LIMIT CYCLES CREATED BY PIECEWISE LINEAR CENTERS

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ABSTRACT. In this paper we study the limit cycles of the discontinuous piecewise linear differential systems in the plane $\mathbb{R}^2$ formed by three arbitrary linear centers separated by the set

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ or } x = 0 \text{ and } y \geq 0\}.$$

We prove that such discontinuous piecewise linear differential systems can have 1, 2 or 3 limit cycles which intersect in a unique point each branch of the set Σ, and that they cannot have more than 3 of such limit cycles.

1. Introduction and statement of the main result

For a differential system in the plane $\mathbb{R}^2$ a periodic orbit which is isolated in the set of all periodic orbits is a *limit cycle*. At the end of the 19th century started the studies of the limit cycles of the differential systems with Poincaré [15]. Limit cycles modelize many phenomena of the real world, see the Belousov–Zhabotinskii reaction [2, 21], or the van der Pol oscillator [16, 17], or the motion of the galaxies [5], and many examples can be found in the survey [13], or in the book [3].

In the book of Andronov, Vitt and Khaikin [1] appeared some of the first studies on the *discontinuous* piecewise linear differential systems in the plane $\mathbb{R}^2$ separated by straight lines. The work on such differential systems has continued up today, mainly due to their applications, for instance in mechanics, economy, electrical circuits, etc, see the surveys [13, 20] and the books [3, 19].

There are two types of limit cycles in the planar discontinuous piecewise linear differential systems, the crossing and sliding ones. The *sliding limit cycles* contain some arc of the lines of discontinuity which

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separate the different linear differential systems (more precise definition can be found in [14]). The *crossing limit cycles* only contain isolated points of the lines of discontinuity. In this paper we only consider the crossing limit cycles of some planar discontinuous piecewise linear differential systems separated by pieces of straight lines. From now on we only shall work with crossing limit cycles, but we simply call them limit cycles instead of crossing limit cycles.

The easiest discontinuous piecewise linear differential systems in the plane are the discontinuous piecewise linear differential systems separated by a unique straight line. It is known that such differential systems can have 3 limit cycles, see [4, 6, 7, 8, 9, 11]. But at this moment it is an open problem to know if 3 is the maximum number of limit cycles that such discontinuous piecewise linear differential systems can have.

Here our objective is to study the number of limit cycles which can exhibit the planar discontinuous piecewise linear differential systems separated by pieces of straight lines such that all their linear differential systems are formed by centers. There are some results on these piecewise linear differential systems. Thus in Theorem 4 of [10] it is proved:

Theorem 1. *A discontinuous piecewise linear differential system separated by one straight line formed by two linear centers has no limit cycles.*

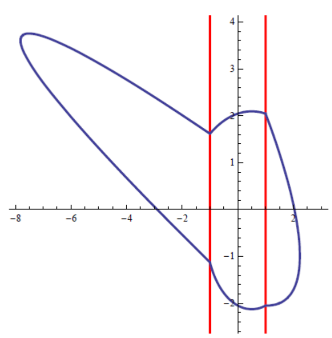

FIGURE 1. The limit cycle of the discontinuous piecewise linear differential system (1). The two straight lines of separation between the differential systems are drawn in red color.

But in [12] it is proved the existence of discontinuous piecewise linear differential system separated by two parallel straight lines and formed by three linear centers, which exhibit one limit cycle. As far as we know this discontinuous system was the first example that only centers in a piecewise linear differential system can create limit cycles. More precisely, consider the discontinuous piecewise linear differential system with three pieces separated by the two parallel straight lines $x = \pm 1$ defined by

$$
\begin{array}{llll}
 & \dot{x} = 2(-4+4x+5y), & \dot{y} = -8(1+x+y), & \text{if } x > 1,\\
(1) & \dot{x} = 1+2y, & \dot{y} = 1-2x, & \text{if } -1 < x < 1,\\
 & \dot{x} = (16+32x+65y)/2, & \dot{y} = -8(x+2y), & \text{if } x < -1.
\end{array}
$$

These three linear differential systems are centers, and the discontinuous piecewise linear differential system has the limit cycle of Figure 1.

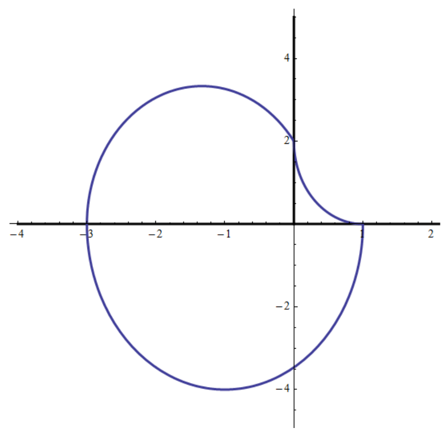

FIGURE 2. The unique limit cycle of the discontinuous piecewise linear differential system (2).

Our main result is the following:

Theorem 2. *Consider three linear differential systems formed by centers and separated by the set*

$$\Sigma = \{(x, y) : y = 0 \text{ or } x = 0 \text{ and } y \geq 0\}.$$

Such discontinuous piecewise linear differential systems can have at most 3 limit cycles intersecting the three branches of $\Sigma \setminus \{(0,0)\}$ in one point. Moreover, inside this class of discontinuous piecewise linear differential systems there are systems with exactly either 1,or 2, or 3 limit cycles.

Theorem 2 is proved in section 2.

We remark that in general it is not easy to provide an explicit upper bound for the maximum number of limit cycles in a class of differential systems, and such that this bound be reached.

