LIMIT CYCLES OF LINEAR VECTOR FIELDS ON MANIFOLDS

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Abstract. It is well known that linear vector fields on the manifold $\mathbb{R}^n$ cannot have limit cycles, but this is not the case for linear vector fields on other manifolds. We study the periodic orbits of linear vector fields on different manifolds, and motivate and present an open problem on the number of limit cycles of linear vector fields on a class of $\mathcal{C}^1$ connected manifold.

1. Introduction and statement of the results

In the qualitative theory of the ordinary differential equations or vector fields the periodic orbits play an important role in the study of their dynamics. Inside the periodic orbits there is the class of limit cycles, a *limit cycle* is a periodic orbit isolated in the set of all periodic orbits of the differential equation or vector field.

Many works have been done on the limit cycles of many different differential equations (see for instance [5, 6, 7, 8, 9, 10] and the references quoted there), but as far as we know nobody put attention on the limit cycles of linear vector fields, probably because the vector fields in $\mathbb{R}^n$ have no limit cycles. But there are interesting questions on the linear vector fields on other manifolds. To show some of these questions is the objective of this paper.

We deal with $\mathcal{C}^1$ connected manifolds and $\mathcal{C}^1$ vector fields on them. If M is a $\mathcal{C}^1$ connected manifold and TM is its tangent bundle, here a vector field X on M is a $\mathcal{C}^1$ map $X : M \to TM$ such that $X(x) \in T_xM$, where T_xM is the tangent space to M at the point x.

A *linear vector field in* $\mathbb{R}^n$ is a vector field X defined as $X(x) = Ax + b$, where $x, b \in \mathbb{R}^n$ and A is a real $n \times n$ matrix.

Since the solutions of a linear vector field in $\mathbb{R}^n$ are well known, see for instance [2, 14], it follows that when one of these vector fields has a periodic orbit it is not isolated in the set of all periodic orbits, consequently linear vector fields on $\mathbb{R}^n$ cannot have limit cycles.

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In this paper we only consider $\mathcal{C}^1$ connected manifolds M diffeomorphic to $\mathbb{R}^n \times (\mathbb{S}^1)^m$, where $\mathbb{S}^1$ denotes the circle $\mathbb{R}/(2\pi\mathbb{R})$. Then we say that a $\mathcal{C}^1$ vector field X on M is called *linear* if the expression of X in the coordinates $z = (x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \in M$ is of the form $X(z) = Az + b$ with $b \in M$ and A is a real $(n+m) \times (n+m)$ matrix.

Probably the easiest example that a linear differential system on a $\mathcal{C}^1$ connected manifold can have limit cycles is the following. Consider the cylinder $\mathbb{R} \times \mathbb{S}^1$ with coordinates $r \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$, and on it the linear vector field X defined by

$$X(r,\theta) = (r,1) \in T_{(r,\theta)}(\mathbb{R} \times \mathbb{S}^1) \simeq \mathbb{R}^2.$$

The differential system associated to the linear vector field X is

$$\dot{r} = r, \qquad \dot{\theta} = 1, \tag{1}$$

where the dot denotes derivative with respect to the independent variable t. Then clearly $r = 0$ is a limit cycle of the linear vector field X.

Let $\mathbb{R}^+$ be the set of all positive real numbers. We consider the vector field X on the connected manifold $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \approx \mathbb{R}^2 \times \mathbb{S}^1$ with coordinates $r \in \mathbb{R}^+$, $z \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$ associated to the differential system

$$\dot{r} = -z, \qquad \dot{z} = r - 1, \qquad \dot{\theta} = 1. \tag{2}$$

The solution $(r(t), z(t), \theta(t))$ of the linear differential system (2) such that $(r(0), z(0), \theta(0)) = (r_0, z_0, \theta_0)$ is

$$\begin{array}{l} r(t) = 1 + (r_0 - 1)\cos t - z_0 \sin t, \\ z(t) = z_0 \cos t + (r_0 - 1)\sin t, \\ \theta(t) = t + \theta_0. \end{array} \tag{3}$$

Therefore the periodic orbits of system (2) fill all invariant tori $z^2 + (r-1)^2 = a^2$ with $a \in (0,1)$, and have period 2π. Let V be the submanifold of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1$ formed by the union of all above invariant tori together with the periodic orbit $z^2 + (r-1)^2 = 0$. So V is a submanifold formed by 2π–periodic orbits.

