LIMIT CYCLES OF THE CLASSICAL LIÉNARD DIFFERENTIAL SYSTEMS: A SURVEY ON THE LINS NETO, DE MELO AND PUGH'S CONJECTURE

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Abstract. In 1977 Lins Neto, de Melo and Pugh [Lectures Notes in Math. **597**, 335–357] conjectured that the classical Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -x,$$

with $F(x)$ a real polynomial of degree n, has at most $[(n-1)/2]$ limit cycles, where $[\cdot]$ denotes the integer part function. In this paper we summarize what is known and what is still open on this conjecture. For the known results on this conjecture we present a complete proof.

1. Introduction and statement of the main results

The classical Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{1}$$

with $F(x)$ a real polynomial of degree n, has been extensively studied (see for instance [2, 11, 20, 25, 31, 32, 39, 42, 43], and references therein). In 1977 Lins Neto, de Melo and Pugh [25] proved that there exist systems (1) of degree n having $[(n-1)/2]$ limit cycles, and stated the following:

Conjecture *System* (1) *has at most* $[(n-1)/2]$ *limit cycles, where n is the degree of the real polynomial $F(x)$.*

Here $[x]$ denotes the integer part function of x.

In this paper we summarize what is known and what is still open on this conjecture. Moreover for the known results on this conjecture we present a complete proof.

The conjecture was based in the following result of Lins Neto, de Melo and Pugh [25]:

Theorem 1. *If the real polynomial $F(x)$ has degree n, then there are Liénard differential systems* (1) *having at least* $[(n-1)/2]$ *limit cycles.*

2010 *Mathematics Subject Classification.* Primary 34C05, 34C23; secondary 34C25, 34C29.

Key words and phrases. Liénard system, limit cycle, conjecture of Lins Neto, de Melo and Pugh.

Here we shall present a shorter and different proof of Theorem 1 from the one given in [25], this new proof also provides information about the stability of the limit cycles.

The known results on the conjecture are the following:

Theorem 2. *For the Liénard differential system* (1) *the following statements hold.*

- (a) *For* $n = 1, 2$*, system* (1) *has no limit cycles.*
- (b) *For* $n = 3, 4$*, system* (1) *has at most one limit cycle, and there exist systems* (1) *having one limit cycle.*
- (c) *For any* $n \geq 6$*, there exist systems* (1) *having at least* $n - 2$ *limit cycles.*

Theorem 2 says that for $n = 1, 2, 3, 4$ the conjecture holds, while for $n \geq 6$ it does not hold. At this moment only remains to know if the conjecture holds or not for $n = 5$.

The result that the conjecture holds for $n = 1, 2, 3$ already was proved by Lins Neto, de Melo and Pugh [25]. Here, for $n = 1, 2$ we shall present the orignal proofs, but for $n = 3$ we shall present two new different and shorter proofs. In 2012, thirty five years after the statement of the conjecture, it was proved by Li and Llibre [23] that the conjecture also holds for $n = 4$, but that proof is long and considers several cases, and the whole paper has 20 pages. We will not repeat this proof.

Statement (c) of Theorem 2 shows that the conjecture is not correct for $n \geq 6$. Through the conjecture remains unchanged for more than thirty years in 2007 Dumortier, Panazzolo and Roussarie [11] shown that the conjecture is not true for $n \geq 7$ providing one additional limit cycle to the ones predicted by the conjecture. In 2011 De Maesschalck and Dumortier [8] proved that the conjecture is not true for $n \geq 6$ providing two additional limit cycles to the ones predicted by the conjecture. Finally, in 2015 De Maesschalck and Huzak [9] proved that the number of limit cycles is at least $n - 2$ if $n \geq 6$, i.e. showing that the Liénard differential systems of degree $n \geq 6$ have essentially at least $n/2$ more limit cycles than the number conjectured by Lins Neto, de Melo and Pugh. Summarizing the above results we state the following question:

Open problem. *What is the maximum number of limit cycles for the Liénard differential systems* (1) *when* $n \geq 5$*?*

This paper is organized as follows. In section 2 we prove Theorem 1. In section 3 statement (a) of Theorem 2 is proved. In section 4 we prove statement (b) for $n = 3$ of Theorem 2. Finally, the proof of statement (c) is presented in section 5.

2. Proof of Theorem 1

The proof presented here of Theorem 1, is shorter, different and provides information about the kind of stability of the limit cycles, it comes from the paper [26], see also [27].

For doing the proof of Theorem 1 we need to recall some basic results from the averaging theory of first order, for a proof of these results see, for instance, Theorems 11.5 and 11.6 of the book of Verhulst [37].

The averaging theory says: If the function

$$f(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, r)\, d\theta$$

has k simple real roots, $0 < r_1 < \cdots < r_k$, then the differential equation in polar coordinates (r, θ)

$$\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r, \varepsilon) \tag{2}$$

has k limit cycles tending to the circles $r = r_i$ for $i = 1, \ldots, k$ when $\varepsilon \to 0$, where F_1 and F_2 are periodic of period 2π in θ and C^2 smoothness. Moreover, the limit cycle tending to the circle $r = r_i$ is stable if $f'(r_i) < 0$, and unstable if $f'(r_i) > 0$.

