Integrable polynomial differential systems and their perturbations

Adam Mahdi

In this work we are mainly interested in the *planar polynomial differential systems* or simply *polynomial systems* of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in x and y and the maximum degree of P and Q is m.

Certainly one of the most celebrated problems in the qualitative theory of dynamical systems on the plane is the Hilbert's 16^{th} problem concerning limit cycles. To be more precise the second part of this problem asks: *What is the maximum number of limit cycles of any planar polynomial differential system of degree at most m?* Recall that a *limit cycle* of system (1) is a closed orbit isolated in the set of all closed orbits of the system. The progress in this particular area of mathematics is very slow. Even the existence of a uniform upper bound of limit cycles in function of the degree is not solved. In other words the answer to the following question (see [6]) is not yet known: Is there a bound K on the number of limit cycles of system (1) of the form $K \leq m^q$ where m is the maximum of the degrees of P and Q and q is a universal constant? Until now mathematicians have not been able to answer it even for $m = 2$.

A classical way to obtain limit cycles is perturbing the periodic orbits of a center. Most of the methods are based on the Poincaré return map, the Poincaré-Melnikov integral and the Abelian integral. In fact in the plane the second and the third method are essentially equivalent. Many authors have studied the limit cycles which bifurcate from periodic orbits of a center [11, 17, 21]. For example, Chicone and Jacobs in [9] studied the linear isochronous center and four families of quadratic isochronous centers in the so called *Sibirsky form* and their perturbations inside the class of all quadratic polynomial differential systems. They developed techniques for treating the bifurcations of all orders, and applied them to proving the following result. For the linear isochronous center the maximum number of continuous families of limit cycles which can emerge is three, and for a class of nonlinear isochronous center, at most one continuous family of limit cycles can emerge, whereas for all other nonlinear isochronous centers, at most two continuous families of limit cycles can emerge. Moreover, they proved that for each isochronous center of these classes there are small perturbations such that the indicated maximum number of continuous families of limit cycles can be made to emerge from a corresponding number of arbitrarily prescribed periodic orbits within the period annulus of the isochronous center.

The perturbations of the quadratic isochronous systems inside the class of all polynomials were studied by Chengzhi Li, Weigu Li, J. Llibre and Zhifen Zhang in [13]. The technique that they used is a classical one. It consists in writing the non–Hamiltonian quadratic isochronous center in a Hamiltonian form, multiplying the non–Hamiltonian system by an integrating factor, and then to use the method based on computing the zeros of an Abelian integral to determine the limit cycles bifurcating from periodic orbits of the center. They use Green Theorem, to transform the Abelian integral to a double integral in order to simplify the calculations.

In [14] the same authors studied the perturbations of cubic polynomial differential systems having a rational first integral of degree 2 whose phase portraits correspond to the phase portraits $P1$, $P3$ and $P4$ of Figure 2.1. These systems were denoted in [14] by (B), (A) and (C), respectively. They proved that all the centers of these systems are reversible and isochronous, see [14, p.314]. One of their main results provides upper bounds for the maximum number of isolated zeros of the associated Abelian integral of systems (A), (B) and (C) when they are perturbed inside the class of all polynomial differential systems of degree three. To be more precise they proved that if we perturb the three cubic reversible isochronous systems (A), (B) and (C) inside the class of all polynomial differential systems of degree three an upper bound for the maximum number of zeros of the Abelian integrals associated with these systems is 5, 4 and 4, respectively.

In Chapter 2 we complete their result by giving lower bounds for the maximum number of zeros of the associated Abelian integrals. The following theorem is one of the main results of this chapter.

Theorem 2.2. *When we perturb the three cubic reversible isochronous systems (A), (B) and (C) inside the class of all polynomial differential systems of degree three, a lower bound for the maximum number of zeros of the Abelian integrals associated to these systems is* 4, 4 *and* 3, *respectively.*

The proof of Theorem 2.2 is based on the explicit calculation of the associated Abelian integrals and then studying their zeros by applying Lemma 2.13. In addition, we study the degenerate system S_4, which can be thought of as a limit case of system S_3. For the definitions of systems S_1–S_6 see page 18 of the thesis. It has a line of singular points which appears when four invariant straight lines of system S_3 coincide as shown in Chapter 3 (see config. 28 in Figure **??**). For systems S_4 and S_2 we also give a lower bound for the maximum number of isolated zeros of Abelian integral when they are perturbed inside the class of cubic polynomial differential systems.