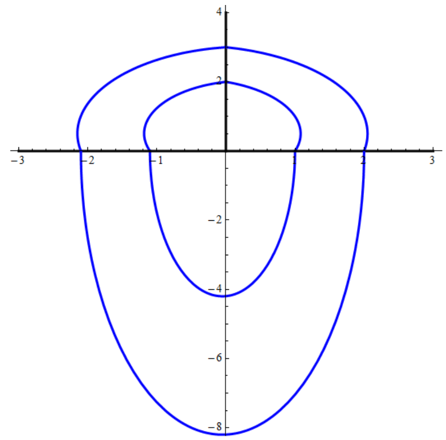

FIGURE 3. The two limit cycles of the discontinuous piecewise linear differential system (3).

The three components of $\mathbb{R}^2 \setminus \Sigma$ are the positive or first quadrant $Q_1 = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$, the second quadrant $Q_2 = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y > 0\}$, and the half-plane $H = \{(x, y) \in \mathbb{R}^2 : y < 0\}$.

Proposition 3. *Consider the discontinuous piecewise linear differential system with three pieces separated by the set Σ defined by*

$$
\begin{aligned}
&\dot{x} = -\frac{1}{4}y - 1, && \dot{y} = x - 1, && \textit{in } Q_1,\\
&\dot{x} = -\frac{1}{4}y, && \dot{y} = x + \frac{4}{3}, && \textit{in } Q_2,\\
&\dot{x} = -\frac{1}{4}y, && \dot{y} = x + 1, && \textit{in } H.
\end{aligned} \tag{2}
$$

These three linear differential systems are centers, and the discontinuous piecewise linear differential system has exactly one limit cycle, see it in Figure 2.

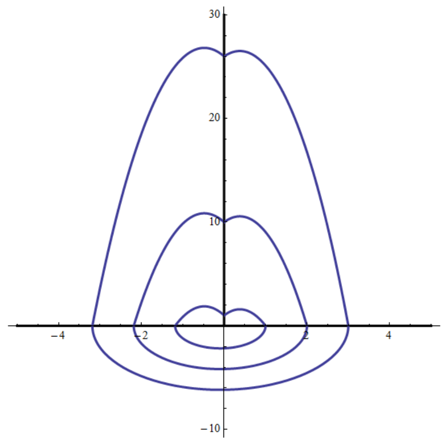

FIGURE 4. The three limit cycles of the discontinuous piecewise linear differential system (4).

Proposition 4. *Consider the discontinuous piecewise linear differential system with three pieces separated by the set Σ defined by*

$$
\begin{aligned}
&\dot{x} = -y + \frac{1}{2}, && \dot{y} = x + \frac{1}{2}, && \textit{in } Q_1,\\
&\dot{x} = -\frac{479}{1000}y + \frac{97}{400}, && \dot{y} = \frac{1}{2}x - \frac{31}{200}, && \textit{in } Q_2,\\
&\dot{x} = -\frac{1}{8}y, && \dot{y} = 2x + \frac{1}{10}, && \textit{in } H.
\end{aligned}
\tag{3}
$$

These three linear differential systems are centers, and the discontinuous piecewise linear differential system has exactly two limit cycles, see them in Figure 3.

Proposition 5. *Consider the discontinuous piecewise linear differential system with three pieces separated by the set Σ defined by*

$$
\begin{aligned}
&\dot{x} = -\frac{2}{1565}y - \frac{379}{1565}, && \dot{y} = 2x - \frac{237}{313}, && \textit{in } Q_1,\\
&\dot{x} = -\frac{4}{1565}y - \frac{11566}{10955}, && \dot{y} = 8x + \frac{4\sqrt{4430533}}{2191}, && \textit{in } Q_2,\\
&\dot{x} = -2y, && \dot{y} = 8x + \frac{2\left(\sqrt{4430533} - 1299\right)}{2191}, && \textit{in } H.
\end{aligned}
\tag{4}
$$

These three linear differential systems are centers, and the discontinuous piecewise linear differential system has exactly three limit cycles, see them in Figure 4.

Propositions 3, 4 and 5 are proved in section 3.

2. Proof of Theorem 2

It is known that a normal form for a linear differential system having a center is given in the next result. Since the proof is short we provide it.

Lemma 6. *The equations of a linear differential system with a center are*

$$\dot{x} = -bx - \frac{4b^2+\omega^2}{4a}y + d, \qquad \dot{y} = ax + by + c,$$

with $a > 0$ and $\omega > 0$.

Proof. Consider an arbitrary linear differential system

$$\dot{x} = Ax + By + d, \qquad \dot{y} = ax + by + c,$$

in the plane and suppose that it has a center. Then the eigenvalues of this system are

$$\frac{A + b \pm \sqrt{4aB + (A-b)^2}}{2}.$$

If this system has a center then $A + b = 0$ and $4aB + (A - b)^2 = -\omega^2$ for some $\omega > 0$ and $aB < 0$, i.e. if $A = -b$, $B = -(4b^2 + \omega^2)/(4a)$ and $a > 0$. Therefore the lemma is proved. $\square$

We remark that the normal form in Lemma 6 is independent of the change of coordinates, so in the next proof of Theorem 2 we can use this normal form in each of the three regions Q_1, Q_2 and H for the three centers.

Now we start the proof of Theorem 2. Suppose that we have a discontinuous piecewise linear differential system separated by the set Σ formed by three linear centers. By Lemma 6 we can write such a discontinuous piecewise linear differential system as

$$\begin{aligned}
&\dot{x} = -\beta x - \frac{4\beta^2+\omega^2}{4\alpha}y + \delta, \qquad \dot{y} = \alpha x + \beta y + \gamma, \quad \text{in } Q_1,\\
&\dot{x} = -bx - \frac{4b^2+w^2}{4a}y + d, \qquad \dot{y} = ax + by + c, \quad \text{in } Q_2,\\
&\dot{x} = -Bx - \frac{4B^2+W^2}{4A}y + D, \quad \dot{y} = Ax + By + C, \quad \text{in } H.
\end{aligned} \tag{5}$$

