Physica D 251 (2013) 60–62

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

On the nonintegrability of magnetic field lines

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ARTICLE INFO

Article history:
Received 3 September 2012
Received in revised form
28 January 2013
Accepted 3 February 2013
Available online 9 February 2013
Communicated by G. Stepan

Keywords:
Holomorphic integrability
Stefanescu conjecture
Magnetic field

ABSTRACT

We prove the existence of a magnetic field created by a planar configuration of piecewise rectilinear wires which is not holomorphically integrable when considered as a vector field in $\mathbb{C}^3$. This is a counterexample to the S. Stefanescu conjecture (1986) in the holomorphic setting. In particular the method of the proof gives an easy way of showing that the corresponding real vector field does not admit a real polynomial first integral which provides also an alternative way of contradicting the Stefanescu conjecture in the polynomial setting.

1. Introduction

Magnetic fields created by current flows appear in several branches of sciences such as electrical engineering [1], spectroscopy [2], medicine [3]. In order to define a magnetic field mathematically consider a smooth curve $L \subset \mathbb{R}^3$, parameterized by the map $l : I \ni \tau \to l(\tau) \in \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an interval, L represents the electric wire and J is the current intensity associated with it. Using the *Biot–Savart law* [4] we can compute the magnetic field $\mathbf{B}$ generated by a steady current associated with a *current distribution* (L, J) as follows

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 J}{4\pi} \int_I \frac{l'(\tau) \times (\mathbf{r} - l(\tau))}{|\mathbf{r} - l(\tau)|^3} \mathrm{d}\tau, \tag{1}$$

where μ_0 is a magnetic constant which is the value of the magnetic permeability in a classical vacuum, $l'(\tau) = \mathrm{d}l/\mathrm{d}\tau$, $|\cdot|$ represents the Euclidean norm in $\mathbb{R}^3$ and $\times$ represents the vector product. A magnetic field $\mathbf{B}$ created by a configuration $(L_1, J_1), \ldots, (L_n, J_1)$ is obtained via linear superposition, that is $\mathbf{B} = \mathbf{B}_1 + \cdots + \mathbf{B}_n$, where each $\mathbf{B}_i$ is obtained from the Biot–Savart law (1). Consequently the resulting vector field $\mathbf{B}$ is defined everywhere in $\mathbb{R}^3 \setminus (\bigcup_{i=0}^n L_i)$.

In this work we are concerned with the integrability of magnetic fields, i.e. the existence of a function which is constant on the magnetic lines. It is known that for certain configurations of wires with the constant current, the resulting magnetic field is integrable. For example the Biot–Savart magnetic field created by a straight line wire perpendicular to the $z = 0$ plane, has two independent polynomial first integrals: $F_1(x, y, z) = z$ and $F_2(x, y, z) = x^2 + y^2$. Also, a magnetic field created by two rectilinear wires, admits at least one polynomial first integral. Similar observations together with some computations motivated S. Stefanescu to state in 1986 the following conjecture ([5], see also [6,7]): there exists an *algebraic* first integral for any magnetic field originated by a configuration of piecewise rectilinear wires.

In other words the conjecture is asking to prove that there is always a *polynomial* or a *rational* first integral for a magnetic field generated by a rectilinear configuration of wires. Numerical simulations suggest that for the nonplanar configuration of rectilinear wires the resulting magnetic field can be very difficult, even chaotic [8,7]. Therefore one does not expect those cases to be integrable. On the other hand magnetic fields created by planar configuration of wires always possess two independent smooth first integrals in a sufficiently small tubular neighborhood of each current line, provided that the tubular neighborhood does not enclose any non-regular point [7]. This is only a *local* result, and does not say anything about the existence of the *global* first integral. Thus, the more precise statement of the Stefanescu conjecture could be:

The Stefanescu conjecture: there exists an *algebraic* first integral for any magnetic field originated by a *planar* configuration of piecewise rectilinear wires.

The conjecture was motivated by the belief that the magnetic field induced by rectilinear wires cannot be too complicated [8,9]. It became clear that even quite simple configurations of wires can

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0167-2789/$ – see front matter
doi:10.1016/j.physd.2013.02.002