Theorem 2.3. *When we perturb systems S_2 and S_4 inside the class of all polynomial systems of degree three, a lower bound for the maximum number of zeros of the Abelian integrals associated to these systems is* 4 *and* 3, *respectively.*

The results of Chapter 2 are contained in:

`J. Llibre and A. Mahdi`, *Lower bound for the number of zeros of Abelian integral for some cubic isochronous centers,* preprint 2008.

Another major problem in the qualitative theory of differential equations in dimension two is to give, for a concrete family of differential systems, a topological classification of the phase portraits for all the systems in the family. Roughly speaking, we say that two planar differential systems are *topologically equivalent* if there exists a homeomorphism of the plane which maps the solution curves of one system onto the solution curves of the other. Thus we wish to partition the parameter space corresponding to the family of systems under consideration into subsets having the property that the phase portraits of any two elements of the family corresponding to elements of the same subset are topologically equivalent.

In a series of remarkable works Markus [18], Neumann [19] and Peixoto [20] considered the topological classification problem for general differential systems on the plane. They identified certain key orbits called *separatrices* that divide the plane into connected components called *canonical regions.*

They introduced the notion of the separatrix configuration of a planar differential system (the union of all separatrices plus one representative orbit of each canonical region) and proved that two planar differential systems are topologically equivalent if and only if there exists a homeomorphism sending the separatrix configuration of one system onto the separatrix configuration of the other. Their results gave a good starting point for obtaining all topologically equivalent classes for particular families of planar differential systems.

The study of global phase portraits has been carried out for many years. One of the oldest papers on the subject was published in 1904 by W. Büchel, *Zur Topologie der durch eine gewhnliche Differentialgleichung erster Ordnung und ersten Grades definierten Kurvenschar. (On the topology of the curves defined by an ordinary differential equation of the first order and the first degree)*, Mitteil. der Math. Gesellsch. in Hamburg, Band IV, 4, 33-68. (25 figures). The author states the possible combinations in number and character of singular points in the finite part of the plane and at infinity. The list of mathematicians that contributed to the topological classification of quadratic systems is very long and includes: J.C. Artes, A. Gasull, J. Llibre, N. Vulpe, D. Schlomiuk, K.S. Sibirsky and many others. For a long list of publications on the qualitative theory of quadratic systems of differential equations in the plane see [22]. On the other hand relatively little was done regarding topological classification of cubic systems.

In Chapter 3 we give all the topologically distinct phase portraits of cubic systems having a rational first integral of degree two. For a planar differential system or a vector field defined on the plane $\mathbb{R}^2$ the existence of a first integral completely determines its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral, the following natural question arises: *Given a vector field on $\mathbb{R}^2$, how to recognize if this vector field has a first integral*? One of the easiest planar vector fields having a first integral are the Hamiltonian ones. The integrable planar vector fields which are not Hamiltonian are, in general, very difficult to detect. In [3] L. Cairó and J. Llibre classified all quadratic systems having a rational first integral of degree two. We will consider similar problem for cubic systems. To be more precise one of the main results of Chapter 3 is the following.

Theorem 3.1. *The phase portrait of a non–degenerate planar cubic poly-*

nomial differential system with a rational first integral of degree 2 *is topologically equivalent to one of the* 27 *phase portraits described in Figure 3.1.*

Furthermore we will show that a real cubic system having a rational first integral of degree two has either a finite number of invariant straight lines, real or complex, of total multiplicity 6, or it has infinitely many of them. We also study the configuration of the invariant straight lines that these kind of systems exhibit. If at least four invariant straight lines coincide the system becomes degenerate. The results of Chapter 3 form the following article:

`J. Llibre, A. Mahdi and N. Vulpe`, *Phase portraits and invariant straight lines of cubic polynomial vector fields having a quadratic rational first integral,* preprint.

In Chapter 4 we study the global dynamics of all planar polynomial differential systems having all their orbits imbedded in conics. To be more precise we say that system (1) has the orbit γ *imbedded in a conic* if there exists a polynomial of degree two $F(x,y) \in \mathbb{R}[x,y]$ such that $\gamma \subset \{F(x,y) = 0\}$. Although real conics are very simple curves and there are only nine different types of them up to an affine transformation, the differential polynomial systems having their orbits contained in conics give rise to a rich dynamics as it is shown in the following result.