In the first part of this paper we study the periodic orbits of three different kinds of linear perturbations of the linear vector field (2).

First we consider the Hamiltonian

$$\begin{array}{rl} H = & H(r,z,w,\theta) \\ = & \dfrac{1}{2}(z^2 + (r-1)^2) + w + \varepsilon(a_1 r + a_2 z + a_3 w + \\ & a_4 r^2 + a_5 z^2 + a_6 w^2 + a_7 rz + a_8 rw + a_9 zw + a_{10}\theta w), \end{array}$$

defined on $(r,z,w,\theta) \in \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{S}^1$. Then the Hamiltonian system of 2 degrees of freedom associated to this Hamiltonian is the linear differential system

$$
\begin{array}{rrl}
\dot r = & -H_z = & -z - \varepsilon(a_2 + a_7 r + 2a_5 z + a_9 w), \\
\dot z = & H_r = & r - 1 + \varepsilon(a_1 + 2a_4 r + a_7 z + a_8 w), \\
\dot w = & -H_\theta = & -\varepsilon a_{10} w, \\
\dot\theta = & H_w = & 1 + \varepsilon(a_3 + a_8 r + a_9 z + 2a_6 w + a_{10}\theta).
\end{array} \tag{4}
$$

For $\varepsilon = 0$ we have that H and w are independent first integrals in involution for the Hamiltonian system (4). Hence this system is completely integrable in the Liouvillian sense. The unperturbed Hamiltonian system (i.e. system (4) with $\varepsilon = 0$) is not generic in the sense that their invariant tori are all filled of periodic orbits with period 2π, when the generic completely integrable Hamiltonian systems with 2 degrees of freedom have invariant tori filled either of periodic orbits or of quasi–periodic orbits (i.e. orbits dense on the tori) depending on the rotation number of the tori which varies in some interval of $\mathbb{R}$. When the rotation number is rational the corresponding invariant torus is fulfilled of periodic orbits, otherwise is fulfilled of quasi–periodic orbits. For more details on Hamiltonian systems and on their Liouvillian integrability see [1, 3].

We note that Hamiltonian system (4) has the hyperplane $w = 0$ invariant, i.e. if an orbit of system (4) has a point on that hyperplane all the orbit is contained in it. So in what follows we will restrict our attention to Hamiltonian system (4) restricted to the invariant hyperplane $w = 0$, i.e. to the system

$$
\begin{aligned}
\dot r &= -z - \varepsilon(a_2 + a_7 r + 2a_5 z), \\
\dot z &= r - 1 + \varepsilon(a_1 + 2a_4 r + a_7 z), \\
\dot\theta &= 1 + \varepsilon(a_3 + a_8 r + a_9 z + a_{10}\theta).
\end{aligned} \tag{5}
$$

We observe that system (5) when $\varepsilon = 0$ coincides with system (2). For system (5) we have the following result.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small the linear vector field associated to the perturbed Hamiltonian system* (4) *restricted to the invariant hyperplane $w = 0$ (i.e. system* (5)*) has a limit cycle bifurcating from the periodic orbits of system* (2) *if $(a_4 + a_5)a_{10} \neq 0$. Moreover this limit cycle bifurcates from the periodic orbit $z^2 + (r-1)^2 = 0$ of system* (2)*.*

Theorem 1 is proved in Section 3.

Now we consider a second kind of linear perturbation of the linear vector field associated to system (2), namely

$$
\begin{aligned}
\dot r &= -z + \varepsilon a_1 z, \\
\dot z &= r - 1 + \varepsilon a_2 (1 - r), \\
\dot\theta &= 1 + \varepsilon(a_3 + a_4 r + a_5 z + a_6\theta).
\end{aligned} \tag{6}
$$

We note (as we shall prove later on) that system (6) has the first integral $H = z^2 + (r-1)^2$. So for system (6) all the invariant tori of the unperturbed system (2) persist. Clearly since H is a first integral of system (6), the periodic orbit $H = 0$ exists for every system (6). In what follows, for $\varepsilon \neq 0$ sufficiently small and if $(a_1 + a_2)a_6 \neq 0$, we prove that the periodic orbit $H = 0$ is a limit cycle of system (6), and that there are no limit cycles on the invariant tori $H = a$ with $a \in (0, 1)$.