Proof of Theorem 1. We shall prove that $[(n-1)/2]$ is a lower bound for the maximum number of limit cycles that Liénard polynomial differential systems (1) of degree n can have. More precisely, we shall show that there are differential systems of the form

$$\dot{x} = y + \varepsilon F(x), \qquad \dot{y} = -x, \tag{3}$$

with $F(x) = a_0 + a_1 x + \ldots + a_n x^n$ and $a_n \neq 0$ having $[(n-1)/2]$ limit cycles.

As usual polar coordinates (r, θ) are defined as $x = r\cos\theta$ and $y = r\sin\theta$. In polar coordinates the differential system (3) is

$$\dot{r} = \varepsilon \cos\theta\, F(r\cos\theta), \qquad \dot{\theta} = -1 - \varepsilon\frac{1}{r}\sin\theta\, F(r\cos\theta). \tag{4}$$

Choosing the variable θ as the new independent variable, system (4) becomes

$$\frac{dr}{d\theta} = -\varepsilon\cos\theta\, F(r\cos\theta) + O(\varepsilon^2) = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r, \varepsilon). \tag{5}$$

Applying the averaging theory described for equation (2) to equation (5), we obtain

$$
\begin{aligned}
f(r) = & \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, r)d\theta \\
= & -\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, F(r\cos\theta)d\theta \\
= & -\frac{1}{2\pi}\sum_{i=0}^{n} a_i r^i \int_0^{2\pi} \cos^{i+1}\theta d\theta \\
= & -\frac{1}{2\pi}\sum_{j=0}^{[(n-1)/2]} a_{2j+1} r^{2j+1} \int_0^{2\pi} \cos^{2j+2}\theta d\theta \\
= & \sum_{j=0}^{[(n-1)/2]} a_{2j+1} b_{2j+1} r^{2j+1}
\end{aligned}
$$

where

$$
b_{2j+1} = -\frac{1}{2\pi}\int_0^{2\pi} \cos^{2j+2}\theta d\theta \neq 0,
$$

for $j = 0, 1, \ldots, [(n-1)/2]$.

Since the monomials of the polynomial $f(r)$ are $r, r^3, \ldots, r^{2[(n-1)/2]+1}$, and the coefficient a_{2j+1} in the monomial r^{2j+1} can be chosen arbitrarily, we can obtain that the roots of the polynomial $f(r)$ are 0 and $\pm r_1, \ldots$, $\pm r_{[(n-1)/2]}$ with $0 < r_1 < \ldots < r_{[(n-1)/2]}$. Note that all these roots are simple, i.e. $f'(r_k) \neq 0$ for $k = 1, 2, \ldots, [(n-1)/2]$. Therefore the averaging theory says that for ε sufficiently small the differential equation (5), and consequently the differential system (3) have $[(n-1)/2]$ limit cycles near the circles of radius r_k for $k = 1, 2, \ldots, [(n-1)/2]$. This completes the proof of the theorem. □

3. Proof of statement (a) of Theorem 2

The materials of this section follows from [25]. The first one characterizes the structure of system (1) at the infinity in the so called Poincaré disc, see Theorem 1 of [25]. Since it is not a result properly on limit cycles we do not prove it here.

Theorem 3. *Let $F = a_1x + a_2x^2 + \ldots + a_nx^n$. The topological phase portrait of system* (1) *at infinity is given in Fig. 1.*

The second one is on the non–existence of periodic orbits and consequently on limit cycles of system (1), see Proposition 1 of [25] for a proof. We reproduce here that proof.

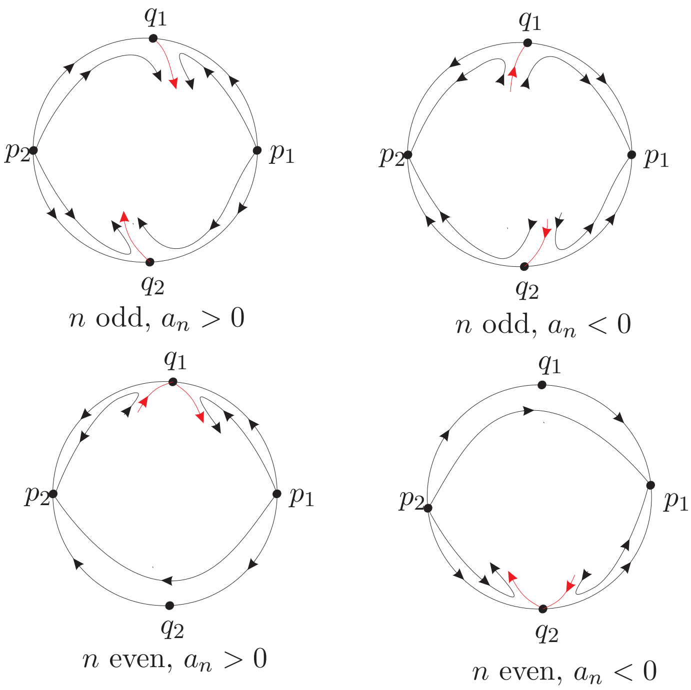

FIGURE 1. The topological phase portraits of the Liénard differential system (1) in a neighborhood of the infinity.