The linear centers in Q_1, Q_2 and H have the first integrals H_1, H_2 and H_3, respectively, where

$$
\begin{aligned}
H_1(x, y) &= 4(\alpha x + \beta y)^2 + 8\alpha(\gamma x - \delta y) + \omega^2 y^2,\\
H_2(x, y) &= 4(ax + by)^2 + 8a(cx - dy) + w^2 y^2,\\
H_3(x, y) &= 4(Ax + By)^2 + 8A(Cx - Dy) + W^2 y^2.
\end{aligned}
$$

Suppose that this discontinuous piecewise differential system with three linear centers has some limit cycle intersecting each branch of $\Sigma \setminus \{(0, 0)\}$ in one point, namely $(x_+, 0)$ with $x_+ > 0$, $(0, y_+)$ with $y_+ > 0$, and $(x_-, 0)$ with $x_- < 0$. Then the first integrals H_1, H_2 and H_3 must satisfy the following three equations

$$
\begin{aligned}
H_1(x_+, 0) - H_1(0, y_+) &= 0,\\
H_2(0, y_+) - H_2(x_-, 0) &= 0,\\
H_3(x_-, 0) - H_3(x_+, 0) &= 0,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&4\alpha^2 x_+^2 - (4\beta^2 + \omega^2)\, y_+^2 + 8\alpha\gamma x_+ + 8\alpha\delta y_+ = 0,\\
&4a^2 x_-^2 - (4b^2 + w^2)\, y_+^2 + 8acx_- + 8ady_+ = 0,\\
&-4A(x_+ - x_-)(Ax_+ + Ax_- + 2C) = 0.
\end{aligned}
\tag{6}
$$

Since $x_+ - x_- > 0$, by Bezout Theorem (see for instance [18]), the system of the three polynomial equations (6) of degrees 2, 2 and 1 have at most 4 real solutions (x_+, y_+, x_-). So the discontinuous piecewise linear differential system (2) can have at most 4 limit cycles.

Since $x_+ > 0$, $x_- < 0$ and $A \neq 0$ the third equation of system (6) reduces to $Ax_+ + Ax_- + 2C = 0$. Isolating x_- from this equation and substituting it into the second equation of system (6), we get the system

$$
\begin{aligned}
&4\alpha^2 x_+^2 - (4\beta^2 + \omega^2)\, y_+^2 + 8\alpha\gamma x_+ + 8\alpha\delta y_+ = 0,\\
&4a^2 x_+^2 - (4b^2 + w^2)\, y_+^2 + \frac{8a(2aC - Ac)}{A} x_+\\
&\qquad\qquad + 8ady_+ + \frac{16aC(aC - Ac)}{A^2} = 0.
\end{aligned}
\tag{7}
$$

Note that we can assume $C(aC - Ac) \neq 0$, otherwise the polynomial system (7) has the solution $(0, 0)$ which cannot contribute with a limit cycle having a unique point in each branch of $\Sigma \setminus \{(0, 0)\}$. So, in this case system (7) only can have at most 3 solutions which can provide at most 3 limit cycles for the discontinuous piecewise linear differential system (2), and the theorem is proved if $C(aC - Ac) = 0$.

In order that the polynomial system (6) can have eventually four real solutions, (x_+^k, y_+^k, x_-^k) for $k = 1, 2, 3, 4$, producing four limit cycles for the discontinuous piecewise linear differential system (2) it is necessary that $x_+^1,\ x_+^2,\ x_+^3$ and x_+^4 have the same order as that of $y_+^1,\ y_+^2,\ y_+^3$ and y_+^4. For instance,

$$x_+^1 < x_+^2 < x_+^3 < x_+^4 \quad \text{and} \quad y_+^1 < y_+^2 < y_+^3 < y_+^4, \tag{8}$$

otherwise the solutions of the piecewise linear differential system (2) connecting the points $(x_+^k, 0)$ and $(0, y_+^k)$ in the quadrant Q_1 would intersect, in contradiction with the Uniqueness Theorem for the solutions of an ordinary differential equation.

In what follows, but only inside the rest of this section, in order to simplify the notation we shall write x_k and y_k instead of x_+^k and y_+^k, respectively.

We consider the following two conics

$$\begin{aligned} F_1(x, y) &= ax^2 + by^2 + cx + dy = 0, \\ F_2(x, y) &= Ax^2 + By^2 + Cx + Dy + E = 0. \end{aligned} \tag{9}$$

where $abABE \neq 0$. We claim that the polynomial systems (9) cannot have 4 solutions (x_k, y_k) with $x_1,\ x_2,\ x_3,\ x_4$ and $y_1,\ y_2,\ y_3,\ y_4$ having the same order as illustrated in (8). Since the systems (9) contain the systems (7), it follows that systems (7) cannot have 4 solutions (x_k, y_k) with $x_1,\ x_2,\ x_3,\ x_4$ and $y_1,\ y_2,\ y_3,\ y_4$ having the same order. Hence the discontinuous piecewise linear differential system (2) cannot have 4 limit cycles, and this completes the proof of the theorem after proving Propositions 3, 4 and 5.

For proving the claim we consider that the two conics of (9) share four points (x_k, y_k) with $x_1,\ x_2,\ x_3,\ x_4$ and $y_1,\ y_2,\ y_3,\ y_4$ having the same order, and we shall arrive to a contradiction. Since the first conic (9) has the four distinct points (x_k, y_k), these points satisfy the system

$$\begin{aligned} ax_1^2 + by_1^2 + cx_1 + dy_1 &= 0, \\ ax_2^2 + by_2^2 + cx_2 + dy_2 &= 0, \\ ax_3^2 + by_3^2 + cx_3 + dy_3 &= 0, \\ ax_4^2 + by_4^2 + cx_4 + dy_4 &= 0. \end{aligned} \tag{10}$$

The determinant of this linear system in the variables a, b, c and d is the determinant of the matrix

$$M = \begin{pmatrix} x_1^2 & y_1^2 & x_1 & y_1 \\ x_2^2 & y_2^2 & x_2 & y_2 \\ x_3^2 & y_3^2 & x_3 & y_3 \\ x_4^2 & y_4^2 & x_4 & y_4 \end{pmatrix}, \tag{11}$$

which is zero, otherwise the coefficients of the first conic of (9) will be all zero, in contradiction with the fact that $ab \neq 0$. So the rank of the matrix M is at most 3.