Theorem 2. *For $\varepsilon \neq 0$ sufficiently small the linear vector field associated to system* (6) *has a limit cycle bifurcating from the periodic orbits of system* (2) *if $(a_1 + a_2)a_6 \neq 0$. Moreover this limit cycle bifurcates from the periodic orbit $z^2 + (r-1)^2 = 0$ of system* (2)*.*

Theorem 2 is proved in Section 4.

Finally we deal with a third kind of linear perturbation of the linear vector field associated to system (2), namely

$$\begin{aligned} \dot{r} &= -z + \varepsilon(a_0 + a_1 r + a_2 z + a_3\theta), \\ \dot{z} &= r - 1 + \varepsilon(b_0 + b_1 r + b_2 z + b_3\theta), \\ \dot{\theta} &= 1 + \varepsilon(c_0 + c_1 r + c_2 z + c_3\theta). \end{aligned} \tag{7}$$

That is, we consider the more general linear perturbation of system (2). For this system we have the following result.

Theorem 3. *For $\varepsilon \neq 0$ sufficiently small the linear vector field associated to system* (7) *has a limit cycle bifurcating from the periodic orbits of system* (2) *if $((a_2-b_1)^2+(a_1+b_2)^2)c_3 \neq 0$. Moreover this limit cycle bifurcates from one of the periodic orbits contained on the invariant torus $z^2+(r-1)^2 = a^2$ of system* (2) *if $a \in (0, 1)$ and*

$$a^2 = \frac{4(a_3^2 + b_3^2)}{(a_2 - b_1)^2 + (a_1 + b_2)^2},$$

or from the periodic orbit $z^2 + (r-1)^2 = 0$ if $a = 0$.

Theorem 3 is proved in Section 5.

In the second part of this note we study the periodic orbits of linear vector fields on the connected manifold $\mathbb{R} \times (\mathbb{S}^1)^2$ with coordinates $r \in \mathbb{R}$ and $(\theta, \varphi) \in (\mathbb{S}^1)^2$ associated to the differential systems

$$\begin{aligned} \dot{r} &= r - 1 + \varepsilon(a_0 + a_1 r + a_2\theta + a_3\varphi), \\ \dot{\theta} &= 1 + \varepsilon(b_0 + b_1 r + b_2\theta + b_3\varphi), \\ \dot{\varphi} &= 1 + \varepsilon(c_0 + c_1 r + c_2\theta + c_3\varphi). \end{aligned} \tag{8}$$

Note that the topology of the connected manifold where systems (5), (6) and (7) are defined is equal and different from the topology of the connected manifold where system (8) is defined.

We remark that system (8) is the more general linear perturbation of system (8) with $\varepsilon = 0$. Clearly all periodic orbits of system (8) with $\varepsilon = 0$ fill the invariant torus $r = 1$, and have period 2π. Let $\mathcal{Z}$ be this torus. Then $\mathcal{Z}$ is a submanifold of dimension 2 of the 3–dimensional manifold $\mathbb{R} \times (\mathbb{S}^1)^2$. For system (8) we have the next result.

Theorem 4. *For $\varepsilon \neq 0$ sufficiently small the linear vector field associated to system* (8) *has a limit cycle bifurcating from the periodic orbits of system* (8) *with $\varepsilon = 0$ if $b_2c_3 - b_3c_2 \neq 0$. Moreover if $a_0 + a_1 = a_3 = a_4 = 0$, then the torus $r = 1$ is invariant by the flow of system* (8)*, and such a limit cycle is on this torus when it exists.*

Theorem 4 is proved in Section 6.

The key tool for proving Theorems 1, 2, 3 and 4 is the averaging theory. For a general introduction to the averaging theory see the books of Sanders, Verhulst and Murdock [13] and of Verhulst [15]. But the results on the averaging theory that we shall use here are presented in Section 2.

As we shall see in the proofs of Theorems 1, 2, 3 and 4 our method based on the averaging theory applied to linear vector fields at most can produce one limit cycle.

We have provide the existence of at least one limit cycle for some linear vector fields over $\mathbb{R} \times \mathbb{S}^1$ (see system (1)), over $\mathbb{S}^1 \times \mathbb{S}^1$ (see Theorem 4), over $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1$ (see Theorems 1, 2 and 3) and over $\mathbb{R} \times (\mathbb{S}^1)^2$. In the first two cases we also have proved that such linear vector fields have at most one limit cycle. From these results a natural question is the following.