Proposition 4. *Let $F(x) = \mathcal{E}(x) + \mathcal{O}(x)$ with $\mathcal{E}(x)$ an even polynomial and $\mathcal{O}(x)$ an odd polynomial. If 0 is the unique root of $\mathcal{O}(x)$, then the Liénard differential system (1) has no periodic orbits.*

Proof. Consider the differential system

$$\dot{x} = y - \mathcal{E}(x), \quad \dot{y} = -x, \tag{6}$$

Let a_k be the coefficient of the highest order term of $\mathcal{E}(x)$. Since this system is invariant under the symmetry $(x, y, t) \to (-x, y, -t)$ the origin of coordinates is a center. Theorem 3 implies that system (6) has the two phase portraits given in Fig. 2 for $n = k$ even depending on $a_k > 0$ and $a_k < 0$, respectively.

We study the case $a_k > 0$. For $a_k < 0$ the arguments are completely same as those of $a_k > 0$. Since each periodic orbit of system (6) intersects the negative y–axis in a unique point, we define a function $H : \mathbb{R}^2 \to \mathbb{R}$ as follows: for each $p \in \mathbb{R}^2$ the value of $H(p)$ is the y–coordinate of the intersection point of the negative y–axis with the orbit passing through p. Then H is an analytic function and $H(0) = 0$ is the unique maximum. By definition H is a first integral of system (6), so there exists an integrating

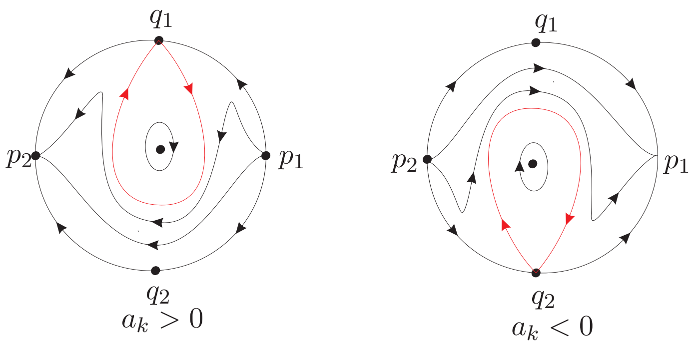

FIGURE 2. The phase portrait of system (6).

factor $R(x, y)$ such that

$$\frac{\partial H}{\partial x} = xR(x, y), \quad \frac{\partial H}{\partial y} = (y - \mathcal{E}(x))R(x, y).$$

Furthermore we have $R(x, y) < 0$ for $(x, y) \neq (0, 0)$ because the origin is the unique maximum and H monotonically decreases in $x > 0$.

Direct calculations on the orbits of the differential system (1) show that

$$\left.\frac{dH}{dt}\right|_{(1)} = -\mathcal{O}(x)\frac{\partial H}{\partial x} = -x\mathcal{O}(x)R(x, y).$$

By assumption we get that the derivative of H along an orbit of (1) vanishes if and only if $x = 0$, and that for $x \neq 0$ the derivative is either always positive or always negative. This implies that system (1) has no periodic orbits. □

Proof of statement (a) *of Theorem 2.* When $n = 1$ the differential system (1) is a linear differential system in $\mathbb{R}^2$, and consequently it has no limit cycles, because when a linear differential system has a periodic orbit this is not isolated in the set of all periodic orbits of the system. This proves statement (a) of Theorem 2 for $n = 1$.

Assume $n = 2$. Then, applying Proposition 4 to system (1) we get that $\mathcal{O}(x) = a_1 x$. So the unique root of $\mathcal{O}(x)$ is $x = 0$, and by applying Proposition 4 the system has no limit cycles. This completes the proof of statement (a) of Theorem 2. □

4. PROOF OF STATEMENT (b) OF THEOREM 2

We shall use the following well-known result, the Green's theorem, for a proof see for instance [29].

Theorem 5. *Let γ be a piecewise smooth, simple closed curve in $\mathbb{R}^2$, and let R be the open region bounded by γ. If $P = P(x, y)$ and $Q = Q(x, y)$*

are functions defined on an open region containing R and have continuous partial derivatives there, then

$$\oint_\gamma (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where the integration path along γ is in counterclockwise sense.

The divergence of a $\mathcal{C}^1$ differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{7}$$

is the function

$$\operatorname{div}(x,y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Proposition 6. *Let $\gamma = \gamma(t) = (x(t), y(t))$ be a periodic orbit of a $\mathcal{C}^1$ differential system* (7) *of period T. Define*

$$\sigma = \int_\gamma \operatorname{div}(x,y)\,dt = \int_0^T \operatorname{div}(x(t),y(t))\,dt. \tag{8}$$

Then, if $\sigma < 0$ the periodic orbit γ is a stable limit cycle, and if $\sigma > 0$ the periodic orbit γ is an unstable limit cycle.

For a proof of Proposition 6 see for instance Theorem 1.23 of [10].

The limit cycles for which the value σ defined in (8) is non-zero are called *hyperbolic* limit cycles.

