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ISOCHRONOUS FOCI FOR ANALYTIC DIFFERENTIAL SYSTEMS

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Consider the two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = \lambda x - y + P(x, y), \quad \dot{y} = x + \lambda y + Q(x, y),$$

where $P(x, y)$ and $Q(x, y)$ are analytic functions. The origin is a strong focus of this system if $\lambda \neq 0$ and is either a weak focus or its center if $\lambda = 0$. In this work we provide some sufficient conditions to have an isochronous focus at the origin.

Keywords: Nonlinear differential equations; isochronicity.

1. Introduction

One of the basic differences between linear and nonlinear autonomous systems is that if a nonlinear system has a family of periodic orbits, then the period of these orbits is, in general, not constant and it depends on the orbit. Contrarily, for the simple harmonic oscillator $\ddot{x} + x = 0$, that is the linear differential system $\dot{x} = -y$, $\dot{y} = x$, the whole plane xy is rotating around the origin like a wheel with the same angular velocity, implying that the points far from the origin have very large speeds; the speed tends to infinity as the distance from the origin tends to infinity. This is an instance of *isochronism*, characteristic of autonomous linear systems. Note that for the nonlinear differential equation of the mathematical pendulum without damping $\ddot{x} + \sin x = 0$ the period of the periodic solutions depends on the amplitude of the oscillation.

The problem of isochronism has already been treated by Galilei in the 16th century in connection with the study of the pendulum. The basic problem was to find the curve in a vertical plane with the following property: a weighing body (in the gravitational field) released from rest (with zero initial velocity) at any point of the curve and moving along the curve should always consume the same amount of time to reach a fixed horizon. The problem was solved by Huygens in 1673 who proved that the *cycloid* is the curve with this isochronous property. He applied this result in constructing clocks with cycloidal pendulums.

One of the most important problems in the construction of these clocks has always been the damping. We assume that the damping is proportional to the velocity $\dot{x}$; this is sometimes called *viscous damping*. Let us observe that in fact, the friction generates simple differential systems with isochronism and therefore, at first sight, good systems are to be used when constructing clocks. The simple harmonic oscillator with damping $\ddot{x} + \dot{x} + x = 0$, that is the linear differential system $\dot{x} = -y$, $\dot{y} = x - y$, is a normalizable system (see below Theorem 5) and therefore has constant angular velocity and isochronism. The following question arises: Why these systems are not used in the construction of clocks? The answer is easy, due to friction, the oscillation becomes undetectable. The harmonic oscillator with damping has a strong focus at the origin