Since the second conic (9) also has the four points (x_k, y_k), these points satisfy the system

$$\begin{aligned} Ax_1^2 + By_1^2 + Cx_1 + Dy_1 &= -E, \\ Ax_2^2 + By_2^2 + Cx_2 + Dy_2 &= -E, \\ Ax_3^2 + By_3^2 + Cx_3 + Dy_3 &= -E, \\ Ax_4^2 + By_4^2 + Cx_4 + Dy_4 &= -E. \end{aligned} \tag{12}$$

Case 1. Assume that the rank of the matrix M is 3. So a row of this matrix is a linear combination of the other three, without loss of generality assume that it is the fourth arrow. Then there exist λ_i for $i = 1, 2, 3$ not all zero such that

$$\begin{aligned} x_4^2 &= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2, \\ y_4^2 &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2, \\ x_4 &= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \\ y_4 &= \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3. \end{aligned}$$

We define $K = x_1^2 x_2 y_3^2 - x_1^2 x_3 y_2^2 - x_1 x_2^2 y_3^2 + x_1 x_3^2 y_2^2 + x_2^2 x_3 y_1^2 - x_2 x_3^2 y_1^2$, and we consider two subcases.

Subcase 1.1: $K \neq 0$. Under this assumption we can solve the first three equations of system (12) with respect the variables A, B and C obtaining

$$A = \frac{A_1 D + A_2 E}{K}, \quad B = \frac{B_1 D + B_2 E}{K}, \quad C = \frac{C_1 D + C_2 E}{K},$$

where

$$\begin{aligned}
A_1 = &\ -x_1y_2^2y_3 + x_1y_2y_3^2 + x_2y_1^2y_3 - x_2y_1y_3^2 - x_3y_1^2y_2 + x_3y_1y_2^2,\\
A_2 = &\ -x_1y_2^2 + x_1y_3^2 + x_2y_1^2 - x_2y_3^2 - x_3y_1^2 + x_3y_2^2,\\
B_1 = &\ -x_1^2x_2y_3 + x_1^2x_3y_2 + x_1x_2^2y_3 - x_1x_3^2y_2 - x_2^2x_3y_1 + x_2x_3^2y_1,\\
B_2 = &\ (x_2 - x_1)(x_1 - x_3)(x_2 - x_3),\\
C_1 = &\ x_1^2y_2^2y_3 - x_1^2y_2y_3^2 - x_2^2y_1^2y_3 + x_2^2y_1y_3^2 + x_3^2y_1^2y_2 - x_3^2y_1y_2^2,\\
C_2 = &\ x_1^2y_2^2 - x_1^2y_3^2 - x_2^2y_1^2 + x_2^2y_3^2 + x_3^2y_1^2 - x_3^2y_2^2.
\end{aligned}$$

Substituting A, B and C in the fourth equation of (12) we obtain $E(\lambda_1 + \lambda_2 + \lambda_3 - 1) = 0$. Since $E \neq 0$ we have $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Now we solve the system

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 - 1 = 0,\\
&(\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3)^2 - \lambda_1x_1^2 - \lambda_2x_2^2 - \lambda_3x_3^2 = 0,\\
&(\lambda_1y_1 + \lambda_2y_2 + \lambda_3y_3)^2 - \lambda_1y_1^2 - \lambda_2y_2^2 - \lambda_3y_3^2 = 0.
\end{aligned} \tag{13}$$

with respect to the variables λ_1, λ_2 and λ_3, and we get

$$\begin{aligned}
\lambda_1 = &\ \frac{1}{L}(-x_1y_2 + x_1y_3 - x_2y_1 + 2x_2y_2 - x_2y_3 + x_3y_1 - x_3y_2)\\
&\ (-x_1y_2 + x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 + x_3y_2 - 2x_3y_3),\\
\lambda_2 = &\ -\frac{1}{L}(-2x_1y_1 + x_1y_2 + x_1y_3 + x_2y_1 - x_2y_3 + x_3y_1 - x_3y_2)\\
&\ (-x_1y_2 + x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 + x_3y_2 - 2x_3y_3),\\
\lambda_3 = &\ 1 - \lambda_1 - \lambda_2,
\end{aligned}$$

provided that $L = (x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1))^2 \neq 0$. We note that if $L = 0$ then taken

$$x_3 = \frac{-x_1y_2 + x_1y_3 + x_2y_1 - x_2y_3}{y_1 - y_2},$$

and solving the system (13) we get

$$\lambda_1 = \frac{-2\lambda_3y_1 + y_1 - y_2 + 2\lambda_3y_3 + R}{2y_1 - 2y_2},$$

$$\lambda_2 = \frac{-2\lambda_3y_1 + y_1 - y_2 + 2\lambda_3y_3 - R}{2y_1 - 2y_2},$$

where $R = \pm\sqrt{y_1^2 - 2y_1(-2\lambda_3y_2 + y_2 + 2\lambda_3y_3) + y_2^2 - 4\lambda_3y_2y_3 + 4\lambda_3y_3^2}$. If R is complex we are done. If R is real then λ_1 and λ_2 depend on λ_3, and the rank of the matrix M is not 3 as we are considering, so L cannot be zero.