First proof of statement (*b*) *of Theorem 2 for $n = 3$.* Set $\mathcal{E}(x) = a_2x^2$ and $\mathcal{O}(x) = a_1x + a_3x^3$. If $a_1a_3 \geq 0$, we have either $a_1a_3 > 0$, or $a_1 = 0$ and $a_3 \neq 0$, or $a_1 \neq 0$ and $a_3 = 0$, or $a_1 = a_3 = 0$. In the last case system (1) is symmetric with respect to the y–axis, and the origin of coodinates is a center, so it has no limit cycles. In the other three cases the odd function $\mathcal{O}(x)$ has the unique root $x = 0$. Therefore, by Proposition 4, system (1) has no periodic orbits, and consequently no limit cycles. Hence in what follows we assume that $a_1a_3 < 0$.

For $a_1a_3 < 0$ we can assume without loss of generality that $a_1 > 0$ and $a_3 < 0$, otherwise doing the change of variables $(x, y, t) \to (-x, y, -t)$ in system (1) we obtain the wanted assumptions. Since $a_1 > 0$ the singular point at the origin of coordinates of the differential system (1) is a stable focus or node.

Let γ be a periodic solution of the differential system (1), and $-a_1 - 2a_2x - 3a_3x^2$ is the divergence of that differential system. We consider the integral of the divergence along the periodic orbit γ as in (8), i.e.

$$\begin{aligned} I = & -\oint_\gamma (a_1 + 2a_2x + 3a_3x^2)dt \\ = & \oint_\gamma \frac{a_1 + 2a_2x + 3a_3x^2}{x} dy, \end{aligned} \tag{9}$$

where we have used the second equation of the differential system (1). Since the integral of the first line of the expressions (9) is well defined, also it is well defined the integral of the second line of (9).

In order to apply the Green's theorem to the integral of the second line of (9), we shall split such an integral as limit of two integrals as follows. We add to the periodic orbit γ the segment S of the y-axis contained in the region bounded by γ, now we split this segment as limit of two parallel segments $S_-(\varepsilon)$ and $S_+(\varepsilon)$ contained in $x < 0$ and $x > 0$ and at a distance $\varepsilon > 0$ of S, respectively, and such that a piece $\gamma_-(\varepsilon)$ of γ contained in $x < 0$ together with $S_-(\varepsilon)$ forms an oval $O_-(\varepsilon)$. Similarly, we consider a piece $\gamma_+(\varepsilon)$ of γ contained in $x > 0$ such that together with $S_+(\varepsilon)$ forms another oval $O_+(\varepsilon)$, in such a way that the union of these ovals tends to $\gamma \cup S$ when $\varepsilon \mapsto 0$.

Since the orbit γ and consequently the ovals $O_\pm(\varepsilon)$ are run in clockwise, and later on we want to apply the Green's Theorem to these ovals, we orient the orbit γ and both ovals in counterclockwise sense and denote them with these new orientations by $\widetilde{\gamma}$ and $\widetilde{O}_\pm(\varepsilon)$ respectively. Clearly the two integrals

$$\oint_{\widetilde{O}_-(\varepsilon)} \frac{a_1 + 2a_2x + 3a_3x^2}{x} dy \quad \text{and} \quad \oint_{\widetilde{O}_+(\varepsilon)} \frac{a_1 + 2a_2x + 3a_3x^2}{x} dy$$

are well defined, and the integral

$$\oint_{\widetilde{\gamma}} \frac{a_1 + 2a_2x + 3a_3x^2}{x} dy$$

is the limit when $\varepsilon \mapsto 0$ of

$$I_\varepsilon = \oint_{\widetilde{O}_-(\varepsilon)} \frac{a_1 + 2a_2x + 3a_3x^2}{x} dy + \oint_{\widetilde{O}_+(\varepsilon)} \frac{a_1 + 2a_2x + 3a_3x^2}{x} dy. \tag{10}$$

Applying the Green's theorem (Theorem 5) to both integrals of (10) we obtain that

$$I_\varepsilon = \oint_{R_-(\varepsilon)} \left(-\frac{a_1}{x^2} + 3a_3\right)\, dx\, dy + \oint_{R_+(\varepsilon)} \left(-\frac{a_1}{x^2} + 3a_3\right)\, dx\, dy, \tag{11}$$

where $R_{\pm(\varepsilon)}$ are the open regions bounded by the ovals $\widetilde{O}_{\pm(\varepsilon)}$. Now, from (9), (10), (11), taking into account the change of orientation from γ to $\widetilde{\gamma}$, and taking the limit of I_ε given in (11) when $\varepsilon \mapsto 0$ we obtain that

$$I = -\iint_R \left(-\frac{a_1}{x^2} + 3a_3\right)\, dx\, dy > 0,$$

because $a_1 > 0 > a_3$, where R is the open region bounded by γ.

By Proposition 6, this implies that all the periodic orbits γ surrounding the origin of the differential system (1) are hyperbolic and unstable, consequently at most there is one periodic orbit surrounding the origin, and when it exists is hyperbolic. This completes the proof of statement (b) of Theorem 2 for $n = 3$. □

Our second proof on the uniqueness of limit cycles when $n = 3$ is again different from the original one and more simple.