Open Question *Let n and m be two non–negative integers. Is it true that every linear vector field on the manifold $\mathbb{R}^n \times (\mathbb{S}^1)^m$ has at most one limit cycle*?

2. Basic results

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential systems

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{9}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}), \tag{10}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the system (10) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{11}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (11), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Theorem 5. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha))\,,\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of (10) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (11) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k\times(n-k)$ zero matrix, and in the lower right corner a $(n-k)\times(n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$ defined by

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))dt\right). \tag{12}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\alpha\right)(a)\right) \neq 0$, then there is a limit cycle $\varphi(t, \varepsilon)$ of period T of system (9) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

Theorem 5 goes back to Malkin [11] and Roseau [12], for a shorter proof see [4].

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T–periodic. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (9); i.e. it is a set formed only by periodic orbits all of them having the same period. Then an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $\mathrm{Cl}(V)$ for system (9) is given in the following result.

Theorem 6 ((Perturbations of an isochronous set))**.** *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t, \mathbf{z}, 0)$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$ defined by*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}, 0))dt. \tag{13}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(a)\right) \neq 0$, then there exists a limit cycle $\varphi(t, \varepsilon)$ of period T of system (9) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

For a proof of Theorem 6 see Corollary 1 of [4].

3. Proof of Theorem 1

Since $r > 0$ the periodic orbits (3) of the linear differential system (2) fill all invariant tori $f_a(r, z, \theta) = z^2 + (r-1)^2 - a^2 = 0$ with $a \in (0, 1)$, and have period 2π. The torus $f_a(r,z,\theta)=0$ is invariant by the flow of system (2), because for every periodic solution (3) on $f_a(r,z,\theta)=0$ we have that

$$\frac{df_a}{dt}(r(t),z(t),\theta(t)) = \frac{\partial f_a}{\partial r}\dot r(t) + \frac{\partial f_a}{\partial z}\dot z(t) + \frac{\partial f_a}{\partial \theta}\dot\theta(t) = 0.$$

Let V be the submanifold of $\mathbb{R}^+\times\mathbb{R}\times\mathbb{S}^1$ formed by the union of all above invariant tori together with the periodic orbit $z^2+(r-1)^2=0$. So V is an open and bounded submanifold formed by 2π–periodic orbits.

By using Theorem 6 we want to study the limit cycles of the perturbed system (5) for $\varepsilon\neq 0$ sufficiently small, which bifurcate from the periodic orbits of system (2). Thus for applying Theorem 6 to the differential system (5) we take

$$\begin{aligned} k =& \ \ n=3,\\ \mathbf{x} =& \ \ (r,z,\theta),\\ \mathbf{z} =& \ \ (r_0,z_0,\theta_0),\\ \mathbf{x}(t,\mathbf{z},0) =& \ \ (r(t),z(t),\theta(t)) \text{ given by (3)},\\ F_0(t,\mathbf{x}) =& \ \ (-z,r-1,1),\\ F_1(t,\mathbf{x}) =& \ \ (a_2+a_7r+2a_5z, a_1+2a_4r+a_7z,\\ & \ \ a_3+a_8r+a_9z+a_{10}\theta),\\ F_2(t,\mathbf{x},\varepsilon) =& \ \ \mathbf{0},\\ \Omega =& \ \ \mathbb{R}^+\times\mathbb{R}\times\mathbb{S}^1,\\ T =& \ \ 2\pi. \end{aligned} \tag{14}$$

We note that since $\mathbb{R}\times\mathbb{S}^1$ can be thought as an open annulus, Ω can be thought as an open subset of $\mathbb{R}^3$. Of course the open and bounded subset V in the statement of Theorem 6 coincides with the submanifold V defined in the previous paragraph.

For the function F_0 and the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ given in (14), easy computations show that the fundamental matrix $M_{\mathbf{z}}(t)$ of the differential system (11) such that $M_{\mathbf{z}}(0)$ is the identity matrix of $\mathbb{R}^3$ is

$$M_{\mathbf{z}}(t) = \begin{pmatrix} \cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{15}$$

We remark that for system (2) the fundamental matrix $M_{\mathbf{z}}(t)$ does not depend on the particular periodic orbit $\mathbf{x}(t,\mathbf{z},0)$; i.e. it is independent of the initial conditions $\mathbf{z}$.