In short, we have that

$$F_2 = D(d_1x^2+d_2y^2+d_3x+d_4y)+E(e_1x^2+e_2y^2+e_3x+e_4y+e_5), \tag{14}$$

where

$$\begin{aligned}
d_1 &= -x_1y_2^2y_3 + x_1y_2y_3^2 + x_2y_1^2y_3 - x_2y_1y_3^2 - x_3y_1^2y_2 + x_3y_1y_2^2,\\
d_2 &= -x_1^2x_2y_3 + x_1^2x_3y_2 + x_1x_2^2y_3 - x_1x_3^2y_2 - x_2^2x_3y_1 + x_2x_3^2y_1,\\
d_3 &= x_1^2y_2^2y_3 - x_1^2y_2y_3^2 - x_2^2y_1^2y_3 + x_2^2y_1y_3^2 + x_3^2y_1^2y_2 - x_3^2y_1y_2^2,\\
d_4 &= x_1^2x_2y_3^2 - x_1^2x_3y_2^2 - x_1x_2^2y_3^2 + x_1x_3^2y_2^2 + x_2^2x_3y_1^2 - x_2x_3^2y_1^2,\\
e_1 &= -x_1y_2^2 + x_1y_3^2 + x_2y_1^2 - x_2y_3^2 - x_3y_1^2 + x_3y_2^2,\\
e_2 &= (x_2-x_1)(x_1-x_3)(x_2-x_3),\\
e_3 &= x_1^2y_2^2 - x_1^2y_3^2 - x_2^2y_1^2 + x_2^2y_3^2 + x_3^2y_1^2 - x_3^2y_2^2,\\
e_4 &= 0,\\
e_5 &= x_1^2x_2y_3^2 - x_1^2x_3y_2^2 - x_1x_2^2y_3^2 + x_1x_3^2y_2^2 + x_2^2x_3y_1^2 - x_2x_3^2y_1^2.
\end{aligned}$$

From the computations for obtaining the expression of F_2 given in (14) it follows immediately the expression of F_1, i.e.

$$F_1 = d(d_1x^2 + d_2y^2 + d_3x + d_4y). \tag{15}$$

Now solving the system $F_1(x, y) = 0$ and $F_2(x, y) = 0$, we obtain the four points (x_k, y_k) for $k = 1, 2, 3, 4$ with

$$\begin{aligned}
x_4 &= -\frac{2(x_1-x_3)(x_3-x_2)(y_1-y_2)}{N} + x_1 + x_2 - x_3,\\
y_4 &= \frac{2(y_1-y_3)(y_3-y_2)(x_1-x_2)}{N} + y_1 + y_2 - y_3,
\end{aligned} \tag{16}$$

where $N = x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) \neq 0$ because $L = N^2$ and we have proved that $L \neq 0$.

We can label the points (x_k, y_k) in order that

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3. \tag{17}$$

Now we claim that the following four systems of inequalities do not hold:

$$\begin{aligned}
&x_1 < x_2 < x_3 < x_4 \quad \text{and} \quad y_1 < y_2 < y_3 < y_4,\\
&x_1 < x_2 < x_4 < x_3 \quad \text{and} \quad y_1 < y_2 < y_4 < y_3,\\
&x_1 < x_4 < x_2 < x_3 \quad \text{and} \quad y_1 < y_4 < y_2 < y_3,\\
&x_4 < x_1 < x_2 < x_3 \quad \text{and} \quad y_4 < y_1 < y_2 < y_3.
\end{aligned} \tag{18}$$

We prove this claim. We can assume without loss of generality that $N > 0$. Using the expressions of x_4 and y_4 given in (16) we obtain

$$\begin{aligned} x_4 - x_1 =& \frac{x_2 - x_3}{N}(2(x_1 - x_3)(y_1 - y_2) \\ &+ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)) \end{aligned} \tag{19}$$

Since $x_2 > x_1$ we have

$$\begin{aligned} &2(x_1 - x_3)(y_1 - y_2) + x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &\quad > 2(x_1 - x_3)(y_1 - y_2) + x_1(y_2 - y_1) + x_3(y_1 - y_2) \\ &\quad = (x_1 - x_3)(y_1 - y_2) > 0. \end{aligned}$$

Then from (17) and (19) it follows that $x_4 - x_1 < 0$. Now note that

$$\begin{aligned} y_4 - y_1 =& \frac{y_2 - y_3}{N}(-2(x_1 - x_2)(y_1 - y_3) \\ &+ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)). \end{aligned} \tag{20}$$

Since $x_3 > x_1$ we have

$$\begin{aligned} &-2(x_1 - x_2)(y_1 - y_3) + x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &\quad < -2(x_1 - x_2)(y_1 - y_3) + x_1(y_1 - y_3) + x_2(y_3 - y_1) \\ &\quad = -(x_1 - x_2)(y_1 - y_3) < 0. \end{aligned}$$

Therefore from (17) and (20) we obtain $y_4 - y_1 > 0$.

Doing similar computations for $x_4 - x_2$ and $y_4 - y_2$ we get

$$\begin{aligned} x_4 - x_2 =& \frac{x_1 - x_3}{N}(2(x_2 - x_3)(y_1 - y_2) \\ &+ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)) \\ <& \frac{(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)}{N} < 0, \\ y_4 - y_2 =& \frac{y_1 - y_3}{N}(-2(x_1 - x_2)(y_2 - y_3) \\ &+ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)) \\ >& \frac{(-2(x_1 - x_2)(y_1 - y_3)(y_2 - y_3)}{N} > 0. \end{aligned} \tag{21}$$

Finally

$$
\begin{aligned}
x_4 - x_3 =& -\frac{2(x_1 - x_3)(x_3 - x_2)(y_1 - y_2)}{N} + x_1 + x_2 - 2x_3 \\
=& \left(\frac{(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)}{N} + x_1 - x_3\right) \\
& + \left(\frac{(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)}{N} + x_2 - x_3\right) \\
<& 0 + 0 = 0, \\
y_4 - y_3 =& \frac{2(y_1 - y_3)(y_3 - y_2)(x_1 - x_2)}{N} + y_1 + y_2 - 2y_3 \\
=& \left(\frac{(y_1 - y_3)(y_3 - y_2)(x_1 - x_2)}{N} + y_1 - y_3\right) \\
& + \left(\frac{(y_1 - y_3)(y_3 - y_2)(x_1 - x_2)}{N} + y_2 - y_3\right) \\
>& 0 + 0 = 0.
\end{aligned} \tag{22}
$$

Summarizing these results we obtain

$$
(x_4 - x_1)(y_4 - y_1) < 0, \quad (x_4 - x_2)(y_4 - y_2) < 0, \quad (x_4 - x_3)(y_4 - y_3) < 0.
$$

This proves the claim.