Theorem 4.1. *The phase portrait of a real non–degenerate planar polynomial differential system having its orbits imbedded in conics is topologically equivalent to one of the* 49 *phase portraits given in Figure 4.1.*

All quadratic polynomial differential systems having a rational first integral of degree two were classified in [3] and [1]. These systems have their orbits contained in conics. The cubic polynomial systems of Lotka-Voltera type having a rational first integral of degree two were characterized in [2]. Finally all cubic differential system having a rational first integral of degree two were classified in [15]. All these results are particular cases of Theorem 4.1.

Our proof of Theorem 4.1 is based on the real affine classification of the pencils of conics. A *pencil of conics* is a 1-dimensional linear system of plane curves of degree two. Given two distinct conics $F = 0$ and $G = 0$ there is a unique pencil containing them, formed by all conics of the pencil $\lambda F + \mu G = 0$, $(\lambda : \mu) \in \mathbb{P}^1$, where $\mathbb{P}^n$ denotes the n–dimensional projective space. Thus a pencil of conics can be identified with a line in the space $\mathbb{P}^5$ which parameterizes all conics. There are two main types of pencils, *non–degenerate* and *degenerate*, according to whether they contain non–degenerate conics or not. The projective classification of the pencils of conics, both over the real or the complex numbers can be found in the literature on projective geometry, see for instance [12]. We shall be dealing with pencils of affine conics defined over the real numbers. For each affine real non–degenerate equisingularity type we shall give *normal forms*. Two projective pencils are real *equisingular* if there is a bijection between the sets of real singular elements (real base points, real singular points and real components of fibers) preserving the types.

The degree of a non–degenerate planar polynomial differential system having its orbits embedded in conics is at most three. This is a corollary of our following result.

Theorem 4.5. *Assume that system* (1) *is non–degenerate and all its orbits are contained in algebraic curves of degree d. Then the degree of system* (1) *is at most* $2d-1$.

The results of Chapter 4 are contained in:

`J. Llibre, A. Mahdi and J. Roé`, *The geometry of the real planar polynomial differential systems having their orbits imbedded in conics,* preprint.

Chapter 5 deals with the integrable polynomial vector fields $\mathcal{X} = (P, Q)$ defined either over $\mathbb{C}^2$ or $\mathbb{R}^2$. The main concern of this chapter is a relationship between the form of the first integral and its integrating factor, or inverse integrating factor. One of the main open problems in the qualitative theory of planar polynomial vector fields $\mathcal{X}$ is to characterize the integrable ones. One way to study integrable vector fields is through the inverse integrating factor V, for more details see [4]. Moreover, if $\mathcal{X}$ is real and $V : U \to \mathbb{R}$ is an inverse integrating factor of $\mathcal{X}$ on the open subset U of $\mathbb{R}^2$, then V becomes very important because $\{V = 0\}$ contains the limit cycles of $\mathcal{X}$ which are in U, see [10, 16]. Moreover, if V is polynomial, then it is defined on the whole $\mathbb{R}^2$ and consequently we can control all limit cycles of $\mathcal{X}$, see for instance [16].

There are several known relationships between the nature of the first integrals and its associated inverse integrating factors. If, for example, $\mathcal{X}$ has a Liouvillian first integral, then it has a Darboux inverse integrating factor. If $\mathcal{X}$ has a Darboux first integral, then it has a rational inverse integrating factor, and if $\mathcal{X}$ has a polynomial first integral, then it has a polynomial integrating factor. The proof of the following proposition will be given in Chapter 5 and states the missing relationship between the polynomial first integral and the polynomial inverse integrating factor for polynomial vector fields $\mathcal{X}$.

Proposition 5.2. *If $\mathcal{X}$ has a polynomial first integral, then it has a polynomial inverse integrating factor.*

Looking at the relationship between a first integral and the inverse integrating factor of polynomial vector field a natural question arises: *Suppose that the polynomial vector field $\mathcal{X}$ has a rational first integral. When does $\mathcal{X}$ have a polynomial inverse integrating factor?* Using the notion of critical remarkable values these vector fields were characterized by J. Chavarriga, H. Giacomini, J. Giné and J. Llibre in [5].

