s1}(x)$ is a first integral of the vector field $\mathcal{X}$ defined on $\mathcal{D}_2$. Similar arguments show that C_{sj} for $j = 2, \dots, k$ are also first integrals of system (1). This completes the proof of statement (b). $\square$

4. Proof of Theorem 1

The "if" part of Theorem 1 is obvious. In what follows we shall prove the "only if" part.

Let $\{f_i(x) = 0\}$ for $i = 1, \dots, \begin{pmatrix} d+n-1 \\ n \end{pmatrix} + n$ be invariant algebraic hypersurfaces of the polynomial vector field $\mathcal{X}$ with the cofactor $k_i(x)$. Then $\deg k_i(x) \leq d-1$. We note that each polynomial $k_i(x)$ is uniquely determined by its coefficients and so it is a vector of the vectorial space $\mathcal{V}$ formed by all polynomials of $\mathbb{C}[x]$ of degree less than or equal to $d-1$. It is easy to check that $N = \begin{pmatrix} d+n-1 \\ n \end{pmatrix}$ is the dimension of the vectorial space $\mathcal{V}$ over the field $\mathbb{C}$.

Let p be the dimension of the vectorial subspace of $\mathcal{V}$ generated by $\{k_1(x), \dots, k_{N+n}(x)\}$. Then we have $p \leq N$. Now in order to simplify the proof and the notation we shall assume that $p = N$ and that $k_1(x), \dots, k_N(x)$ are linearly independent in $\mathcal{V}$. If $p < N$ the proof would follows exactly equal using the same arguments.

For each $s \in \{1, \dots, n\}$ there exists a vector $(\sigma_{s1}, \dots, \sigma_{sN}, 1) \in \mathbb{C}^{N+1}$ such that

$$\sigma_{s1} k_1(x) + \ldots + \sigma_{sN} k_N(x) + k_{N+s}(x) = 0. \tag{6}$$

From the definition of the invariant algebraic hypersurface $\{f_i = 0\}$ we get that $k_i = \mathcal{X} f_i / f_i$. Now equation (6) can be written as $\mathcal{X}\left(\log\left(f_1^{\sigma_{s1}} \ldots f_N^{\sigma_{sN}} f_{N+s}\right)\right) = 0$. This means that the functions $H_s = \log\left(f_1^{\sigma_{s1}} \ldots f_N^{\sigma_{sN}} f_{N+s}\right)$ for $s = 1, \dots, n$ are holomorphic first integrals of the vector field $\mathcal{X}$, defined on a convenient full Lebesgue measurable subset $\mathcal{D}_3$ of $\mathbb{C}^n$.

We claim that the n first integrals H_i's are functionally dependent on any positive Lebesgue measurable subset of $\mathcal{D}_3$. Otherwise there exists a positive Lebesgue measurable subset $\mathcal{D}_4$ of $\mathcal{D}_3$ where they are functionally independent, then from the definition of first integral we have

$$\frac{\partial H_i(x)}{\partial x_1} P_1(x) + \ldots + \frac{\partial H_i(x)}{\partial x_n} P_n(x) = 0, \qquad \text{for } i = 1, \dots, n \text{ and for all } x \in \mathcal{D}_4,$$

and from the functionally independence this last homogeneous linear system of dimension n only has the trivial solution $P_i(x) = 0$ for $i = 1, \dots, n$ on $\mathcal{D}_4$, and consequently the vector field $\mathcal{X} \equiv 0$ in $\mathbb{C}^n$, in contradiction with the fact that $\mathcal{X}$ has degree $d > 0$. So the claim is proved.

We define

$$r(x) = \text{rank}\{\nabla H_1(x), \dots, \nabla H_n(x)\} \quad \text{and} \quad m = \max\{r(x) : x \in \mathcal{D}_3\}.$$

Then there exists an open subset $\mathcal{O}$ of $\mathcal{D}_3$ such that $m = r(x)$ for each $x \in \mathcal{O}$ and $m < n$. Without loss of generality we can assume that $\{\nabla H_1(x), \dots, \nabla H_m(x)\}$ has the rank m for all $x \in \mathcal{O}$. Therefore, by Theorem 2(a) for each $x \in \mathcal{O}$ there exist $C_{k1}(x), \dots, C_{km}(x)$ such that

$$\nabla H_k(x) = C_{k1}(x) \nabla H_1(x) + \ldots + C_{km}(x) \nabla H_m(x), \quad k = m+1, \dots, n. \tag{7}$$

By Theorem 2(b) it follows that the function $C_{kj}(x)$ (if not a constant) for $j \in \{m+1, \dots, n\}$ is a first integral of the vector field $\mathcal{X}$ defined on $\mathcal{O}$.

From the construction of H_i's we know that each ∇H_i is a vector of rational functions. Since the vectors $\{\nabla H_1(x), \ldots, \nabla H_m(x)\}$ are linearly independent for each $x \in \mathcal{O}$, solving system (7) we get a unique solution $(C_{k1}(x), \ldots, C_{km}(x))$ on $\mathcal{O}$ for every $k = m+1, \ldots, n$. Clearly each function $C_{kj}(x)$ for $j \in \{1, \ldots, m\}$ is rational and by Theorem 2(b) it satisfies

$$\frac{\partial C_{kj}}{\partial x_1} P_1 + \ldots + \frac{\partial C_{kj}}{\partial x_n} P_n = 0 \quad \text{on } \mathcal{O}.$$

Since $\mathcal{O}$ is an open subset of $\mathbb{C}^n$ and $C_{kj}(x)$ is rational, it should satisfy the last equation in $\mathbb{C}^n$ except possibly a subset of Lebesgue measure zero where C_{kj} is not defined. Hence if some of the functions $C_{kj}(x)$'s is not a constant, it is a rational first integral of the vector field $\mathcal{X}$.

Now we shall prove that some function C_{kj} is not a constant. Equation (7) implies that if all functions $C_{k1}, \ldots, C_{km}$ are constants, then $H_k(x) = C_{k1}H_1(x) + \ldots + C_{km}H_m(x) + \log C_k$, where C_k is a constant. So we have $f_1^{\sigma_{k1}} \ldots f_N^{\sigma_{kN}} f_{N+k} = C_k \left(f_1^{\sigma_{11}} \ldots f_N^{\sigma_{1N}} f_{N+1}\right)^{C_{k1}} \ldots \left(f_1^{\sigma_{m1}} \ldots f_N^{\sigma_{mN}} f_{N+m}\right)^{C_{km}}$ for $k \in \{m+1, \ldots, n\}$. This is in contradiction with the fact that the polynomials $f_1, \ldots, f_{N+m}$ are irreducible and pairwise different. Hence we must have a non–constant function $C_{k_0 j_0}(x)$ for some $j_0 \in \{1, \ldots, m\}$ and some $k_0 \in \{m+1, \ldots, n\}$. This completes the proof of Theorem 1.

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