In short all the assumptions of Theorem 6 are satisfied. Therefore we must study the zeros in V of the system $\mathcal{F}(\mathbf{z})=0$ of three equations and three unknowns, where $\mathcal{F}$ is given by (13). More precisely after some tedious but easy computations we have

$$\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(r_0,z_0,\theta_0),\mathcal{F}_2(r_0,z_0,\theta_0),\mathcal{F}_3(r_0,z_0,\theta_0)),$$

where

$$\begin{aligned}\mathcal{F}_1 &= -2\pi(a_4 + a_5)z_0,\\ \mathcal{F}_2 &= 2\pi(a_4 + a_5)(r_0 - 1),\\ \mathcal{F}_3 &= 2\pi(a_3 + a_8 + a_{10}(\theta_0 + \pi)).\end{aligned}$$

So, since by assumptions $(a_4+a_5)a_{10} \neq 0$, the unique solution of the previous system is

$$r_0 = 1, \quad z_0 = 0, \quad \theta_0 = -\frac{a_3 + a_8 + \pi a_{10}}{a_{10}}. \tag{16}$$

Moreover the determinant

$$\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(r_0, z_0, \theta_0)}\right|_{(16)} \right) = 8\pi^3(a_4 + a_5)^2 a_{10} \neq 0.$$

Hence applying Theorem 6 there is a periodic solution $(r(t,\varepsilon), z(t,\varepsilon), \theta(t,\varepsilon))$ of the differential system (5) such that

$$(r(0,\varepsilon), z(0,\varepsilon), \theta(0,\varepsilon)) \mapsto \left(1, 0, -\frac{a_3 + a_8 + \pi a_{10}}{a_{10}}\right)$$

when $\varepsilon \mapsto 0$. Therefore Theorem 1 is proved.

4. Proof of Theorem 2

In this section we want to study the limit cycles of the system (6) for $\varepsilon \neq 0$ sufficiently small, which bifurcate from the periodic orbits of system (2).

First we note that $H = z^2 + (r - 1)^2$ is a first integral of system (6), because over the solutions $(r(t), z(t), \theta(t))$ of system (6) we have

$$\frac{df}{dt}(r(t), z(t), \theta(t)) = \frac{\partial f}{\partial r}\dot r(t) + \frac{\partial f}{\partial z}\dot z(t) + \frac{\partial f}{\partial \theta}\dot\theta(t) = 0.$$

For applying Theorem 6 to the differential system (6) we take equalities (14) but now F_1 is

$$F_1(t, \mathbf{x}) = (a_1 z, a_2(1 - r), a_3 + a_4 r + a_5 z + a_6\theta).$$

As the function F_0 is the same than for system (5) the fundamental matrix $M_{\mathbf{z}}(t)$ is also given by (15).

In short all the assumptions of Theorem 6 are satisfied. Therefore we must study the zeros in V of the system $\mathcal{F}(\mathbf{z}) = 0$ given by (13). More precisely, after the same computations than for system (5), we have

$$\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(r_0, z_0, \theta_0), \mathcal{F}_2(r_0, z_0, \theta_0), \mathcal{F}_3(r_0, z_0, \theta_0)),$$

where

$$\begin{aligned}\mathcal{F}_1 &= \pi(a_1 + a_2)z_0,\\ \mathcal{F}_2 &= -\pi(a_1 + a_2)(r_0 - 1),\\ \mathcal{F}_3 &= 2\pi(a_3 + a_4 + a_6(\theta_0 + \pi)).\end{aligned}$$

Hence, since by assumptions $(a_1+a_2)a_6 \neq 0$, the unique solution of the previous system is

$$r_0 = 1, \quad z_0 = 0, \quad \theta_0 = -\frac{a_3+a_4+\pi a_6}{a_6}. \tag{17}$$

Moreover the determinant

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)}{\partial(r_0,z_0,\theta_0)}\right|_{(17)}\right) = 2\pi^3(a_1+a_2)^2 a_6 \neq 0.$$

Therefore applying Theorem 6 there is a periodic solution $(r(t,\varepsilon), z(t,\varepsilon), \theta(t,\varepsilon))$ of the differential system (6) such that

$$(r(0,\varepsilon), z(0,\varepsilon), \theta(0,\varepsilon)) \mapsto \left(1, 0, -\frac{a_3+a_4+\pi a_6}{a_6}\right)$$

when $\varepsilon \mapsto 0$. Therefore Theorem 2 is proved.

