Analytic non-integrability of the Suslov problem

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In this work, we consider the Suslov problem, which consists of a rotation motion of a rigid body, whose center of mass is located at one axis of inertia, around a fixed point O in a constant gravity field restricted to a nonholonomic constraint. The integrability and non-integrability has been established by a number of authors for the nongeneric values of $\mathbf{b} = (b_1, b_2, b_3)$ which is the unit vector along the line connecting the point O with the center of mass of the body. Here, we prove the analytic non-integrability for the remaining (generic) values of $\mathbf{b}$. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4763464]

I. INTRODUCTION

The Suslov problem is one of the most famous problems in nonholonomic dynamics with no shape space and was formulated in Ref. 6. It is a generalized rigid body with some of its body angular velocity components set equal to zero, i.e., it consists of a rotational motion of a rigid body around a fixed point O in a constant gravity field when restricted by a nonholonomic constraint

$$\langle \mathbf{n}, \omega \rangle = 0,$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the body angular velocity, $\mathbf{n}$ is a vector fixed in the body and $\langle \cdot, \cdot \rangle$ denotes the standard metric in $\mathbb{R}^3$. To be more precise, the equations of motion of the Suslov problem are

$$\mathbf{I}\,\dot{\omega} = \mathbf{I}\,\boldsymbol{\omega} \times \boldsymbol{\omega} + \varepsilon \boldsymbol{\gamma} \times \mathbf{b} + \lambda \mathbf{n}, \qquad \dot{\gamma} = \gamma \times \omega, \quad \langle \mathbf{n}, \omega \rangle = 0, \tag{1}$$

where λ is the Langrange multiplier; the diagonal matrix $\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$ represents the inertia of the body; $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vertical vector, and $\mathbf{b} = (b_1, b_2, b_3)$ is the unit vector along the line connecting the point O with the center of mass of the body; ε is the product of the mass of the body and the gravity constant. We assume that $\mathbf{n} = (0, 0, 1)$, i.e., we assume that the center of mass is located at the third axis of inertia, and thus the equation of the constraint is $\omega_3 = 0$. The Suslov equations can be written as

$$\begin{aligned} \dot{\omega}_1 &= \frac{\varepsilon}{I_1}(\gamma_2 b_3 - \gamma_3 b_2), \qquad \dot{\omega}_2 = \frac{\varepsilon}{I_2}(\gamma_3 b_1 - \gamma_1 b_3), \\ \dot{\gamma}_1 &= -\omega_2\gamma_3, \qquad \dot{\gamma}_2 = \omega_1\gamma_3, \qquad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2, \end{aligned} \tag{2}$$

where $I_1, I_2 > 0$. System (2) has two polynomial first integrals

$$\mathcal{F}_1 = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) + \varepsilon(b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3) \quad \text{and} \quad \mathcal{F}_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2. \tag{3}$$

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