Since at least one of the four inequalities in (18) must hold in order that the piecewise linear differential system (2) has 4 limit cycles, it follows that in this subcase the system has at most 3 limit cycles.

Subcase 1.2: $K = 0$. Then solving the system $F_1(x, y) = 0$ and $F_2(x, y) = 0$ we get that $y = ((y_1 - y_2)x - x_2 y_1 + x_1 y_2)/(x_1 - x_2)$, So we have a continuum of solutions, and consequently no limit cycles.

Case 2: Assume that the rank of the matrix M is 2. Now two rows of the matrix M are linear combination of the other two rows. Without loss of generality we can suppose that the rows three and four are combination of the rows one and two, i.e.

$$
x_3 = \lambda_1 x_1 + \lambda_2 x_2, \;\; y_3 = \lambda_1 y_1 + \lambda_2 y_2, \;\; x_3^2 = \lambda_1 x_1^2 + \lambda_2 x_2^2, \; y_3^2 = \lambda_1 y_1^2 + \lambda_2 y_2^2. \tag{23}
$$

A similar expression we have for x_4, y_4, x_4^2 and y_4^2 but with μ's instead of λ's.

Again we assume that the conics $F_1 = 0$ and $F_2 = 0$ share four points (x_k, y_k) for $k = 1, 2, 3, 4$. Substituting them in F_2 we obtain

$$
\begin{aligned}
&Ax_1^2 + By_1^2 + Cx_1 + Dy_1 + E = 0,\\
&Ax_2^2 + By_2^2 + Cx_2 + Dy_2 + E = 0,\\
&E(1 - \lambda_1 - \lambda_2) = 0,\\
&E(1 - \mu_1 - \mu_2) = 0.
\end{aligned}
\tag{24}
$$

respectively.

From (23) and (16) we get that either $(\lambda_1, \lambda_2) = (1, 0)$, or $(\lambda_1, \lambda_2) = (0, 1)$. Therefore, the third row is either equal to the first row, or two the second. In contradiction with the fact that the four points (x_k, y_k) are distinct.

Case 3: The rank of the matrix M is one. But then the four points (x_k, y_k) are collinear in contradiction with the fact that the two hyperbolas (7) cannot intersect in four collinear points.

In summary, the discontinuous piecewise linear differential system (2) has at most 3 limit cycles. So the proof of Theorem 2 is completed, modulo the proof of the Propositions 3, 4 and 5.

3. Proofs of Propositions 3, 4 and 5

Proof of Proposition 3. We shall prove that the discontinuous piecewise linear differential system (2) having a center in each of the pieces Q_1, Q_2 and H has exactly 1 limit cycle.

Since $\pm 4i$ are the eigenvalues of the matrices of the three linear differential systems of (2), these systems are centers.

The three linear differential systems of (3) have the following first integrals

$$
\begin{aligned}
H_1(x, y) &= 4x^2 + y^2 - 8x - 4y,\\
H_2(x, y) &= 4x^2 + y^2 + \frac{32}{3}x,\\
H_3(x, y) &= 4x^2 + y^2 + 8x,
\end{aligned}
$$

in Q_1, Q_2 and H, respectively. Then for the discontinuous piecewise linear differential system (2) the system (6) becomes

$$
\begin{aligned}
&4x_+^2 - y_+^2 - 8x_+ + 4y_+ = 0,\\
&4x_-^2 - y_+^2 + \frac{32}{3}x_- = 0,\\
&(x_+ - x_-)(2 + x_- + x_+) = 0.
\end{aligned}
$$

Taking into account that we are only interested in the solutions (x_+, y_+, x_-) satisfying $x_+ > 0$, $x_- < 0$ and $y_+ > 0$, the unique solution of the previous system is $(x_+^1, y_+^1, x_-^1) = (1, 2, -3)$.

The first linear differential system of (2) has the solution

$$
\begin{aligned}
x_1(t) &= \sin^2(4t) + \cos^2(4t) - \cos(4t),\\
y_1(t) &= 2\left(\sin^2(4t) + \cos^2(4t) - \sin(4t)\right),
\end{aligned}
$$

satisfying the initial conditions $x_1(0) = 1$ and $y_1(0) = 0$. The second linear differential system of (2) has the solution

$$
\begin{aligned}
x_2(t) &= -\frac{1}{3}\left(4\sin^2(4t) + 4\cos^2(4t) + 5\cos(4t)\right),\\
y_2(t) &= \frac{10}{3}\sin(4t),
\end{aligned}
$$

satisfying the initial conditions $x_2(0) = 0$ and $y_2(0) = 2$. And the third linear differential system of (2) has the solution

$$
\begin{aligned}
x_3(t) &= 2\cos(4t) - \sin^2(4t) - \cos^2(4t),\\
y_3(t) &= -4\sin(4t),
\end{aligned}
$$

satisfying the initial conditions $x_3(0) = -3$ and $y_3(0) = 0$.

The time that the solution $(x_1(t), y_1(t))$ contained in Q_1 needs to reach the point $(0, 2)$ is $t_1 = 0.3926990954490209..$ The time that the solution $(x_2(t), y_2(t))$ contained in Q_2 needs to reach the point $(-3, 0)$ is $t_2 = 0.624522886199127..$ The time that the solution $(x_3(t), y_3(t))$ contained in H needs to reach the point $(1, 0)$ is $t_3 = 0.785398163397448..$

Drawing the the orbit $(x_k(t), y_k(t))$ for the times $t \in [0, t_k]$ for $k = 1, 2, 3$, we obtain the limit cycle of Figure 2. $\square$

Proof of Proposition 4. We shall prove that the discontinuous piecewise linear differential system (3) having a center in each of the pieces Q_1, Q_2 and H has exactly 2 limit cycles.