Second proof of statement (b) of Theorem 2 for $n = 3$. In a similar way to the first proof we can assume that $a_1 > 0$ and $a_3 < 0$. Under this assumption the origin of coordinates is stable, and the infinity is also stable, see Fig. 1. Hence it follows from the Poincaré–Bendixson Theorem (see for instance Corollary 1.30 of [10]) that system (1) has at least one periodic orbit.

First we claim that any periodic orbit of system (1) intersects the straight lines $x = \pm\sqrt{-a_1/a_3}$. Take

$$V(x,y) = \begin{cases} e^{-2a_2 y}\left(y - a_2x^2 + \frac{1}{2a_2}\right) & \text{if } a_2 \neq 0, \\ x^2 + y^2 & \text{if } a_2 = 0, \end{cases} \tag{12}$$

which is the first integral of system (1) with $a_1 = a_3 = 0$, i.e. of the differential system

$$\dot{x} = y - a_2x^2, \qquad \dot{y} = -x. \tag{13}$$

Then the derivative of V along an orbit of the differential system (1) is

$$\dot{V} = \left.\frac{dV}{dt}\right|_{(1)} = Lx^2\left(a_1 + a_3x^2\right), \tag{14}$$

where $L = -2$ if $a_2 = 0$ or $L = 2a_2e^{-2a_2y}$ if $a_2 \neq 0$. This shows that $\dot{V}$ does not change its sign inside the vertical strip $-\sqrt{-a_1/a_3} \leq x \leq \sqrt{-a_1/a_3}$. On the other hand $V(x,y)$ is a first integral of system (13), so near the origin the level curves are closed. Moreover, we get from Fig. 2 and the invariance of V under the symmetry $(x,y) \to (-x,y)$ that there exists a closed level curve of V which contains the origin in its interior and is tangent to both straight lines $x = \pm\sqrt{-a_1/a_3}$. Let D be the region enclosed by this closed level curve of V. Then a periodic orbit cannot intersect D, otherwise there is a contradiction with the fact that $\dot{V}$ does not change its sign inside D. This proves the claim.

Next we prove that system (1) has at most one periodic orbit. By contradiction, we assume that $\widetilde{\Gamma}$ and Γ are two different periodic orbits of system (1) with $\widetilde{\Gamma}$ in the interior of Γ and the origin of coordinates in the interior of $\widetilde{\Gamma}$. From the last claim we have Fig. 3 which shows the separation of the two periodic orbits $\widetilde{\Gamma}$ and Γ by the straight lines $x = \pm\sqrt{-a_1/a_3}$.

For the function $V(x,y)$ defined in (12) we have that

$$\widetilde{I} = \int\limits_{\widetilde{\Gamma}} dV(x,y) = 0, \qquad I = \int\limits_{\Gamma} dV(x,y) = 0.$$

On the other hand we will prove that $\widetilde{I} \neq I$. This contradiction implies that system (1) cannot have more than one periodic orbit.

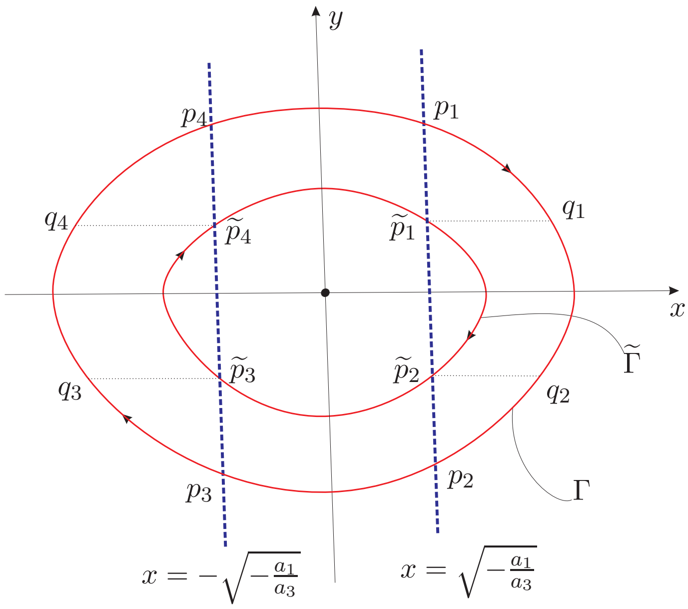

FIGURE 3. The graph of the two periodic orbits $\widetilde{\Gamma}$ and Γ separated by the vertical straight lines $x = \pm\sqrt{-\frac{a_1}{a_3}}$.

From (14) we have

$$\widetilde{I} = \int\limits_{\widetilde{\Gamma}} x^2\left(a_1 + a_3x^2\right)L(y)dt, \quad I = \int\limits_{\Gamma} x^2\left(a_1 + a_3x^2\right)L(y)dt,$$

where $L(y) = -2$ if $a_2 = 0$ or $L(y) = 2a_2e^{-2a_2y}$ if $a_2 \neq 0$. We claim that $\widetilde{I} < I$ for $a_2 > 0$, and $\widetilde{I} > I$ for $a_2 \le 0$.

