The criticality of centers of potential systems at the outer boundary

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Abstract. The number of critical periodic orbits that bifurcate from the outer boundary of a potential center is studied. We call this number the criticality at the outer boundary. Our main results provide sufficient conditions in order to ensure that this number is exactly 0 and 1. We apply them to study the bifurcation diagram of the period function of $X = -y\partial_x + ((x+1)^p - (x+1)^q)\partial_y$ with $q < p$. This family was previously studied for $q = 1$ by Y. Miyamoto and K. Yagasaki.

1 Introduction and setting of the problem

In this paper we study planar differential systems

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \end{cases}$$

where f and g are analytic functions on some open subset U of $\mathbb{R}^2$. A singular point $p \in U$ of the vector field $X = f(x,y)\partial_x + g(x,y)\partial_y$ is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of the center and it will be denoted by $\mathscr{P}$. Henceforth $\partial\mathscr{P}$ will denote the boundary of $\mathscr{P}$ after embedding it into $\mathbb{RP}^2$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi := \partial\mathscr{P} \setminus \{p\}$. Note that Π is a non-empty compact subset of $\mathbb{RP}^2$. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parametrize this set. This can be done by taking an analytic transverse section to X on $\mathscr{P}$, for instance an orbit of the orthogonal vector field $X^{\perp}$. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is an analytic map that provides the qualitative properties of the period function that we are concerned about. In particular the existence of *critical periods*, which are isolated critical points of this function, i.e. $\hat{s} \in (0,1)$ such that $T'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can readily see that this definition does not depend on the particular parametrization of the set of periodic orbits used. Critical periodic orbits play in the study of the period function an equivalent role to limit cycles, which is a fundamental notion in qualitative theory of differential systems in the plane.

Suppose now that the vector field X depends on a parameter $\mu \in \Lambda$, where Λ is an open set of $\mathbb{R}^d$. Thus, for each $\mu \in \Lambda$, we have an analytic vector field X_μ, defined on some open subset U_μ of $\mathbb{R}^2$, with

2010 *Mathematics Subject Classification.* 34C07; 34C23; 37C25.
Key words and phrases: Center, period function, critical periodic orbit, bifurcation, criticality.
All the authors are partially supported by the MEC/FEDER grant MTM2014-52209-C2-1-P.
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a center at p_μ. Concerning the regularity with respect to the parameter, we shall assume that $\{X_\mu\}_{\mu\in\Lambda}$ is a continuous family of planar vector fields, meaning that the map $(x,y,\mu)\longmapsto X_\mu(x,y)$ is continuous on the subset $\{(x,y,\mu);\mu\in\Lambda \text{ and } (x,y)\in U_\mu\}$ of $\mathbb{R}^{d+2}$. Fix $\hat\mu\in\Lambda$ and, following the notation introduced previously, let $\Pi_{\hat\mu}$ be the outer boundary of the period annulus $\mathscr{P}_{\hat\mu}$ of the center at $p_{\hat\mu}$ of $X_{\hat\mu}$. The aim of the present paper is to provide tools in order to study the following bifurcation problem: which is the number of critical periodic orbits that can emerge or disappear from $\Pi_{\hat\mu}$ as we move slightly the parameter $\mu\approx\hat\mu$? We shall call this number the criticality of the outer boundary of the period annulus. In order to define it precisely we adapt the notion of cyclicity (cf. [2,26]), which is its counterpart in the study of limit cycles.

Definition 1.1. We define the *criticality* of the pair $(\Pi_{\hat\mu},X_{\hat\mu})$ with respect to the deformation X_μ to be $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big):=\inf_{\delta,\varepsilon}N(\delta,\varepsilon)$, where

$$N(\delta,\varepsilon)=\sup\{\text{number of critical periodic orbits } \gamma \text{ of } X_\mu\colon\, d_H(\gamma,\Pi_{\hat\mu})\leqslant\varepsilon \text{ and } \|\mu-\hat\mu\|\leqslant\delta\},$$

with d_H being the Hausdorff distance between compact sets of $\mathbb{RP}^2$. □

In other words, what we call the criticality $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)$ is the maximal number of critical periodic orbits that tend to $\Pi_{\hat\mu}$ in the Hausdorff topology of the non-empty compact subsets of $\mathbb{RP}^2$ as $\mu\to\hat\mu$.

Definition 1.2. We say that $\hat\mu\in\Lambda$ is a *local regular value of the period function at the outer boundary* of the period annulus if $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)=0$. Otherwise we say that it is a *local bifurcation value of the period function at the outer boundary.* □

At this point it is to be quoted some previous results on the period function closely related to the ones we are concerned about. The aim of the series of papers [17–19,22–24] is also to study the bifurcation of critical periodic orbits from the outer boundary in a family of centers. However there are some striking differences with our approach due to the fact that we deal with non-polynomial vector fields. Recall that a polynomial vector field X on $\mathbb{R}^2$ can be extended to a vector field $\hat X$ on the two-dimensional sphere $\mathbb{S}^2$ by means of the *Poincaré compactification*. The compactified vector field $\hat X$ is meromorphic on the equator of $\mathbb{S}^2$, which corresponds to the line at infinity in the original coordinates. Thus, even in case that the center has an unbounded period annulus, one can use this meromorphic extension $\hat X$ to study the bifurcation of critical periodic orbits from its outer boundary Π, which becomes a polycycle in $\mathbb{S}^2$. The polycycle consists of regular trajectories and singular points with a hyperbolic sector, which after the desingularization process give rise to saddles and saddle-nodes. It is here where the use of normal forms of such singular points permit to obtain an asymptotic development of the period function near Π. Computing the first non-vanishing coefficient in this development is the key tool in the mentioned series of papers in order to determine which parameters are local regular values of the period function at Π. On the contrary, the vector fields that we deal with in the present paper are not polynomial, but only analytic on some open subset U of $\mathbb{R}^2$. We compactify the set $\mathscr{P}$ in order to define its outer boundary Π in case that $\mathscr{P}$ is unbounded, but we can not compactify the vector field X itself. Furthermore, even in the case of a bounded period annulus, it may happen that the vector field X is not defined at all the points in Π. For this reason the approach that we follow must be completely different. It is also to be noted that once we have determined the local bifurcation values of the period function at the outer boundary, we aim to bound its criticality. This is also a novelty with respect to the quoted papers previously.

The notions that we have introduced so far are general. In the present paper we shall develop tools in order to study them in case that the differential system is potential, i.e.,

$$\begin{cases}\dot x=-y,\\ \dot y=V'(x).\end{cases}$$

The corresponding Hamiltonian function is given by $H(x,y)=\frac{1}{2}y^2+V(x)$, where we set $V(0)=0$. Suppose that the origin is a non-degenerated center (i.e., $V'(0)=0$ and $V''(0)>0$) and let (x_ℓ,x_r) be the projection of its period annulus $\mathscr{P}$ on the x-axis. Let us also fix that $H(\mathscr{P})=(0,h_0)$ with $h_0\in\mathbb{R}^+\cup\{+\infty\}$, in other words, that the energy level of the outer boundary Π is $H = h_0$. It turns out that the period $T(h)$ of the periodic orbit γ_h inside the energy level $h \in (0, h_0)$ is given by

$$T(h) = \int_{\gamma_h} \frac{dx}{y} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\big(\sqrt{h}\sin\theta\big)d\theta,$$

where the definite integral follows by using the polar coordinates that brings the oval $\gamma_h \subset \{\frac{1}{2}y^2 + V(x) = h\}$ to the circle of radius $\sqrt{h}$. Suppose now that the function V depends on a parameter $\mu \in \Lambda$, so that we deal with a family of differential systems given by $X_\mu = -y\partial_x + V_\mu'(x)\partial_y$. Then the bifurcation problem that we are interested in is to compute $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big)$ for some fixed $\hat{\mu} \in \Lambda$. To this end, following the obvious notation, we compute the derivative with respect to h of the above definite integral

$$T_\mu'(h) = \frac{1}{2\sqrt{h}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\mu'\big(\sqrt{h}\sin\theta\big)\sin\theta d\theta.$$