Let $H = f/g$ be a rational first integral of $\mathcal{X}$. we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable* value of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x, y]$. Note that for all $c \in \mathbb{C}$ the curve $f + cg = 0$ is an invariant algebraic curve. Here, if $c = \infty$, then $f + cg$ denotes g. In [5] it is proved that there are finitely many remarkable values for a given rational first integral H. Now suppose that $c \in \mathbb{C}$ is a remarkable value of a rational first integral H and that $u_1^{\alpha_1} \cdots u_r^{\alpha_r}$ is the factorization of the polynomial $f + cg$ into irreducible factors in $\mathbb{C}[x, y]$. If some of the α_i for $i = 1, \ldots, r$ is larger than 1, then we say (following again Poincaré) that c is a *critical remarkable value* of H, and that $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable invariant algebraic curve* of $\mathcal{X}$ with *exponent* α_i. They proved the following. Suppose that $H = f/g$ is a canonical rational first integral of $\mathcal{X}$ and it does not have polynomial first integrals. Canonicity means that f and g have the same degree and the degree of H is minimal among all the degrees of the rational first integral of the system. Then $\mathcal{X}$ has a polynomial inverse integrating factor if and only if H has at most two critical remarkable values. As far as we know this statement was not complete in the sense that there were no examples of polynomial vector fields satisfying its assumptions and without a polynomial inverse integrating factor. In what follows we provide such an example.

Proposition 5.4. *The polynomial vector field*

$$\mathcal{X}(x, y) = 2x(5 + 30x + 40x^2 + 8y^2)\frac{\partial}{\partial x} + y(5 + 44x + 80x^2 + 16y^2)\frac{\partial}{\partial y},$$

has a rational first integral, and has neither a polynomial first integral, nor a polynomial inverse integrating factor.

Most of the known examples of polynomial differential systems having an isochronous center admit a polynomial inverse integrating factor (see for example [7]). We can ask if this is always the case. In another words: Does a polynomial differential system with an isochronous center always admits a polynomial inverse integrating factor? We will show that the answer to this question is negative. Even *uniformly isochronous* systems that is, systems with an isochronous center whose trajectories rotate with constant angular velocity do not always admit a polynomial inverse integrating factor. The results of Chapter 5 have been published in the paper:

A. Ferragut, J. Llibre and A. Mahdi, *Polynomial inverse integrating factors for polynomial vector fields*, Discrete and Continuous Dynamical Systems **17** (2007), 387–395.

In Chapter 6 we present a result on the number of singular points of the radial projection of polynomial gradient vector fields of $\mathbb{R}^3$ on the sphere $\mathbb{S}^2$. To be more precise let $r = (x, y, z)$ and we consider the radial projection of the homogeneous polynomial gradient vector field

$$\nabla_r H(r) = (H_x, H_y, H_z)$$

of degree $m-1$ in $\mathbb{R}^3$ over the 2–dimensional sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$; i.e. we consider the polynomial vector field $\mathcal{X}$ of degree $m+1$ on $\mathbb{S}^2$ defined by

$$\mathcal{X}(r) = \nabla_r H(r) - \langle r, \nabla_r H(r) \rangle r \tag{2}$$

for $r \in \mathbb{S}^2$. Here $\langle r, s \rangle$ denotes the inner product of the two 3–dimensional vectors r and s. Since $\mathcal{X}(r)$ for every $r \in \mathbb{S}^2$ is a vector of the tangent plane to $\mathbb{S}^2$ at the point r, $\mathcal{X}$ is a vector field on $\mathbb{S}^2$. In [8, p.55] C. Chicone asked the following question. *What is the number of singular points of the polynomial vector field $\mathcal{X}$ on $\mathbb{S}^2$?* The next theorem provides the answer to Chicone's question.

Proposition 6.1. *Let H be a homogeneous polynomial of degree $m \geq 1$ in the variables (x, y, z). If the polynomial vector field $\mathcal{X}$ has not a continuous of singular points, then it has at most $(m-1)^2 + (m-1) + 1$ pairs of diame-*

trally opposite singular points on $\mathbb{S}^2$ taking into account their multiplicities.

This result is the first step toward the solution of the following problem. Let H denote the space of homogeneous cubic polynomials in three variables and let A denote the subset of H consisting of those elements whose gradients (orthogonally) project to a vector field on the unit sphere such that the corresponding dynamical system has no saddle connections. Prove or disprove that A is a dense subset of H with the coefficient topology.

The results of Chapter 6 have been published in the article:

`J. Llibre and A. Mahdi`, *On the number of singular points of the radial projection of polynomial gradient vector fields of* $\mathbb{R}^3$ *on the sphere* $\mathbb{S}^2$, Communications on Applied Nonlinear Analysis **14** (2007), 77–84.

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