We only prove the claim for $a_2 > 0$, the proof for the other case follows using the same arguments than for the case $a_2 > 0$. From Fig. 3 we have

$$\begin{aligned}
\widetilde{\Gamma} &= \widehat{\widetilde{p}_1\widetilde{p}_2} \cup \widehat{\widetilde{p}_2\widetilde{p}_3} \cup \widehat{\widetilde{p}_3\widetilde{p}_4} \cup \widehat{\widetilde{p}_4\widetilde{p}_1},\\
\Gamma &= \widehat{p_1q_1} \cup \widehat{q_1q_2} \cup \widehat{q_2p_2} \cup \widehat{p_2p_3} \cup \widehat{p_3q_3} \cup \widehat{q_3q_4} \cup \widehat{q_4p_4} \cup \widehat{p_4p_1}.
\end{aligned}$$

On $\gamma = \widehat{p_1q_1} \cup \widehat{q_2p_2} \cup \widehat{p_3q_3} \cup \widehat{q_4p_4}$, we have $a_1 + a_3x^2 \ge 0$ with the equality only at the points p_1, p_2, p_3 and p_4. Since $L(x, y) > 0$, it follows that

$$I_0 = \int\limits_{\gamma} x^2\left(a_1 + a_3x^2\right)L(x, y)dt > 0.$$

For convenience to express the integrals, we denote $q_i = (x_i, y_i)$ and $\widetilde{p}_i = (\widetilde{x}_i, \widetilde{y}_i)$ for $i = 1, \ldots, 4$.

For comparing the integrals on $\widehat{q_1q_2} \subset \Gamma$ and $\widehat{\widetilde{p}_1\widetilde{p}_2} \subset \widetilde{\Gamma}$ we parameterize the two orbit arcs as $(x_1(y), y)$ and $(\widetilde{x}_1(y), y)$ for $y \in [y_2, y_1]$, respectively.

Then we have

$$I_1 = \int\limits_{\widehat{q_1q_2}} x^2\left(a_1+a_3x^2\right)L(y)dt = \int\limits_{y_1}^{y_2} \left.\frac{x^2\left(a_1+a_3x^2\right)L(y)}{-x}\right|_{x=x_1(y)} dy$$

$$> \int\limits_{y_1}^{y_2} \left.\frac{x^2\left(a_1+a_3x^2\right)L(y)}{-x}\right|_{x=\widetilde{x}_1(y)} dy = \int\limits_{\widehat{\widetilde{p}_1\widetilde{p}_2}} x^2\left(a_1+a_3x^2\right)L(y)dt = \widetilde{I}_1,$$

where we have used $y_1 > y_2$, $L(y) > 0$ and $x_1(y)(a_1+a_3x_1(y)^2) > \widetilde{x}_1(y)(a_1+a_3\widetilde{x}_1(y)^2) > 0$ for $y \in [y_2, y_1]$.

Parameterizing the orbit arcs $\widehat{p_2p_3}$ and $\widehat{\widetilde{p}_2\widetilde{p}_3}$ by $(x, y_2(x))$ and $(x, \widetilde{y}_2(x))$ for $x \in [\widetilde{x}_3, \widetilde{x}_2]$ respectively, then we have

$$I_2 = \int\limits_{\widehat{p_2p_3}} x^2\left(a_1+a_3x^2\right)L(y)dt = \int\limits_{\widetilde{x}_2}^{\widetilde{x}_3} \left.\frac{x^2\left(a_1+a_3x^2\right)L(y)}{y-F(x)}\right|_{y=y_2(x)} dx$$

$$> \int\limits_{\widetilde{x}_2}^{\widetilde{x}_3} \left.\frac{x^2\left(a_1+a_3x^2\right)L(y)}{y-F(x)}\right|_{y=\widetilde{y}_2(x)} dx = \int\limits_{\widehat{\widetilde{p}_2\widetilde{p}_3}} x^2\left(a_1+a_3x^2\right)L(y)dt = \widetilde{I}_2,$$

where we have used $a_1 + a_3x^2 \leq 0$ with equality only at $x = \widetilde{x}_2$ and $x = \widetilde{x}_3$, and $y_2(x) < \widetilde{y}_2(x)$ and $L(y_2(x)) > L(\widetilde{y}_2(x))$ for $x \in [\widetilde{x}_3, \widetilde{x}_2]$.

Similarly we have $I_3 > \widetilde{I}_3$ on the orbit arcs $\widehat{q_3q_4}$ and $\widehat{\widetilde{p}_3\widetilde{p}_4}$, and $I_4 > \widetilde{I}_4$ on the orbit arcs $\widehat{p_4p_1}$ and $\widehat{\widetilde{p}_4\widetilde{p}_1}$. Summarizing the above proof we have $I = I_0 + I_1 + I_2 + I_3 + I_4 > \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 + \widetilde{I}_4 = \widetilde{I}$. This proves the claim, and consequently statement (b) of Theorem 2 for $n = 3$. □

The proof of statement (b) of Theorem 2 for $n = 4$ was given in [23]. This proof considers several cases and contains 20 pages, and since we cannot provide a new and shorter proof of this statement we do not prove it here.

5. Proof of statement (c) of Theorem 2

Here the proof mainly follows from that of [9] by De Maesschalck and Huzak, who proved the result using slow divergence integrals.