This leads us to an integral operator that depends on a parameter and our aim is to study its asymptotic behaviour as h tends to h_0. We tackle this problem of mathematical analysis in an abstract setting and Section 2 is devoted to obtain the theoretical results in this regard. These general results are then applied in Section 3 to the specific definite integral that gives the derivative of the period function for potential systems. For simplicity we only consider two situations: the case in which $h_0 = +\infty$ for all $\mu \approx \hat{\mu}$ and the case in which $h_0 < +\infty$ for all $\mu \approx \hat{\mu}$. Theorems A and B give, respectively, sufficient conditions for $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$ and $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$ in the first case, whereas Theorems C and D provide, respectively, sufficient conditions for $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$ and $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$ in the second case. These results are of course related with the finiteness problem of the number of critical periodic orbits in a given family of centers, and the reader is referred to the papers of Chicone and Dumortier [7] and Mardešić and Saavedra [21] in this regard. Concerning the applicability of our results, we note that, among others, Loud's centers and quadratic-like Hamiltonian centers can be brought to a potential system by means of a coordinate transformation (see [9, 12, 30]).

As an application of the previous results, in Section 4 we study the family of potential systems given by

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = (x+1)^p - (x+1)^q, \end{cases} \tag{1}$$

where p and q are *real* numbers. This differential system is analytic on $U = \{(x, y) \in \mathbb{R}^2 : x > -1\}$. Note in addition that the singular point at the origin is a hyperbolic saddle for $p < q$ and a non-degenerated center for $p > q$. The period function of this center in case that $q = 1$ was previously studied by Miyamoto and Yagasaki in [25, 31]. Following the notation just introduced, we define $\Lambda = \{(q, p) \in \mathbb{R}^2 : p > q\}$ and $X_\mu = -y\partial_x + ((x+1)^p - (x+1)^q)\partial_y$ with $\mu = (q, p)$. In order to state our result concerning this family of potential system let us denote

$$\Gamma_B := \{\mu \in \Lambda : q = 0\} \cup \{\mu \in \Lambda : p = 1, q \leqslant -1\} \cup \{\mu \in \Lambda : p + 2q + 1, q \geqslant -1\}$$

and

$$\Gamma_U := \{\mu \in \Lambda : (2q+1)(3q+1)(q+1)(p+1) = 0\}.$$

Here the subscripts B and U stand for bifurcation and unspecified, respectively. The curve Γ_B splits the parameter space Λ into three connected components, see Figure 1. We denote by I_B the union of the two grey components and by D_B the white component.

Theorem E. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of vector fields in (1) and consider the period function of the center at the origin. Then the open set $\Lambda \setminus (\Gamma_B \cup \Gamma_U)$ corresponds to local regular values of the period function at the outer boundary of the period annulus. In addition,*

(a) *If $\hat{\mu} \in I_B \setminus \Gamma_U$ then the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.*

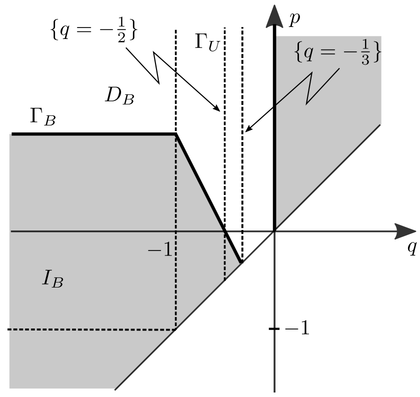

Figure 1: Bifurcation diagram of the period function of (1) at the outer boundary.

(b) If $\hat{\mu} \in D_B \setminus \Gamma_U$ then the period function of $X_{\hat{\mu}}$ is decreasing near the outer boundary.

Moreover the parameters in Γ_B are local bifurcation values of the period function at the outer boundary of the period annulus. Finally, $\mathrm{Crit}((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu) = 1$ *for all $\hat{\mu} = (\hat{q}, 1)$ with $\hat{q} < -3$ and $\hat{\mu} = (\hat{q}, -2\hat{q} - 1)$ with $\hat{q} \in (-\frac{5}{3}, -\frac{1}{3}) \setminus \{-\frac{1}{2}\}$.*

We have not determined the character of the parameters in Γ_U. We conjecture that they are not bifurcation values at the outer boundary. Besides the criticality of the outer boundary of $\mathscr{P}$, Theorem E provides information about the monotonicity of the period function there. The reason for this is because we can combine this information with the behaviour of the period function near the center in order to obtain a global conjecture. Indeed, one can easily show (see [13] for instance) that the first period constant of the center at the origin for (1) is given by

$$\Delta_1(q, p) = 2p^2 + 2q^2 + 7pq - p - q - 1.$$

The parameters outside the hyperbola $\{\Delta_1 = 0\}$ are local regular values of the period function at the *inner boundary* of $\mathscr{P}$ (i.e., the center). The hyperbola consists of local bifurcation values and in a forthcoming paper [15] we will prove that its criticality is exactly one. The sign of Δ_1 outside the hyperbola determines weather the period function is increasing or decreasing near the center. The combination of this information with the monotonicity near the outer boundary given by Theorem E lead us to formulate a conjecture for the global behaviour of the period function, see Figure 2. This conjecture claims in particular that there are no parameters for which two critical periodic orbits collapse disappearing in the "interior" of the period annulus.

Questions related to the behaviour of the period function have been extensively studied by a number of authors. Let us quote for instance the problems of isochronicity (see [9,14,20]), monotonicity (see [5,6,28]) or bifurcation of critical periodic orbits (see [8,10,27,29]). Most of the work on qualitative theory of differential systems in the plane, including the present paper, is related to the questions surrounding Hilbert's 16th problem (see [3,11,26,32] and references there in) and its various weakened versions.

2 Previous technical machinery

As we will see in Section 3, to study the criticality at the outer boundary in a family of potential systems $X_\mu = -y\partial_x + V_\mu'(x)\partial_y$, it is necessary to investigate the asymptotic behaviour of a certain family of functions defined by means of integrals depending on parameters. In case that the outer boundary Π_μ is reached with infinite energy this leads to study the behaviour at infinity of a family of functions $\{F_\mu\}$ defined by

$$F_\mu(s) = \int_{-\pi/2}^{\pi/2} f_\mu(s \sin\theta) d\theta,$$

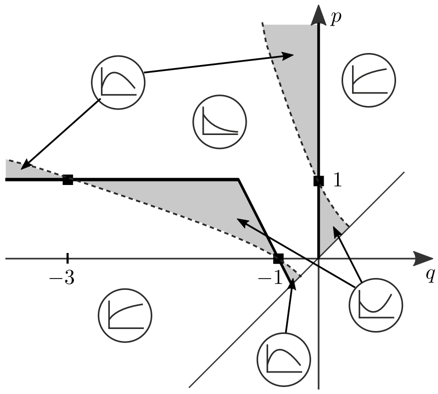

Figure 2: Conjectural bifurcation diagram for the period function of the differential system (1). The solid and dashed curves consist of local bifurcation values at the inner and outer boundary of $\mathscr{P}$, respectively. The parameters in the grey region correspond to systems with exactly one critical periodic orbit, and the three squares at $(-3,1)$, $(-1,0)$ and $(0,1)$ to the isochronous centers.

where f_μ is a function obtained from the potential V_μ. In this section we introduce some technical results that relate the asymptotic behavior of f_μ and F_μ, that plays an essential role to prove Theorems A and B.

