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On the polynomial vector fields on $\mathbb{S}^2$

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Let $\mathcal{X}$ be a polynomial vector field of degree n on M, $M = \mathbb{R}^m$. The dynamics and the algebraic–geometric properties of the vector fields $\mathcal{X}$ have been studied intensively, mainly for the case when $M = \mathbb{R}^2$, and especially when $n = 2$. Several papers have been dedicated to the study of the homogeneous polynomial vector field of degree n on $\mathbb{S}^2$, mainly for the case where $n = 2$ and $M = \mathbb{S}^2$. But there are very few results on the non-homogeneous polynomial vector fields of degree n on $\mathbb{S}^2$. This paper attempts to rectify this slightly.

1. Introduction and statement of the main results

Let $\mathbb{R}[x, y, z]$ be the ring of all polynomials in the variables x, y and z with real coefficients. The vector field

$$\mathcal{X} = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z} \tag{1.1}$$

is called a *polynomial vector field of degree n in* $\mathbb{R}^3$ if $P, Q, R \in \mathbb{R}[x, y, z]$ and $n = \max\{\deg P, \deg Q, \deg R\}$. For simplicity, sometimes we will write the vector field $\mathcal{X}$ simply as $\mathcal{X} = (P, Q, R)$.

The vector field (1.1) is a *homogeneous polynomial vector field $\mathcal{X}$ of degree n in* $\mathbb{R}^3$ if P, Q and R are homogeneous polynomials of degree n.

Let $f \in \mathbb{R}[x, y, z]$. The algebraic surface $f(x, y, z) = 0$ is an *invariant algebraic surface* of $\mathcal{X}$ if there exists $K \in \mathbb{R}[x, y, z]$ such that

$$\mathcal{X}f = P(x, y, z)\frac{\partial f}{\partial x} + Q(x, y, z)\frac{\partial f}{\partial y} + R(x, y, z)\frac{\partial f}{\partial z} = K(x, y, z)f(x, y, z). \tag{1.2}$$

As usual, $\mathbb{S}^2$ denotes the two-dimensional sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. A *polynomial vector field $\mathcal{X}$ of degree n on* $\mathbb{S}^2$ is a polynomial vector field in $\mathbb{R}^3$ of degree n such that restricting to $\mathbb{S}^2$ defines a vector field on $\mathbb{S}^2$, i.e. it must satisfy the equality

$$xP(x, y, z) + yQ(x, y, z) + zR(x, y, z) = 0 \quad \text{for all } (x, y, z) \in \mathbb{S}^2. \tag{1.3}$$

In particular, $\mathcal{X}$ is called *a quadratic vector field on* $\mathbb{S}^2$ if $n = 2$.