Consider the slow fast Liénard differential system

$$\dot{x} = y - F(x), \quad \dot{y} = -\varepsilon x, \tag{15}$$

with $F(x)$ polynomial and satisfying

$$F(0) = F'(0) = 0, \quad \frac{F'(x)}{x} > 0 \text{ for } x \in \mathbb{R}. \tag{16}$$

Under the assumption (16) the function $y = F(x)$ has the graph shown in Fig. 4. For each $x > 0$ there exists a unique $L(x) < 0$ such that $F(x) =$

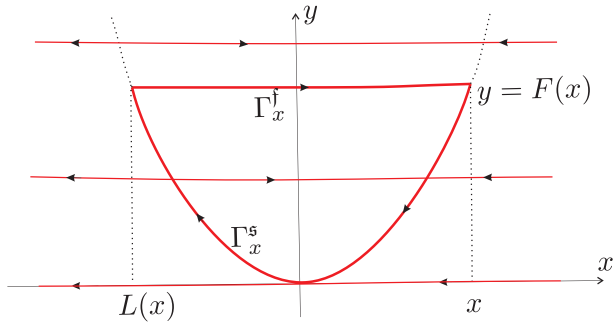

FIGURE 4. Slow fast cycle Γ_x.

$F(L(x))$. The piecewise smooth closed curve

$\Gamma_x = \Gamma_x^{\mathfrak{s}} \cup \Gamma_x^{\mathfrak{f}}, \quad \Gamma_x^{\mathfrak{s}} = \{(s, F(s)) : s \in [L(x), x]\}, \; \Gamma_x^{\mathfrak{f}} = \{(s, F(x)) : s \in (L(x), x)\},$

is called a *slow–fast cycle*, which is formed by the fast orbit $\Gamma_x^{\mathfrak{f}}$ of the layer equation

$$\dot{x} = y - F(x), \quad \dot{y} = 0,$$

and the slow orbit $\Gamma_x^{\mathfrak{s}}$ of the reduced equation

$$0 = y - F(x), \quad y' = -x \qquad \text{with } y' = \frac{dy}{d\tau} \text{ and } \tau = \varepsilon t.$$

Define the slow divergence integral associated to Γ_x

$$I(x) = \int_x^{L(x)} \frac{f(s)^2}{x} ds, \quad x \in (0, \infty), \tag{17}$$

where $f(x) = F'(x)$ with prime the derivative with respect to x.

The next result, due to De Maesschalck and Huzak [9, Theorem 2], characterizes the number of limit cycles of the classical Liénard differential system (15) via slow divergence integral.

Theorem 7. *Under the condition* (16)*, if the slow divergence integral* $I(x)$ *has exactly* k *simple zeros, then there exists a smooth function* $\lambda = \lambda(\varepsilon)$ *with* $\lambda(0) = 0$ *such that the perturbed system*

$$\dot{x} = y - F(x), \quad \dot{y} = \varepsilon(\lambda(\varepsilon) - x), \tag{18}$$

has exactly $k+1$ *limit cycles provided that* $\varepsilon > 0$ *sufficiently small, which are all hyperbolic.*

For computing the slow divergence integral $I(x)$, set

$$F(x) = F_e(x) + \delta F_o(x), \tag{19}$$

where F_e is even and F_o is odd, and δ is a small parameter. In [9] there obtained an asymptotic expression of I as follows.

Proposition 8. *The slow divergence integral $I(x)$ associated to the slow–fast cycle Γ_x with $F(x)$ of the form* (19) *has the asymptotic expression*

$$I(x) = 2\delta I_1(x) + O(\delta^2), \quad I_1(x) = \int_0^x \left(f_e'(s)F_o(s) - f_e(s)F_o'(s)\right) ds, \tag{20}$$

with $f_e(x) = F_e'(x)/x$.

Now we apply Theorem 7 and Proposition 8 to prove statement (c) of Theorem 2. The proof will be manipulated by induction.

Step 1: $n = 6$. Choose

$$F_e(x) = \int_0^x s f_e(s) ds, \quad f_e(x) = 1 + a_1 x^2 + a_2 x^4,$$
$$F_o(x) = b_1 x^3 + b_2 x^5,$$

with $(a_1, a_2) = (-3.1, 2.7)$ and $(b_1, b_2) = (-0.4, 1)$. Then

$$I_1(x) = 0.4x^3 - 1.248x^5 + 1.17429x^7 - 0.3x^9.$$

It has exactly 3 positive zeros $x_1 = 0.824803, x_2 = 0.898793, x_3 = 1.55761$. So for sufficiently small $\delta > 0$ $I(x)$ will also have exactly 3 positive zeros. It follows from Theorem 7 that the classical Liénard differential system of degree 6 could have at least 4 limit cycles.