2.1 Asymptotic study of an integral operator

Given a continuous function $f:[0,\infty)\longrightarrow\mathbb{R}$, in this section we consider

$$F(s):=\int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta, \tag{2}$$

which is a well defined function on $[0,+\infty)$. Our goal is to study under which conditions the asymptotic behaviour of f at infinity is transferred to F after integration. We begin by introducing precisely this notion in a slightly more general context.

Definition 2.1. Let f be a continuous function on $I=(a,b)$. We say that f is *quantifiable at b by α with limit ℓ* in case that:

(i) If $b\in\mathbb{R}$, then $\lim_{x\to b^-} f(x)(b-x)^\alpha=\ell$ and $\ell\neq 0$.

(ii) If $b=+\infty$, then $\lim_{x\to+\infty}\frac{f(x)}{x^\alpha}=\ell$ and $\ell\neq 0$.

We call α the *quantifier* of f at b. We shall use the analogous definition at a. □

The integral $\int_0^{\frac{\pi}{2}}\sin^\alpha\theta d\theta$ is convergent for all $\alpha>-1$. In what follows we shall denote its value by $\mathscr{B}(\alpha)$. It is well known, see for instance [1], that

$$\mathscr{B}(\alpha):=\int_0^{\frac{\pi}{2}}\sin^\alpha\theta d\theta=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}=\frac{1}{2}B\left(\frac{1}{2},\frac{\alpha+1}{2}\right), \tag{3}$$

where Γ and B are respectively the Gamma and Beta functions.

Proposition 2.2. *Let $f:[0,\infty)\longrightarrow\mathbb{R}$ be a continuous function which is quantifiable at $+\infty$ by $\alpha>-1$ with limit a. Then the function F defined in (2) is also quantifiable at $+\infty$ by α with limit $a\mathscr{B}(\alpha)$.*

Proof. Consider a given $\varepsilon > 0$. Since f is quantifiable at $+\infty$ by α with limit a, there exists $M > 0$ such that

$$\left|f(z)z^{-\alpha} - a\right| < \frac{\varepsilon}{2\mathscr{B}(\alpha)} \text{ for all } z > M. \tag{4}$$

Moreover, due to the continuity of f, there exists $K > 0$ such that $|f(z)| < K$ for all $z \in [0, M]$. Then

$$\frac{1}{s^\alpha}\int_0^{\arcsin(M/s)} |f(s\sin\theta)|\, d\theta \leqslant \frac{K}{s^\alpha}\arcsin(M/s).$$

On account of $\alpha > -1$, $\lim_{s\to\infty} \frac{K}{s^\alpha}\arcsin(M/s) = 0$. Hence we can take $s_1 > 0$ such that

$$\left|\frac{1}{s^\alpha}\int_0^{\arcsin(M/s)} f(s\sin\theta)d\theta\right| < \frac{\varepsilon}{4} \text{ for all } s > s_1.$$

Similarly $\lim_{s\to\infty}\int_0^{\arcsin(M/s)} a\,|\sin\theta|^\alpha\, d\theta = 0$, so there exists $s_2 > 0$ such that

$$\left|\int_0^{\arcsin(M/s)} a\sin^\alpha\theta d\theta\right| < \frac{\varepsilon}{4} \text{ for all } s > s_2.$$

Taking $s_3 = \max\{s_1, s_2\}$, from the two previous inequalities we obtain that

$$\left|\int_0^{\arcsin(M/s)}\left(\frac{f(s\sin\theta)}{s^\alpha} - a\sin^\alpha\theta\right)d\theta\right| \leqslant \left|\int_0^{\arcsin(M/s)}\frac{f(s\sin\theta)}{s^\alpha}d\theta\right| + \left|\int_0^{\arcsin(M/s)} a\sin^\alpha\theta d\theta\right| < \frac{\varepsilon}{2}$$

for all $s > s_3$. In addition, due to $s\sin\theta \in (M, s)$ for $\theta \in \big(\arcsin(M/s), \pi/2\big)$, from (4) we get

$$\left|\int_{\arcsin(M/s)}^{\frac{\pi}{2}}\left(\frac{f(s\sin\theta)}{(s\sin\theta)^\alpha} - a\right)\sin^\alpha\theta\right| < \frac{\varepsilon}{2\mathscr{B}(\alpha)}\int_{\arcsin(M/s)}^{\frac{\pi}{2}}\sin^\alpha\theta d\theta < \frac{\varepsilon}{2\mathscr{B}(\alpha)}\int_0^{\frac{\pi}{2}}\sin^\alpha\theta d\theta = \frac{\varepsilon}{2}$$

for all $s > M$. Finally, taking $s_4 = \max\{s_3, M\}$, the combination of the two previous inequalities gives

$$\begin{aligned}\left|\frac{F(s)}{s^\alpha} - a\mathscr{B}(\alpha)\right| &= \left|s^{-\alpha}\int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta - a\mathscr{B}(\alpha)\right| = \left|\int_0^{\frac{\pi}{2}}\left(\frac{f(s\sin\theta)}{s^\alpha} - a\sin^\alpha\theta\right)d\theta\right| \\ &\leqslant \left|\int_0^{\arcsin(M/s)}\left(\frac{f(s\sin\theta)}{s^\alpha} - a\sin^\alpha\theta\right)d\theta\right| + \left|\int_{\arcsin(M/s)}^{\frac{\pi}{2}}\left(\frac{f(s\sin\theta)}{(s\sin\theta)^\alpha} - a\right)\sin^\alpha\theta d\theta\right| < \varepsilon\end{aligned}$$

for all $s > s_4$. This proves the result. ∎

The previous result shows that if f is quantifiable at $+\infty$ by $\alpha > -1$ then F inherits this behaviour. Particularly, when $\alpha > 0$, both functions tend to infinity with the same order. We shall consider next the case $\alpha < -1$, so in particular when f tends to zero at infinity. To this end the following definition is needed:

Definition 2.3. Given a continuous function f on $[0, +\infty)$, setting $f_0 := f$, we define

$$f_n(z) := f_{n-1}(z)z^2 + z\int_0^z f_{n-1}(t)dt \text{ for all } n \geqslant 1.$$

Then, in case that f_{n-1} is quantifiable at $+\infty$ by $\alpha < -1$, we call

$$M_n := \int_0^\infty f_{n-1}(t)dt$$

the n-th *momentum* of f. □

In order that the n-th momentum is well-defined it is necessary that $M_j = 0$ for all $j \in \{1, 2, \ldots, n-1\}$. The following result provides a formula that relates the integrals of f and f_n.

