International Journal of Bifurcation and Chaos, Vol. 11, No. 8 (2001) 2299–2304
© World Scientific Publishing Company

ANALYTIC INTEGRABILITY OF NILPOTENT CUBIC SYSTEMS WITH DEGENERATE INFINITY

JAUME GINÉ*
Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69, 25001 Lleida, Spain

Received June 1, 2000; Revised January 10, 2001

In this paper we study the cubic nilpotent systems having an isolated singular point at the origin with degenerate infinity and give a characterization of those that accept a certain type of analytical first integrals. For these systems we give, also, a characterization of the singular point at the origin; in particular we see that the origin is never a center.

1. Introduction

One of the most important problems related with the plane differential systems

$$\begin{aligned} \dot{x} &= P(x, y)\,, \\ \dot{y} &= Q(x, y)\,, \end{aligned} \tag{1}$$

where P and Q are coprimes, is to determinate when a differential system has an analytic first integral defined in a neighborhood of an isolated singular point. In the case of analytic integrability in a neighborhood of an isolated singular point the other question is when this singular point is a center, i.e. when does a punctured neighborhood U exists such that every orbit in U exists is a cycle surrounding the singular point. This last problem is known as the *center problem.*

It is known that system (1) can have a center at the origin if and only if either it has a linear part of center type or it has a nilpotent linear part or it has a zero linear part.

Poincaré developed an important technique for the resolution of the center problem, see [Chavarriga, 1994], for systems which have linear part of the center type, i.e. systems of the form

$$\begin{aligned} \dot{x} &= -y + X(x, y)\,, \\ \dot{y} &= x + Y(x, y)\,, \end{aligned} \tag{2}$$

and consists of finding a formal power series of the form

$$H(x, y) = \sum_{n=2}^{\infty} H_n(x, y)\,, \tag{3}$$

where $H_2(x, y) = (x^2 + y^2)/2$, and $H_n(x, y)$ is an homogeneous polynomial of degree n, so that $\dot{H}(x, y) = \sum_{k=2}^{\infty} V_{2k}(x^2 + y^2)^k$, where V_{2k} are the *Lyapunov constants.* The formal power series (3) in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ is expressed as $H(r, \varphi) = (r^2/2) + \sum_{n=3}^{\infty} H_n(\varphi) r^n$, and $H_n(\varphi)$ is an homogeneous trigonometric polynomial of degree n, such that the relation $\dot{H}(r, \varphi) = \sum_{k=2}^{\infty} V_{2k} r^{2k}$ is verified. The Liapunov constants are polynomials whose variables are the coefficients of system (2). A necessary condition for the analytic integrability of (2) is the vanishing of all the Liapunov constants and in this case the origin is a center.

*The author is partially supported by a University of Lleida Project P98-207.
E-mail: gine@eup.udl.es