Step 2: $n > 6$ *even*. For any integer $k \geq 3$, we write the Liénard differential system (15) of degree $2k$ in the form

$$F(x) = F_e^{(k)}(x) + \delta F_o^{(k)}(x),$$

with $F_o^{(k)}$ odd of degree $2k-1$ and $F_e^{(k)}$ even of degree $2k$ and $F_e^{(k)}(x) = \int_0^x s f_e^k(s) ds$, where $f_e^{(k)}$ is a polynomial of degree $2k-2$. Correspondingly we have

$$I_1(x) := I_1^{(k)}(x) = \int_0^x \left(f_e^{(k)\prime}(s) F_o^{(k)}(s) - f_e^{(k)}(s) F_o^{(k)\prime}(s)\right) ds.$$

For applying induction through perturbation and using Step 1, we assume that $I_1^{(k)}(x)$ has $2k-3$ simple zeros and $f_e^{(k)}(x) > 0$ for $x \in \mathbb{R}$, and so the classical Liénard differential system (15) of degree $n = 2k$ has at least $n-2$ hyperbolic limit cycles.

Set

$$F_e^{(k+1)}(x) = \int_0^x s f_e^{(k+1)}(s) ds, \quad f_e^{(k+1)}(x) = f_e^{(k)}(x) + 10 a_k x^{2k} \mu^2,$$
$$F_o^{(k+1)}(x) = F_o^{(k)}(x) + b_k x^{2k+1} \mu^2,$$

where a_k is the coefficient of x^{2k-2} in $f_e^{(k)}$ and b_k is the coefficient of x^{2k-1} in $f_o^{(k)}$. We have

$$I_1^{(k+1)}(x) = \int_0^x \left(f_e^{(k+1)\prime}(s) F_o^{(k+1)}(s) - f_e^{(k+1)}(s) F_o^{(k+1)\prime}(s)\right) ds,$$

which has $2k-3$ simple zeros when $\mu=0$ by the inductive assumption. Consequently $I_1^{(k+1)}(x)$ has $2k-3$ simple zeroes near the $2k-3$ simple zeroes of $I_1^{(k)}(x)$ for $\mu>0$ sufficiently small. In addition, $I_1^{(k+1)}(x)$ has other two simple zeros appearing in $O(1/\mu)$ range. Indeed, some calculations show that

$$\begin{aligned} I_1^{(k+1)}(x/\mu) &= \int_0^{x/\mu} \left(f_e^{(k+1)'}(s) F_o^{(k+1)}(s) - f_e^{(k+1)}(s) F_o^{(k+1)'}(s) \right) ds \\ &= \mu^{-4k+3} a_k b_k \left(J_1^{(k+1)}(x) + O(\mu^2) \right), \end{aligned}$$

where

$$J_1^{(k+1)}(x) = \int_0^x (A'_{k+1}(s) B_{k+1}(s) - A_{k+1}(s) B'_{k+1}(s)) ds,$$

with

$$A_{k+1}(x) = x^{2k-2} + 10x^{2k}, \quad B_{k+1}(x) = x^{2k-1} + x^{2k+1}.$$

It is easy to check that $J_1^{(k+1)}(x)$ has exactly 2 positive zeroes, which are simple. Consequently $J_1^{(k+1)}(x)+O(\mu^2)$ has two simple positive zeroes. This proves that $I_1^{(k+1)}(x)$ has two simple zeroes in the range $O(1/\mu)$, and so has $(2k-3)+2 = 2k-1$ positive simple zeroes. By Theorem 7 system (15) of degree $n = 2k+2$ with

$$F(x) = F_e^{(k+1)}(x) + \delta F_o^{(k+1)}(x),$$

has $2k$ hyperbolic limit cycles.

By induction we complete the proof of statement (c) of Theorem 2 for any even degree $n \geq 6$.

Step 3: *$n > 6$ odd.* Set $n = 2k+1$ with $k > 3$. By the proof of Step 2 there exists a polynomial Liénard differential system of degree $2k$ of the form

$$\dot{x} = y - F(x), \quad \dot{y} = \varepsilon_0(\lambda_0 - x), \tag{21}$$

which has $2k-2$ hyperbolic limit cycles. Since these limit cycles are nested and hyperbolic, the largest one should be either stable or unstable, which can be assumed without loss of generality to be unstable. We consider the perturbation of system (21)

$$\dot{x} = y - (F(x) + \rho x^{2k+1}), \quad \dot{y} = \varepsilon_0(\lambda_0 - x), \tag{22}$$

with $\rho \geq 0$ small. Note that the $2k-2$ hyperbolic limit cycles of system (22) when $\rho = 0$ persist for $\rho > 0$ sufficiently small. In addition, the infinity of system (22) is a repeller when $\rho > 0$, and system (22) has a unique finite singularity. By the Poincaré–Bendixson annulus theorem system (22) has an extra limit cycle beside the $2k-2$ limit cycles. Hence there exist classical Liénard differential systems (15) of degree n which have $n-2$ limit cycles. This proves statement (c) of Theorem 2 for any odd degree $n \geq 6$, and consequently statement (c) of Theorem 2. $\square$

Acknowledgements

The first author is partially supported by a MINECO grant MTM2013-40998-P, an AGAUR grant number 2014SGR-568, and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, and from the recruitment program of high–end foreign experts of China.

The second author is partially supported by NNSF of China grant numbers 11271252 and 11671254, by FP7-PEOPLE-2012-IRSES-316338 of Europe, and by innovation program of Shanghai municipal education commission grant 15ZZ012.

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