Lemma 2.4. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Then for any $n \in \mathbb{N}$ we have that*

$$\int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta = \frac{1}{s^{2n}} \int_0^{\frac{\pi}{2}} f_n(s\sin\theta)d\theta \text{ for all } s > 0.$$

Proof. Let us fix $s > 0$ and note that if h is any continuous function on $[0, s]$, then the change of variable $u = s\sin\theta$ gives

$$\int_0^{\pi/2} h(s\sin\theta)d\theta = \int_0^s \frac{h(u)}{\sqrt{s^2-u^2}}du. \tag{5}$$

Set $g(z) := \frac{1}{z}\int_0^z f(t)dt$. Then, integrating by parts,

$$\int_0^s g(u)u^2 \frac{du}{\sqrt{s^2-u^2}} = \int_0^s \left(g'(u)u + g(u)\right)\sqrt{s^2-u^2}du = \int_0^s f(u)\sqrt{s^2-u^2}du. \tag{6}$$

Some easy manipulations show that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \big(f+g\big)(s\sin\theta)\sin^2\theta d\theta &= \frac{1}{s^2}\int_0^s \big(f+g\big)(u)\frac{u^2du}{\sqrt{s^2-u^2}} = \frac{1}{s^2}\int_0^s f(u)\left(\frac{u^2}{\sqrt{s^2-u^2}} + \sqrt{s^2-u^2}\right)du \\ &= \int_0^s \frac{f(u)du}{\sqrt{s^2-u^2}} = \int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta,\end{aligned}$$

where in the first and fourth equalities we use (5) with $h(z) = z^2\big(f+g\big)(z)$ and $h(z) = f(z)$, respectively, while in the second one we use (6). On account of Definition 2.3, this proves the result for $n = 1$. The general case follows recursively. ■

Next result shows that if f is quantifiable at $+\infty$ by $\alpha = -1$, then F is not quantifiable in the sense of Definition 2.1.

Proposition 2.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which is quantifiable at $+\infty$ by $\alpha = -1$ with limit a. Then the function F defined in (2) satisfies $\lim_{s\to+\infty} \frac{sF(s)}{\ln s} = a$.*

Proof. Consider a given $\varepsilon > 0$ and let $M > 0$ be such that $|zf(z) - a| < \varepsilon/6$ for all $z \geqslant M$. Since f is continuous, there exists $K > 0$ such that $|f(z)| \leqslant K$ for all $z \in [0, M]$. Therefore

$$\frac{s}{\ln s}\int_0^{\arcsin(M/s)} |f(s\sin\theta)|d\theta < K\frac{s}{\ln s}\arcsin(M/s) \text{ for all } s > M.$$

This shows that $\lim_{s\to\infty} \frac{s}{\ln s}\int_0^{\arcsin(M/s)} |f(s\sin\theta)|d\theta = 0$ and so there exists $s_0 > M$ satisfying that

$$\frac{s}{\ln s}\int_0^{\arcsin(M/s)} |f(s\sin\theta)|d\theta < \frac{\varepsilon}{3} \text{ for all } s > s_0.$$

On the other hand, since one can verify that $\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{1}{\sin\theta}d\theta = \ln\left(\frac{s+\sqrt{s^2-M^2}}{M}\right)$ for all $s \geqslant M$, we have that $\lim_{s\to\infty} \frac{1}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta} = 1$. Accordingly there exists $s_1 > s_0$ such that

$$\left|\frac{1}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}\right| < 2 \text{ and } \left|\frac{1}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta} - 1\right| < \frac{\varepsilon}{3|a|} \text{ for all } s > s_1.$$

Taking these inequalities into account we get that if $s > s_1$ then

$$\begin{aligned}\left|\frac{s}{\ln s}\int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta - a\right| &< \frac{s}{\ln s}\int_0^{\arcsin(M/s)} |f(s\sin\theta)|d\theta + \left|\frac{s}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} f(s\sin\theta)d\theta - a\right| \\ &< \frac{\varepsilon}{3} + \left|\frac{1}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{f(s\sin\theta)s\sin\theta - a + a}{\sin\theta}d\theta - a\right| \\ &< \frac{\varepsilon}{3} + \frac{1}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{|f(s\sin\theta)s\sin\theta - a|}{\sin\theta}d\theta + |a|\left|\frac{1}{\ln s}\int_{\arcsin(M/s)}^{\frac{\pi}{2}} \frac{1}{\sin\theta}d\theta - 1\right| \\ &< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

This completes the proof of the result. ■

According to the previous results the cases $\alpha = -1$ and $\alpha > -1$ are completely different with regard to the asymptotic behaviour of F at infinity. Following results clarifies that $\alpha = -1$ is a threshold in that respect because to analyse the case $\alpha < -1$ it is required to take the momenta of f into account. Before state the next results we need to introduce some notation. For $\alpha < -1$ let $n > 0$ such that $-2n-1 \leqslant \alpha < -2n+1$. Then for $j = 1, 2, \ldots, n$, we define $\alpha_j := \prod_{i=1}^{j} \frac{\alpha+2i}{\alpha+2i-1}$.

Lemma 2.6. *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function which is quantifiable at $+\infty$ by $\alpha < -1$ with limit a. Let $n \in \mathbb{N}$ be such that $-2n-1 \leqslant \alpha < -2n+1$. Then the following hold:*

(*a*) *If $M_1 = M_2 = \ldots = M_k = 0$ for some $k < n$, then f_j is quantifiable at $+\infty$ by $\alpha + 2j$ with limit $a\alpha_j$ for all $j = 1, 2, \ldots, k$.*

(*b*) *If $M_1 = M_2 = \ldots = M_n = 0$ and $\alpha \neq -2n$, then f_n is quantifiable at $+\infty$ by $\alpha + 2n$ with limit $a\alpha_n$.*

Proof. To show (a) assume that $M_1 = M_2 = \ldots = M_k = 0$ for some $k < n$. We will prove recursively that

$$\lim_{z\to+\infty} \frac{f_j(z)}{z^{\alpha+2j}} = a\alpha_j \text{ for all } j = 1, 2, \ldots, k.$$

We begin with the case $j = 1$. From Definition 2.3 we get

$$\frac{f_1(z)}{z^{\alpha+2}} = \frac{f(z)}{z^{\alpha}} + \frac{1}{z^{\alpha+1}}\int_0^z f(t)dt.$$

The assumption on f implies that $\lim_{z\to\infty} \frac{f(z)}{z^{\alpha}} = a$. Moreover, the hypothesis $M_1 = 0$ and $\alpha < -1$ imply that $\lim_{z\to\infty} \frac{1}{z^{\alpha+1}} \int_0^z f(t)dt$ is a $0/0$ indeterminate form. Thus, by applying Hôpital's Rule, this limit is equal to $\frac{a}{\alpha+1}$. Consequently $\lim_{z\to\infty} \frac{f_1(z)}{z^{\alpha+2}} = a\frac{\alpha+2}{\alpha+1} = a\alpha_1$, which is a real number different from zero because $\alpha \neq -2$. So the case $j = 1$ follows. Suppose now that the result holds for $j < k$ and let us show its validity for $j + 1$. We have

$$\frac{f_{j+1}(z)}{z^{\alpha+2(j+1)}} = \frac{f_j(z)z^2}{z^{\alpha+2(j+1)}} + \frac{z\int_0^z f_j(t)dt}{z^{\alpha+2(j+1)}}.$$

By induction hypothesis, $\lim_{z\to\infty} \dfrac{f_j(z)}{z^{\alpha+2j}} = a\alpha_j$. On the other hand, by assumption, $M_{j+1} = \int_0^{\infty} f_j(t)dt = 0$ and $\alpha + 2j + 1 < 0$, so the second function above is again a $0/0$ indeterminate form as z tends to $+\infty$. Then by applying Hôpital's Rule we get

$$\lim_{z\to\infty} \frac{\int_0^z f_j(t)dt}{z^{\alpha+2j+1}} = \lim_{z\to\infty} \frac{f_j(z)}{(\alpha+2j+1)z^{\alpha+2j}} = \frac{a\alpha_j}{\alpha+2j+1}.$$

Hence $\lim_{z\to\infty} \frac{f_{j+1}(z)}{z^{\alpha+2(j+1)}} = a\alpha_j \frac{\alpha+2(j+1)}{\alpha+2j+1} = a\alpha_{j+1}$, as desired, and this proves (a). To show (b), by using the same arguments we obtain that $\lim_{z\to\infty} \frac{f_n(z)}{z^{\alpha+2n}} = a\alpha_{n-1} \frac{\alpha+2n}{\alpha+2n-1} = a\alpha_n$, which is a number different from zero due to $\alpha \neq -2n$. This completes the proof of the result. ■

