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Short Note

A note on globally periodic maps and integrability

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A strong relation between globally periodic maps was presented in Ref. [1], where it is proved that an analytic and injective map $F:\mathcal{U}\subseteq\mathbb{K}^k\rightarrow\mathcal{U}$ is globally periodic if and only if it is *complete integrable*, that is, there exist k functionally independent analytic first integrals in $\mathcal{U}$. On the other hand, there is a large literature relating the integrability of maps and the properties of its singularities. In this note, we notice that 'singularities' (those points which belong to the forbidden set of the map) can play an essential role in order to construct a Darbouxian-type first integral of some discrete dynamical system (DDS from now on) [2,3]. This is the case of some rational globally periodic difference equations, for instance the ones given by

$$x_{n+2}=\frac{1+x_{n+1}}{x_n},\, x_{n+3}=\frac{1+x_{n+1}+x_{n+2}}{x_n}\quad\text{and}\quad x_{n+3}=\frac{-1+x_{n+1}-x_{n+2}}{x_n},\qquad (1)$$

Given a rational map $F:\mathcal{U}\subseteq\mathbb{K}^k\rightarrow\mathcal{U}$, with $F=(F_1,\ldots,F_k)$, the *singular set of* F is given by $\mathcal{S}(F)=\{\mathbf{x}\in\mathbb{K}^k$ such that $\mathrm{den}(F_i)=0$ for some $i\in\{1,\ldots,k\}\}$. The set $\Lambda(F)=\{\mathbf{x}\in\mathbb{K}^k$ such that there exists $n=n(\mathbf{x})\geq 1$ for which $F^n(\mathbf{x})\in\mathcal{S}(F)\}$, is called the *forbidden set* of F, and it is conformed by the set of the pre-images of the singular set. It is worth noting that the points in $\mathcal{S}(F)$ extend to regular ones when the map is extended to infinity in $\mathbb{K}P^k$ [6].

In this note, we use the functions describing the forbidden set of the maps associated to equations in (1) to generate closed sets of functions, hence Darboux-type first integrals.

Set $F:\mathcal{U}\subseteq\mathbb{K}^k\rightarrow\mathbb{K}^k$. Recall that a set of functions $\mathcal{R}=\{R_i\}_{i\in\{1,\ldots,m\}}$ is *closed under* F if for all $i\in\{1,\ldots,m\}$, there exist functions K_i and constants $\alpha_{i,j}$, such that $R_i(F)=K_i\left(\prod_{j=1}^m R_j^{\alpha_{i,j}}\right)$, with $\prod_{j=1}^m R_j^{\alpha_{i,j}}\neq 1$. Each function K_i is called the *cofactor* of R_i. Very briefly, the method works as follows: If there exist a closed set of functions for F, say $\mathcal{R}=\{R_i\}_{i\in\{1,\ldots,m\}}$, it can be tested if the function $I(\mathbf{x})=\prod_{i=1}^m R_i^{\beta_i}(\mathbf{x})$ gives a first integral for some values β_i, just imposing $I(F)=I$.

Proposition 1. *Consider the maps* $F_1(x,y)=(y,(1+y)/x)$, $F_2(x,y,z)=(y,z,(1+y+z)/x)$, *and* $F_3(x,y,z)=(y,z,(-1+y-z)/x)$ *associated to equations in* (1), *respectively. The following statements hold:*

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