Since $\pm i$, $\pm\dfrac{1}{20}\sqrt{\dfrac{479}{5}}i$ and $\pm\dfrac{1}{2}i$ are the eigenvalues of the matrices of the three linear differential systems of (3), these systems are centers.

The three linear differential systems of (3) have the following first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2+y^2+x-y,\\
H_2(x,y) &= x^2+\frac{479}{500}y^2-\frac{31}{50}x-\frac{97}{100}y,\\
H_3(x,y) &= 16x^2++y^2+\frac{8}{5}x,
\end{aligned}
$$

in Q_1, Q_2 and H, respectively. Then for the discontinuous piecewise linear differential system (3) the system (6) becomes

$$
\begin{aligned}
&(x_+-y_++1)(x_++y_+)=0,\\
&900x_-^2-977y_+^2-1410x_-+955y_+=0,\\
&(x_+-x_-)(1-10x_+-10x_-)=0.
\end{aligned}
\tag{25}
$$

Taking into account that we are only interested in the solutions (x_+,y_+,x_-) satisfying $x_+>0$, $x_-<0$ and $y_+>0$, the unique two solutions of the previous system are

$$
(x_+^1,y_+^1,x_-^1)=\left(2,3,-\frac{19}{10}\right)\quad\text{and}\quad(x_+^2,y_+^2,x_-^2)=\left(1,2,-\frac{9}{10}\right).
$$

The first linear differential system of (3) has the solution

$$
\begin{aligned}
x_1(t) &= \frac{1}{2}((2u+1)\cos t+\sin t-1),\\
y_1(t) &= \frac{1}{2}((2u+1)\sin t-\cos t+1),
\end{aligned}
$$

satisfying the initial conditions $x_1(0)=u$ and $y_1(0)=0$.

The second linear differential system of (3) has the solution

$$
\begin{aligned}
x_2(t) &= \frac{485-958v}{20\sqrt{2395}}\sin\left(\frac{1}{20}\sqrt{\frac{479}{5}}t\right)-\frac{31}{100}\cos\left(\frac{1}{20}\sqrt{\frac{479}{5}}t\right)+\frac{31}{100},\\
y_2(t) &= \left(v-\frac{485}{958}\right)\cos\left(\frac{1}{20}\sqrt{\frac{479}{5}}t\right)-\frac{31}{2\sqrt{2395}}\sin\left(\frac{1}{20}\sqrt{\frac{479}{5}}t\right)+\frac{485}{958},
\end{aligned}
$$

satisfying the initial conditions $x_2(0) = 0$ and $y_2(0) = v$. And the third linear differential system of (3) has the solution

$$
\begin{aligned}
x_3(t) &= \frac{1}{20}\left((20w+1)\cos\left(\frac{t}{2}\right) - 1\right),\\
y_3(t) &= \frac{1}{5}(20w+1)\sin\left(\frac{t}{2}\right),
\end{aligned}
$$

satisfying the initial conditions $x_3(0) = w$ and $y_3(0) = 0$.

Now we consider the solution $(x_k^1(t), y_k^1(t))$ for $k = 1,2,3$ of the discontinuous piecewise linear differential system (3) corresponding to the solution $(x_+^1, y_+^1, x_-^1) = (u, v, w) = (2, 3, -19/10)$ of system (25). Then the time that the solution $(x_1^1(t), y_1^1(t))$ contained in Q_1 needs to reach the point $(0, v)$ is $t_1 = 1.5707963267948966..$ The time that the solution $(x_2^1(t), y_2^1(t))$ contained in Q_2 needs to reach the point $(w, 0)$ is $t_2 = 3.3659397308004984..$ The time that the solution $(x_3^1(t), y_3^1(t))$ contained in H needs to reach the point $(u, 0)$ is $t_3 = 6.283185307179586..$

Let $(x_k^2(t), y_k^2(t))$ for $k = 1,2,3$ be the solution of the discontinuous piecewise linear differential system (3) corresponding to the solution $(x_+^2, y_+^2, x_-^2) = (u, v, w) = (1, 2, -9/10)$ of system (25). Then the time that the solution $(x_1^2(t), y_1^2(t))$ contained in Q_+ needs to reach the point $(0, v)$ is $T_1 = 1.5707963267948966..$ The time that the solution $(x_2^2(t), y_2^2(t))$ contained in Q_2 needs to reach the point $(w, 0)$ is $T_2 = 3.473334629546722..$ The time that the solution $(x_3^2(t), y_3^2(t))$ contained in H needs to reach the point $(u, 0)$ is $T_3 = 6.283185307179586..$

Drawing the two orbits $(x_k^j(t), y_k^j(t))$ for $j = 1,2$ and for the times $t \in [0, t_k]$ and $t \in [0, T_k]$ for $k = 1,2,3$, respectively, we obtain the 2 limit cycles of Figure 3. □

Proof of Proposition 5. We shall prove that the discontinuous piecewise linear differential system (4) having a center in each of the pieces Q_1, Q_2 and H has exactly 3 limit cycles.

Since $\pm\dfrac{2i}{\sqrt{1565}}$, $\pm 4\sqrt{\dfrac{2}{1565}}i$ and $\pm 4i$ are the eigenvalues of the matrices of the three linear differential systems of (4), these systems are centers.