Proposition 2.7. *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function which is quantifiable at $+\infty$ by $\alpha < -1$ with limit a and F defined in (2). Let $n \in \mathbb{N}$ be such that $-2n-1 \leqslant \alpha < -2n+1$. Then the following holds:*

(a) *If $M_1 = M_2 = \cdots = M_{j-1} = 0$ and $M_j \neq 0$ for some $j \leqslant n$, then F is quantifiable at $+\infty$ by $1-2j$ with limit M_j.*

(b) *If $M_1 = M_2 = \cdots = M_n = 0$ and $\alpha \notin \{-2n, -2n-1\}$, then F is quantifiable at $+\infty$ by α with limit $a\alpha_n \mathscr{B}(\alpha + 2n)$.*

(c) *If $M_1 = M_2 = \cdots = M_n = 0$ and $\alpha = -2n-1$, then $\lim_{s\to\infty} \frac{s^{2n+1}}{\ln s} F(s) = a$ and in particular F is not quantifiable at $+\infty$.*

Proof. In order to prove the assertion in (a) let us note first that

$$\frac{f_j(z)}{z} = z f_{j-1}(z) + \int_0^z f_{j-1}(t)dt \longrightarrow M_j \neq 0 \text{ as } z \text{ tends to } +\infty$$

because, by Lemma 2.6, f_{j-1} is quantifiable by $\alpha + 2j - 2 < -1$. Then

$$s^{2j-1}F(s) = s^{2j-1} \int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta = \frac{1}{s}\int_0^{\frac{\pi}{2}} f_j(s\sin\theta)d\theta \longrightarrow M_j \neq 0 \text{ as } z \text{ tends to } +\infty,$$

where the second equality follows by Lemma 2.4 and the limit by applying Proposition 2.2 to f_j. To show (b) we note that, again by Lemma 2.4,

$$s^{2n}F(s) = s^{2n}\int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta = \int_0^{\frac{\pi}{2}} f_n(s\sin\theta)d\theta.$$

Due to $M_1 = \cdots = M_n = 0$ and $\alpha \notin \{-2n, -2n-1\}$, f_n is quantifiable at $+\infty$ by $\alpha + 2n > -1$ with limit $a\alpha_n$ thanks to Lemma 2.6. Thus, by Proposition 2.2, $\int_0^{\frac{\pi}{2}} f_n(s\sin\theta)d\theta$ is also quantifiable at $+\infty$ by $\alpha + 2n$. Accordingly, from the above equality we get that F is quantifiable at $+\infty$ by α, and so (b) follows. Finally let us show (c). By the previous reasoning, f_n is quantifiable at $+\infty$ by $\alpha + 2n = -1$ thanks to Lemma 2.6. Thus, by applying Proposition 2.5,

$$\lim_{s\to\infty} \frac{s}{\ln s}\int_0^{\frac{\pi}{2}} f_n(s\sin\theta)d\theta = a \neq 0,$$

and hence, using Lemma 2.4 once again $\lim_{s\to\infty} \frac{s^{2n+1}}{\ln s}F(s) = a$. This shows (c) and completes the proof. ∎

Remark 2.8. Notice that the previous result deal with all the possible values of α (even when $F(s)$ turns to be not quantifiable) except by the case when $M_1 = M_2 = \cdots = M_n = 0$ and $\alpha = -2n$. The authors want to remark that the hypothesis of f to be quantifiable by $\alpha = -2n$ in this case is not enough to stablish the quantifier of $F(s)$ at infinity. In fact, even it is not possible to say if it is quantifiable or not. For instance, let us consider the following three examples:

$$f(z) = \begin{cases} \frac{1}{z^2} & z \geqslant 1 \\ 4z - 3 & z \in [0,1) \end{cases}, \quad g(z) = \begin{cases} \frac{1}{z^2} + \frac{9}{10z^4} & z \geqslant 1 \\ \frac{32}{5}z - \frac{9}{2} & z \in [0,1) \end{cases}, \quad h(z) = \begin{cases} \frac{1}{z^2} + \frac{1}{z^3} & z \geqslant 1 \\ 7z - 5 & z \in [0,1) \end{cases}.$$

All these functions are quantifiable by $\alpha = -2$ and it is a computation to prove that the first momentum of the three functions vanish. Let us denote $F(s) = \int_0^{\frac{\pi}{2}} f(s\sin\theta)d\theta$, $G(s) = \int_0^{\frac{\pi}{2}} g(s\sin\theta)d\theta$ and $H(s) = \int_0^{\frac{\pi}{2}} h(s\sin\theta)d\theta$. It turns out that $F(s)$ and $H(s)$ are quantifiable by -3 and by -5 respectively, and that $G(s)$ is not quantifiable since

$$\lim_{s\to\infty} \frac{s^3}{\ln s}G(s) = \frac{1}{2}.$$

□

Next result provides a useful tool for the computation of momenta and motivates the terminology.

Lemma 2.9. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function which is quantifiable at $+\infty$ by $\alpha < -1$ with limit a. Let us take $n \geqslant 2$ satisfying $\alpha < -2n+1$ and assume that $M_1 = M_2 = \cdots = M_{n-1} = 0$. Then*

$$M_n = \prod_{k=1}^{n-1}\left(1-\frac{1}{2k}\right)\int_0^\infty t^{2(n-1)}f(t)dt.$$

Proof. By applying Lemma 2.6, the functions $f_{n-(k+1)}$ are quantifiable at $+\infty$ by $\alpha+2(n-k-1)$ because $f_0 = f$ is quantifiable at $+\infty$ by $\alpha < -2n+1$ and $M_1 = M_2 = \cdots = M_{n-1} = 0$. It is also clear that these functions are continuous at the origin. Then, for any $k \in \{1,2,\ldots,n-1\}$, integrating by parts we get

$$\begin{aligned}\int_0^\infty t^{2(k-1)}f_{n-k}(t)dt &= \int_0^\infty \left(t^{2k}f_{n-k-1}(t) + t^{2k-1}\int_0^t f_{n-k-1}(u)du\right)dt\\ &= \left(1-\frac{1}{2k}\right)\int_0^\infty t^{2k}f_{n-(k+1)}(t)dt + \left(\frac{t^{2k}}{2k}\int_0^t f_{n-(k+1)}(u)du\right)\Bigg|_{t=0}^{t=\infty}.\end{aligned}$$

Since $f_{n-(k+1)}$ is quantifiable at $+\infty$ by $\alpha+2(n-k-1)$ and $M_{n-k}=0$, by the Hôpital's Rule we obtain

$$\lim_{t\to\infty}\frac{t^{2k}}{2k}\int_0^t f_{n-(k+1)}(u)du = \lim_{t\to\infty} -\frac{f_{n-(k+1)}(t)}{4k^2t^{-2k-1}} = 0.$$

Therefore

$$\int_0^\infty t^{2(k-1)}f_{n-k}(t)dt = \left(1-\frac{1}{2k}\right)\int_0^\infty t^{2k}f_{n-(k+1)}(t)dt$$

and, using this equality iteratively,

$$M_n = \int_0^\infty f_{n-1}(t)dt = \frac{1}{2}\int_0^\infty t^2 f_{n-2}(t)dt = \ldots = \prod_{k=1}^{n-1}\left(1-\frac{1}{2k}\right)\int_0^\infty t^{2(n-1)}f_0(t)dt.$$

This proves the result. ■

2.2 Parametric results

In this section we generalise the previous results to a family of functions depending on parameters. First of all we extend the previous notion of quantifiable behaviour to this situation.