5. Proof of Theorem 3

Here we want to study the limit cycles of the system (7) for $\varepsilon \neq 0$ sufficiently small, which bifurcate from the periodic orbits of system (2).

For applying Theorem 6 to the differential system (7) we take equalities (20) but now F_1 is

$$F_1(t,\mathbf{x}) = (a_0 + a_1 r + a_2 z + a_3\theta, b_0 + b_1 r + b_2 z + b_3\theta, c_0 + c_1 r + c_2 z + c_3\theta).$$

As the function F_0 is the same than for system (5) the fundamental matrix $M_{\mathbf{z}}(t)$ is given by (21).

In short all the assumptions of Theorem 6 are satisfied. Therefore we must study the zeros in V of the system $\mathcal{F}(\mathbf{z}) = 0$ given by (13). More precisely, after some computations, we have

$$\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(r_0,z_0,\theta_0), \mathcal{F}_2(r_0,z_0,\theta_0), \mathcal{F}_3(r_0,z_0,\theta_0)),$$

where

$$\begin{aligned}
\mathcal{F}_1 &= \pi(-2b_3 + (a_1+b_2)(r_0-1) + (a_2-b_1)z_0),\\
\mathcal{F}_2 &= \pi(a_2 + 2a_3 - b_1 + (b_1-a_2)r_0 + (a_1+b_2)z_0),\\
\mathcal{F}_3 &= 2\pi(c_0 + c_1 + c_3(\pi+\theta_0)).
\end{aligned}$$

Hence, since by assumptions $((a_2-b_1)^2 + (a_1+b_2)^2)c_3 \neq 0$ and

$$0 \leq a^2 = \frac{4(a_3^2+b_3^2)}{(a_2-b_1)^2+(a_1+b_2)^2} < 1,$$

the unique solution of the system $\mathcal{F}(\mathbf{z}) = 0$ is

$$
(18)\qquad
\begin{aligned}
r_0 &= \frac{(a_2 - b_1)(a_2 + 2a_3 - b_1) + (a_1 + b_2)(a_1 + b_2 + 2b_3)}{(a_2 - b_1)^2 + (a_1 + b_2)^2},\\
z_0 &= \frac{2(a_2 - b_1)b_3 - 2a_3(a_1 + b_2)}{(a_2 - b_1)^2 + (a_1 + b_2)^2},\\
\theta_0 &= -\frac{c_0 + c_1 + c_3\pi}{c_3}.
\end{aligned}
$$

Note that $(r_0 - 1)^2 + z_0^2 = a^2$. Moreover the determinant

$$
\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(r_0, z_0, \theta_0)}\right|_{(18)} \right) = 2\pi^3((a_2 - b_1)^2 + (a_1 + b_2)^2)c_3 \neq 0.
$$

Therefore applying Theorem 6 there is a periodic solution $(\theta(t, \varepsilon), z(t, \varepsilon), \theta(t, \varepsilon))$ of the differential system (7) such that

$$
(r(0, \varepsilon), z(0, \varepsilon), \theta(0, \varepsilon)) \mapsto (r_0, z_0, \theta_0)
$$

when $\varepsilon \mapsto 0$, here r_0, z_0 and θ_0 are the ones given in (18). Therefore Theorem 3 is proved.

6. Proof of Theorem 4

First it is easy to check that the solution $(r(t), \theta(t), \varphi(t))$ of the linear differential system (8) with $\varepsilon = 0$ such that $(r(0), \theta(0), \varphi(0)) = (r_0, \theta_0, \varphi_0)$ is

$$
r(t) = 1 + e^t(r_0 - 1), \quad \theta(t) = t + \theta_0, \quad \varphi(t) = t + \varphi_0.
$$

Therefore the periodic solutions of system (8) with $\varepsilon = 0$ are

$$
(19)\qquad r(t) = 1, \quad \theta(t) = t + \theta_0, \quad \varphi(t) = t + \varphi_0.
$$

So the periodic orbits of system (8) with $\varepsilon = 0$ fill the invariant tori $r = 1$, and have period 2π.

For system (8) is easy to check that $\dot{r}|_{r=1} = 0$ if and only if $a_0 + a_1 = a_3 = a_4 = 0$. So the torus $r = 1$ is invariant by the flow of system (8) if and only if $a_0 + a_1 = a_3 = a_4 = 0$.