The three linear differential systems of (4) have the following first integrals

$$\begin{aligned} H_1(x,y) &= 1565x^2 + y^2 - 1185x + 379y, \\ H_2(x,y) &= 21910x^2 + 7y^2 + 10\sqrt{4430533}x + 5783y, \\ H_3(x,y) &= 8764x^2 + 2191y^2 + (2\sqrt{4430533} - 2598)x, \end{aligned}$$

in Q_1, Q_2 and H, respectively. Then for the discontinuous piecewise linear differential system (4) the system (6) becomes

$$\begin{aligned} &1565x_+^2 - y_+^2 - 1185x_+ - 379y_+ = 0, \\ &21910x_-^2 - 7y_+^2 + 10\sqrt{4430533}x_- - 5783y_+ = 0, \\ &(x_+ - x_-)\left(4382x_+ + 4382x_- + \sqrt{4430533} - 1299\right) = 0. \end{aligned} \tag{26}$$

Taking into account that we are only interested in the solutions (x_+, y_+, x_-) satisfying $x_+ > 0$, $x_- < 0$ and $y_+ > 0$, the unique three solutions of the previous system are

$$\begin{aligned} (x_+^1, y_+^1, x_-^1) &= \left(1, \frac{-3083 - \sqrt{4430533}}{4382}, 1\right), \\ (x_+^2, y_+^2, x_-^2) &= \left(2, \frac{-7465 - \sqrt{4430533}}{4382}, 10\right), \\ (x_+^3, y_+^3, x_-^3) &= \left(3, \frac{-11847 - \sqrt{4430533}}{4382}, 26\right). \end{aligned}$$

The first linear differential system of (3) has the solution

$$\begin{aligned} x_1(t) &= \left(u - \frac{237}{626}\right)\cos\left(\frac{2t}{\sqrt{1565}}\right) + \frac{379}{2\sqrt{1565}}\sin\left(\frac{2t}{\sqrt{1565}}\right) + \frac{237}{626}, \\ y_1(t) &= \frac{1}{2}\sqrt{\frac{5}{313}}(237 - 626u)\sin\left(\frac{2t}{\sqrt{1565}}\right) + \frac{379}{2}\cos\left(\frac{2t}{\sqrt{1565}}\right) - \frac{379}{2}, \end{aligned}$$

satisfying the initial conditions $x_1(0) = u$ and $y_1(0) = 0$.

The second linear differential system of (3) has the solution

$$
\begin{aligned}
x_2(t) = \; & \frac{1}{21910}\sin\left(2\sqrt{\frac{2}{1565}}t\right)\left(\sqrt{3130}(14v+5783)\cos\left(2\sqrt{\frac{2}{1565}}t\right)\right. \\
& \left.-10\sqrt{4430533}\sin\left(2\sqrt{\frac{2}{1565}}t\right)\right), \\
y_2(t) = \; & \left(v+\frac{5783}{14}\right)\cos\left(4\sqrt{\frac{2}{1565}}t\right)-\frac{1}{7}\sqrt{\frac{22152665}{626}}\sin\left(4\sqrt{\frac{2}{1565}}t\right) \\
& -\frac{5783}{14},
\end{aligned}
$$

satisfying the initial conditions $x_2(0)=0$ and $y_2(0)=v$. And the third linear differential system of (3) has the solution

$$
\begin{aligned}
x_3(t) &= \frac{1}{8764}\left(8764w\cos(4t)+\left(\sqrt{4430533}-1299\right)\left(\cos(4t)-1\right)\right), \\
y_3(t) &= \frac{\left(1299-8764w-\sqrt{4430533}\right)\sin(4t)}{4382},
\end{aligned}
$$

satisfying the initial conditions $x_3(0)=w$ and $y_3(0)=0$.

Now we consider the solution $(x_k^1(t), y_k^1(t))$ for $k=1,2,3$ of the discontinuous piecewise linear differential system (4) corresponding to the solution (x_+^1, y_+^1, x_-^1) of system (26). Then the time that the solution $(x_1^1(t), y_1^1(t))$ contained in Q_1 needs to reach the point $(0,v)$ is $t_1 = 4.10363864680248..$ The time that the solution $(x_2^1(t), y_2^1(t))$ contained in Q_2 needs to reach the point $(w,0)$ is $t_2 = 1.11762450719575..$ The time that the solution $(x_3^1(t), y_3^1(t))$ contained in H needs to reach the point $(u,0)$ is $t_3 = 0.785398163397448..$

Let $(x_k^2(t), y_k^2(t))$ for $k=1,2,3$ be the solution of the discontinuous piecewise linear differential system (4) corresponding to the solution (x_+^2, y_+^2, x_-^2) of system (26). Then the time that the solution $(x_1^2(t), y_1^2(t))$ contained in Q_+ needs to reach the point $(0,v)$ is $r_1 = 7.93799227264621..$ The time that the solution $(x_2^2(t), y_2^2(t))$ contained in Q_2 needs to reach the point $(w,0)$ is $r_2 = 2.02943545009903..$ The time that the solution $(x_3^2(t), y_3^2(t))$ contained in H needs to reach the point $(u,0)$ is $r_3 = 0.785398163397448..$

Let $(x_k^3(t), y_k^3(t))$ for $k=1,2,3$ be the solution of the discontinuous piecewise linear differential system (4) corresponding to the solution $(x_+^3, y_+^3, x_-^3) =$ of system (26). Then the time that the solution $(x_1^3(t), y_1^3(t))$ contained in Q_+ needs to reach the point $(0,v)$ is

$s_1 = 11.27688306691738..$ The time that the solution $(x_2^3(t), y_2^3(t))$ contained in Q_2 needs to reach the point $(w, 0)$ is $s_2 = 2.88219547492608..$ The time that the solution $(x_3^3(t), y_3^3(t))$ contained in H needs to reach the point $(u, 0)$ is $s_3 = 0.785398163397448..$

Drawing the three orbits $(x_k^j(t), y_k^j(t))$ for $j = 1, 2, 3$ and for the times $t \in [0, t_k]$, $t \in [0, r_k]$ and $t \in [0, s_k]$ for $k = 1, 2, 3$, respectively, we obtain the 3 limit cycles of Figure 4. □

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