Definition 2.10. Let Λ be an open subset of $\mathbb{R}^d$ and suppose that, for each $\mu \in \Lambda$, f_μ is a continuous function on some real interval I_μ. We say that $\{f_\mu\}_{\mu\in\Lambda}$ is a *continuous family* of continuous functions on I_μ if the map $(x,\mu) \longmapsto f_\mu(x)$ is continuous on $\{(x,\mu) \in \mathbb{R}\times\Lambda : x \in I_\mu\}$. □

Definition 2.11. Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of continuous functions defined on an interval I_μ. Assume that $I_\mu = \big(a(\mu), b(\mu)\big)$ where either b (respectively, a) is a continuous function from Λ to $\mathbb{R}$ or $b(\mu) = +\infty$ (respectively, $a(\mu) = -\infty$) for all $\mu \in \Lambda$. Given $\hat{\mu} \in \Lambda$ we shall say that $\{f_\mu\}_{\mu\in\Lambda}$ is *continuously quantifiable* in $\hat{\mu}$ at $b(\mu)$ by $\alpha(\mu)$ with limit ℓ if there exists an open neighbourhood U of $\hat{\mu}$ such that f_μ is quantifiable at $b(\mu)$ by $\alpha(\mu)$ for all $\mu \in U$ and, moreover:

- (i) In case that $b(\hat{\mu}) < +\infty$, then $\lim_{(x,\mu)\to(b(\hat{\mu}),\hat{\mu})} f_\mu(x)\big(b(\mu)-x\big)^{\alpha(\mu)} = \ell$ and $\ell \neq 0$.
- (ii) In case that $b(\hat{\mu}) = +\infty$, then $\lim_{(x,\mu)\to(+\infty,\hat{\mu})} \frac{f_\mu(x)}{x^{\alpha(\mu)}} = \ell$ and $\ell \neq 0$.

We shall use the analogous definition for the left endpoint of I_μ. □

Remark 2.12. Notice that the map $\alpha : U \to \mathbb{R}$ that appears in the above definition must be continuous at $\hat{\mu}$. If not, then there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\alpha(\mu_n) = \alpha(\hat{\mu}) + \kappa$ with $\kappa \neq 0$. Then, for instance in case that $b(\hat{\mu}) = +\infty$, we will have

$$\ell = \lim_{(x,\mu_n)\to(+\infty,+\infty)} \frac{f_{\mu_n}(x)}{x^{\alpha(\mu_n)}} = \lim_{x\to+\infty}\left(\lim_{n\to+\infty}\frac{f_{\mu_n}(x)}{x^{\alpha(\mu_n)}}\right) = \lim_{x\to+\infty}\frac{f_{\hat{\mu}}(x)}{x^{\alpha(\hat{\mu})+\kappa}},$$

which, on account of $\ell \neq 0$, contradicts the fact that, by definition, $\lim_{x\to+\infty}\frac{f_{\hat{\mu}}(x)}{x^{\alpha(\hat{\mu})}}$ is finite and different from zero. □

From now on we shall assume that $\{f_\mu\}_{\mu\in\Lambda}$ is a continuous family of continuous functions on $[0,+\infty)$ which is continuously quantifiable at $+\infty$ by $\alpha : \Lambda \to \mathbb{R}$ at $\hat{\mu} \in \Lambda$ with limit $a(\hat{\mu})$. That is, for all μ in a neighbourhood of $\hat{\mu}$, f_μ is quantifiable by $\alpha(\mu)$ with limit $a(\mu)$ and

$$\lim_{(z,\mu)\to(+\infty,\hat{\mu})}\frac{f_\mu(z)}{z^{\alpha(\mu)}} = a(\hat{\mu}) \neq 0.$$

Let us denote

$$F_\mu(s) := \int_0^{\frac{\pi}{2}} f_\mu(s\sin\theta)d\theta. \tag{7}$$

In the same way as in the previous section, our aim is to investigate if the family $\{F_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable, assuming that $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable, and which is its quantifier. The purpose of this study is essentially the uniformity of the limit with respect to the parameter. The next result is the analogous to Proposition 2.2 for the parameter case and in its statement $\mathscr{B}$ is the function defined in (3).

Theorem 2.13. *Consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of continuous functions defined on $[0,+\infty)$. Suppose that it is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit a and that $\alpha(\hat{\mu}) > -1$. Then the family $\{F_\mu\}_{\mu\in\Lambda}$ defined in (7) is also continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit $a\mathscr{B}(\alpha(\hat{\mu}))$.*

Proof. On account of Remark 2.12 and the fact that $\alpha(\hat{\mu}) > -1$, there exists a compact neighbourhood K_1 of $\hat{\mu}$ such that $\alpha(\mu) > -1$ for all $\mu \in K_1$. Consequently $\int_0^{\frac{\pi}{2}}(\sin\theta)^{\alpha(\mu)}d\theta = \mathscr{B}\big(\alpha(\mu)\big)$ for all $\mu \in K_1$. Let us take $N := \max\{\mathscr{B}\big(\alpha(\mu)\big); \mu \in K_1\}$, which is well defined since $\mu \longmapsto \mathscr{B}\big(\alpha(\mu)\big)$ is continuous. Consider a given $\varepsilon > 0$. Since $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit a, there exists $M > 0$ and a compact neighbourhood $K_2 \subset K_1$ of $\hat{\mu}$ such that

$$\left|f_\mu(z)z^{-\alpha(\mu)} - a\right| < \frac{\varepsilon}{4N} \text{ for all } z > M \text{ and } \mu \in K_2. \tag{8}$$

We have on the other hand

$$\begin{aligned}\left|\int_0^{\frac{\pi}{2}}\left(\frac{f_\mu(s\sin\theta)}{s^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\hat{\mu})}\right)d\theta\right| &\leqslant \left|\int_0^{\frac{\pi}{2}}\left(\frac{f_\mu(s\sin\theta)}{s^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)}\right)d\theta\right| \\ &\quad + \left|\int_0^{\frac{\pi}{2}} a\left((\sin\theta)^{\alpha(\mu)} - (\sin\theta)^{\alpha(\hat{\mu})}\right)d\theta\right|.\end{aligned} \tag{9}$$

Since $\mu \longmapsto \mathscr{B}\big(\alpha(\mu)\big)$ is continuous, there exists a compact neighbourhood $K_3 \subset K_2$ of $\hat{\mu}$ such that

$$\left|\int_0^{\frac{\pi}{2}} a\left((\sin\theta)^{\alpha(\mu)} - (\sin\theta)^{\alpha(\hat{\mu})}\right)d\theta\right| = |a|\left|\mathscr{B}\big(\alpha(\mu)\big) - \mathscr{B}\big(\alpha(\hat{\mu})\big)\right| < \frac{\varepsilon}{2} \text{ for all } \mu \in K_3. \tag{10}$$

Let us take $R := \max\{|f_\mu(z)|\,; (z,\mu) \in [0,M] \times K_3\}$, $\hat{\alpha} := \min\{\alpha(\mu) : \mu \in K_3\}$ and any $s_1 > 1$. Then

$$\frac{1}{s^{\alpha(\mu)}}\int_0^{\arcsin(M/s)}|f_\mu(s\sin\theta)|\,d\theta \leqslant \frac{R}{s^{\alpha(\mu)}}\arcsin(M/s) \leqslant \frac{R}{s^{\hat{\alpha}}}\arcsin(M/s) \text{ for all } s > s_1 \text{ and } \mu \in K_3.$$