By using Theorem 5 we want to study the limit cycles of system (8) for $\varepsilon \neq 0$ sufficiently small, which bifurcate from the periodic orbits of system (8) with $\varepsilon = 0$. Thus for applying Theorem 5 to the differential system (8) we take

$$
\begin{aligned}
k = & \; 2, \\
n = & \; 3, \\
\mathbf{x} = & \; (\theta, \varphi, r), \\
\alpha = & \; (\theta_0, \varphi_0), \\
\beta_0(\alpha) = & \; 1, \\
\mathbf{z}_\alpha = & \; (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, 1), \\
\mathcal{Z} = & \; \{(\theta_0, \varphi_0, r) \in (\mathbb{S}^1)^2 \times \mathbb{R} : r = 1\}, \\
\mathbf{x}(t, \mathbf{z}_\alpha, 0) = & \; (\theta(t), \varphi(t), r(t)) \text{ given by (19)}, \\
F_0(t, \mathbf{x}) = & \; (1, 1, r-1), \\
F_1(t, \mathbf{x}) = & \; (b_0 + b_1 r + b_2\theta + b_3\varphi, c_0 + c_1 r + c_2\theta + c_3\varphi, \\
& \; a_0 + a_1 r + a_2\theta + a_3\varphi), \\
F_2(t, \mathbf{x}, \varepsilon) = & \; \mathbf{0}, \\
\Omega = & \; (\mathbb{S}^1)^2 \times \mathbb{R}, \\
T = & \; 2\pi.
\end{aligned} \tag{20}
$$

We note that the open solid torus $(\mathbb{S}^1)^2 \times \mathbb{R}$ can be thought as an open subset of $\mathbb{R}^3$. Of course the open and bounded subset $V \subset \mathcal{Z}$ in the statement of Theorem 5 can be taken as $\{(\theta_0, \varphi_0) \in (\mathbb{S}^1)^2\}$, i.e. V is diffeomorphic to $\mathcal{Z}$.

For the function F_0 and the periodic solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ given in (20), easy computations show that the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of the differential system (11) such that $M_{\mathbf{z}_\alpha}(0)$ is the identity matrix of $\mathbb{R}^3$ is

$$
M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}. \tag{21}
$$

We remark that for system (8) the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ does not depend on the particular periodic orbit $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$; i.e. it is independent of the initial conditions $\mathbf{z}_\alpha$.

Since the matrix

$$
M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi} \end{pmatrix},
$$

all the assumptions of Theorem 5 are satisfied. Therefore we must study the zeros in V of the system $\mathcal{F}(\mathbf{z}_\alpha) = 0$ of two equations and two unknowns, where $\mathcal{F}$ is given by (12). More precisely after some tedious but easy computations we have

$$
\mathcal{F}(\mathbf{z}_\alpha) = (\mathcal{F}_1(\theta_0, \varphi_0), \mathcal{F}_2(\theta_0, \varphi_0)),
$$

where

$$
\begin{aligned}
\mathcal{F}_1 &= 2\pi(b_0 + b_1 + b_3(\varphi_0 + \pi) + b_2(\theta_0 + \pi)), \\
\mathcal{F}_2 &= 2\pi(c_0 + c_1 + c_3(\varphi_0 + \pi) + c_2(\theta_0 + \pi)).
\end{aligned}
$$

So, since by assumptions $b_2c_3 - b_3c_2 \neq 0$, the unique solution of the previous system is

$$\begin{aligned} \theta_0 &= \frac{b_3(c_0 + c_1 + \pi c_2) - c_3(b_0 + b_1 + b_2\pi)}{b_2c_3 - b_3c_2}, \\ \varphi_0 &= \frac{c_2(b_0 + b_1 + \pi b_3) - b_2(c_0 + c_1 + c_3\pi)}{b_2c_3 - b_3c_2}. \end{aligned} \tag{22}$$

Moreover the determinant

$$\det \left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(\theta_0, \varphi_0)} \right|_{(22)} \right) = 4\pi^2(b_2c_3 - b_3c_2) \neq 0.$$

Hence applying Theorem 5 there is a periodic solution $(\theta(t, \varepsilon), \varphi(t, \varepsilon))$ of the differential system (8) such that

$$(\theta(0, \varepsilon), \varphi(0, \varepsilon)) \mapsto (\theta_0, \varphi_0)$$

when $\varepsilon \mapsto 0$, here θ_0 and φ_0 are the ones given in (22). Therefore Theorem 4 is proved.

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