Due to $\hat{\alpha} > -1$, $\lim_{s\to\infty} \frac{K}{s^{\hat{\alpha}}} \arcsin(M/s) = 0$, so there exists $s_2 > \max\{s_1, M\}$ satisfying that

$$\frac{1}{s^{\alpha(\mu)}} \int_0^{\arcsin(M/s)} |f_\mu(s\sin\theta)|\, d\theta < \frac{\varepsilon}{8} \text{ for all } s > s_2 \text{ and } \mu \in K_3. \tag{11}$$

There exists in addition $s_3 > s_2$ such that

$$\left| a\int_0^{\arcsin(M/s)} (\sin\theta)^{\alpha(\mu)} d\theta \right| \leqslant \left| a\int_0^{\arcsin(M/s)} (\sin\theta)^{\hat{\alpha}} d\theta \right| < \frac{\varepsilon}{8} \text{ for all } s > s_3 \text{ and } \mu \in K_3, \tag{12}$$

where in the first inequality we use that $0 < \sin\theta < 1$, while in the second one we take $\hat{\alpha} > -1$ and $\lim_{s\to\infty} \arcsin(M/s) = 0$ into account. The triangular inequality combined with (11) and (12) yields to

$$\left| \int_0^{\arcsin(M/s)} \left(\frac{f_\mu(s\sin\theta)}{s^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)} \right) d\theta \right| < \frac{\varepsilon}{4} \text{ for all } \mu \in K_3 \text{ and } s > s_3. \tag{13}$$

Note on the other hand that $M < s\sin\theta < s$ for all $\theta \in \big(\arcsin(M/s), \pi/2\big)$. Thus from (8) we get

$$\left| \int_{\arcsin(M/s)}^{\frac{\pi}{2}} \left(\frac{f_\mu(s\sin\theta)}{(s\sin\theta)^{\alpha(\mu)}} - a \right) (\sin\theta)^{\alpha(\mu)} \right| \leqslant \frac{\varepsilon}{4N} \int_{\arcsin(M/s)}^{\frac{\pi}{2}} (\sin\theta)^{\alpha(\mu)} d\theta \leqslant \frac{\varepsilon}{4N} N = \frac{\varepsilon}{4} \tag{14}$$

for all $s > s_0$ and $\mu \in K_3$. The combination of (13) and (14) show that

$$\begin{aligned} \left| \int_0^{\frac{\pi}{2}} \left(\frac{f_\mu(s\sin\theta)}{s^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)} \right) d\theta \right| &\leqslant \left| \int_0^{\arcsin(M/s)} \left(\frac{f_\mu(s\sin\theta)}{s^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)} \right) d\theta \right| \\ &+ \left| \int_{\arcsin(M/s)}^{\frac{\pi}{2}} \left(\frac{f_\mu(s\sin\theta)}{(s\sin\theta)^{\alpha(\mu)}} - a \right) (\sin\theta)^{\alpha(\mu)} d\theta \right| < \frac{\varepsilon}{2} \end{aligned}$$

for all $s > s_0$ and $\mu \in K_3$. By using the above inequality together with (10), from (9) we get

$$\left| s^{-\alpha(\mu)} F(s;\mu) - a\mathscr{B}(\alpha(\hat{\mu})) \right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \text{ for all } s > s_3 \text{ and } \mu \in K_3.$$

This completes the proof of the result. ■

It is clear by Proposition 2.5 that we can not expect $\{F_\mu\}_{\mu\in\Lambda}$ to be continuously quantifiable when $\alpha(\hat{\mu}) = -1$ since $F(s;\hat{\mu})$ is not even quantifiable. So let us study next the case $\alpha(\hat{\mu}) < -1$. With this aim in view we shall first prove some previous results.

Lemma 2.14. *Let $a \in (0, +\infty]$, Λ be an open subset of $\mathbb{R}^d$ and $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of continuous functions defined on the interval $[0, a)$. The following statements hold:*

(*a*) *If $\lim_{x\to a} f_\mu(x) =: f_\mu(a)$ uniformly in μ, then for all $\hat{\mu} \in \Lambda$, $\lim_{(x,\mu)\to(a,\hat{\mu})} f_\mu(x) = f_{\hat{\mu}}(a)$.*

(*b*) *Reciprocally, if $\lim_{(x,\mu)\to(a,\hat{\mu})} f_\mu(x) =: f_{\hat{\mu}}(a)$ exists for all $\hat{\mu} \in \Lambda$, then $\lim_{x\to a} f_\mu(x) = f_\mu(a)$ uniformly on compact subsets of Λ.*

Proof. We prove the result in the case a is finite. (The case $a = +\infty$ follows with the obvious adaptations.) In order to prove (*a*) let us show first the continuity of the function $\mu \longmapsto f_\mu(a)$ at some fixed $\hat{\mu}$. Consider a given $\varepsilon > 0$. The uniformity of the limit $\lim_{x\to a} f_\mu(x) = f_\mu(a)$ implies that there exists $\delta > 0$ such that

$$|f_\mu(x) - f_\mu(a)| < \frac{\varepsilon}{3} \text{ for all } x \in (a-\delta, a) \text{ and } \mu \in \Lambda.$$

On the other hand, since $\mu \longmapsto f_\mu(x)$ is continuous, there exists a neighbourhood U of $\hat{\mu}$ such that

$$|f_\mu(x) - f_{\hat{\mu}}(x)| < \frac{\varepsilon}{3} \text{ for all } \mu \in U.$$

Therefore, on account of the two previous inequalities and taking an auxiliary $x \in (a, a-\delta)$,

$$|f_\mu(a) - f_{\hat\mu}(a)| \leqslant |f_\mu(a) - f_\mu(x)| + |f_\mu(x) - f_{\hat\mu}(x)| + |f_{\hat\mu}(x) - f_{\hat\mu}(a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $\mu \in U$, which proves the continuity of $\mu \longmapsto f_\mu(a)$ at $\hat\mu$. Let us show now that, under the uniformity assumption, $f_\mu(x)$ tends to $f_{\hat\mu}(a)$ as $(x,\mu) \longrightarrow (a,\hat\mu)$. Consider a given $\varepsilon > 0$. Then, since $\mu \longmapsto f_\mu(a)$ is continuous, there exists a neighbourhood U of $\hat\mu$ such that $|f_\mu(a) - f_{\hat\mu}(a)| < \frac{\varepsilon}{2}$ for all $\mu \in U$. Furthermore, thanks to the uniformity assumption, there exists $\delta > 0$ such that $|f_\mu(x) - f_\mu(a)| < \frac{\varepsilon}{2}$ for all $x \in (a-\delta, a)$ and $\mu \in U$. Consequently,

$$|f_\mu(x) - f_{\hat\mu}(a)| \leqslant |f_\mu(x) - f_\mu(a)| + |f_\mu(a) - f_{\hat\mu}(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } x \in (a-\delta, a) \text{ and } \mu \in U$$

and this proves (a). To show (b) let us consider a compact subset K of Λ. By hypothesis $(x,\mu) \longmapsto f_\mu(x)$ extends continuously to $[0,a] \times K$, which is also compact. So the map is uniformly continuous, which clearly implies that $\lim_{x\to a} f_\mu(x) = f_\mu(a)$ is uniform on K. This proves (b) and completes the proof of the result. ■

Following Definition 2.3, for each $\mu \in \Lambda$, we define $f_n(\,\cdot\,;\mu)$ and $M_n(\mu)$ setting $f_0(\,\cdot\,;\mu) := f_\mu$.

Lemma 2.15. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of continuous functions defined on $[0,\infty)$. Suppose that $\{f_{n-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat\mu$ at $+\infty$ by $\beta(\mu)$ and that $\beta(\hat\mu) < -1$. Then $M_n(\mu)$, the n-th momentum of f_μ, is well defined and continuous on some neighbourhood of $\hat\mu$ and, moreover,*

$$\lim_{(z,\mu)\to(+\infty,\hat\mu)} \int_0^z f_{n-1}(t;\mu)dt = M_n(\hat\mu).$$

Proof. We claim that $\lim_{z\to+\infty} \int_0^z f_{n-1}(t;\mu)dt$ converges uniformly to $M_n(\mu)$ in a neighbourhood of $\hat\mu$. Once we prove the claim then the result will follow by (a) in Lemma 2.14. Consider a given $\varepsilon > 0$. On account of Remark 2.12 we can take a compact neighbourhood K_1 of $\hat\mu$ such that $\beta(\mu) < -1$ for all $\mu \in K_1$. Let us denote $\hat\beta := \max\{\beta(\mu); \mu \in K_{\hat\mu}\}$, which is strictly smaller than -1. Since $\{f_{n-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat\mu$ at $+\infty$ by $\beta(\mu)$ with, let us say, limit a, there exist $\hat z > 0$ and a compact neighbourhood $K_2 \subset K_1$ of $\hat\mu$ such that $\left|\frac{f_{n-1}(z;\mu)}{z^{\beta(\mu)}} - a\right| < 1$ for all $z > \hat z$ and $\mu \in K_2$. On the other hand, since the integral $\int_0^\infty t^{\hat\beta}dt$ converges due to $\hat\beta < -1$, there exists $b > \hat z$ such that $\int_b^\infty t^{\hat\beta}dt < \frac{\varepsilon}{1+|a|}$. Therefore,

$$\begin{aligned}\left|\int_c^\infty f_{n-1}(t;\mu)dt\right| &\leqslant \int_c^\infty \left|\frac{f_{n-1}(t;\mu)}{t^{\beta(\mu)}} - a\right| t^{\beta(\mu)}dt + |a|\int_c^\infty t^{\beta(\mu)}dt < (1+|a|)\int_c^\infty t^{\beta(\mu)}dt \\ &< (1+|a|)\int_c^\infty t^{\hat\beta}dt < \varepsilon\end{aligned}$$

for all $c \in (b,\infty)$ and $\mu \in K_2$. This proves the claim and so the result follows. ■

Proposition 2.16. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of continuous functions defined on $[0,\infty)$. Suppose that the family is continuously quantifiable in Λ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)$. Assume also that for some $\hat\mu \in \Lambda$ $\alpha(\hat\mu) < -1$ and take $n \in \mathbb{N}$ such that $-2n-1 \leqslant \alpha(\hat\mu) < -2n+1$. Then, setting $\alpha_j(\mu) := \prod_{i=1}^{j} \frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$ for $j = 1,2,\ldots,n$, the following assertions hold:*

(a) If, for some $k < n$, $M_1(\mu) = M_2(\mu) = \cdots = M_k(\mu) = 0$ for all $\mu \in \Lambda$, then for all $j = 1,2,\ldots,k$, $\{f_j(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat\mu$ at $+\infty$ by $\alpha(\mu)+2j$ with limit $a(\mu)\alpha_j(\mu)$.

(b) If $M_1(\mu) = M_2(\mu) = \cdots = M_n(\mu) = 0$ for all $\mu \in \Lambda$ and $\alpha(\hat\mu) \notin \{-2n, -2n-1\}$, then $\{f_n(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat\mu$ at $+\infty$ by $\alpha(\mu)+2n$ with limit $a(\mu)\alpha_n(\mu)$.

Proof. To show the assertion in (a) assume that, for some $k < n$, $M_1(\mu) = M_2(\mu) = \cdots = M_k(\mu) = 0$ for all $\mu \in \Lambda$. We will prove recursively that there exists a neighbourhood U_j of $\hat{\mu}$ such that

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = a(\bar{\mu})\alpha_j(\bar{\mu}) \text{ for all } \bar{\mu} \in U_j.$$

For $j = 0$ this follows by assumption taking $U_0 = \Lambda$. For the inductive step suppose that it is true for $j-1$. By applying Lemma 2.6 for each fixed $\mu \in U_{j-1}$ we have

$$\lim_{z\to+\infty} \frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = a(\mu)\alpha_j(\mu).$$

Thus, for each fixed $\mu \in U_{j-1}$, the function $f_j(z;\mu)$ is quantifiable at $+\infty$. Let us show that is, indeed, continuously quantifiable. With this aim in view we note that

$$\frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = \frac{f_{j-1}(z;\mu)}{z^{\alpha(\mu)+2(j-1)}} + \frac{\int_0^z f_{j-1}(t;\mu)dt}{z^{\alpha(\mu)+2j-1}}. \tag{15}$$

By the induction hypothesis, $\{f_{j-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in U_{j-1} at $+\infty$ by $\alpha(\mu)+2(j-1)$ with limit $a(\mu)\alpha_{j-1}(\mu)$. Therefore

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_{j-1}(z;\mu)}{z^{\alpha(\mu)+2(j-1)}} = a(\bar{\mu})\alpha_{j-1}(\bar{\mu}) \text{ for all } \bar{\mu} \in U_{j-1}. \tag{16}$$

To obtain the limit of the second summand in (15) we use the uniform Hôpital's Rule in Proposition 4.6. With this aim in view note that the functions $\int_0^z f_{j-1}(t;\mu)dt$ and $z^{\alpha(\mu)+2j-1}$ are differentiable on $(0,\infty)$ for each $\mu \in U_{j-1}$. Moreover, from (16), the limit of the quotient of derivatives is

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_{j-1}(z;\mu)}{(\alpha(\mu)+2j-1)z^{\alpha(\mu)+2j-2}} = \frac{a(\bar{\mu})\alpha_{j-1}(\bar{\mu})}{\alpha(\bar{\mu})+2j-1} \text{ for all } \bar{\mu} \in U_{j-1}$$

and so, by applying Lemma 2.14, there exists a compact neighbourhood K of $\hat{\mu}$ such that

$$\lim_{z\to+\infty} \frac{f_{j-1}(z;\mu)}{(\alpha(\mu)+2j-1)z^{\alpha(\mu)+2j-2}} = \frac{a(\mu)\alpha_{j-1}(\mu)}{\alpha(\mu)+2j-1} \text{ uniformly on } K.$$

Therefore it only remains to check condition (e) in Proposition 4.6, i.e., that there exists $c \in (0,\infty)$ such that, for each $x \in (c,\infty)$,

$$\lim_{z\to+\infty} \frac{z^{\alpha(\mu)+2j-1}}{x^{\alpha(\mu)+2j-1}} = 0 \text{ and } \lim_{z\to+\infty} \frac{\int_0^z f_{j-1}(t;\mu)dt}{x^{\alpha(\mu)+2j-1}} = 0 \text{ uniformly on } \mu.$$

In order to verify this let us take a neighbourhood U_j of $\hat{\mu}$ such that $\hat{\alpha} := \max\{\alpha(\mu)+2j-1 : \mu \in U_j\}$ is strictly smaller than -1. Then, taking $x > 1$,

$$\frac{z^{\alpha(\mu)+2j-1}}{x^{\alpha(\mu)+2j-1}} = \left(\frac{z}{x}\right)^{\alpha(\mu)+2j-1} < z^{\alpha(\mu)+2j-1} < z^{\hat{\alpha}} \longrightarrow 0 \text{ as } z \text{ tends to } +\infty,$$

and so the first limit tends to zero uniformly on U_j. We claim that the second limit is also uniform in a neighbourhood of $\hat{\mu}$. To show this we note that, by Lemma 2.15,

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{\int_0^z f_{j-1}(t;\mu)dt}{x^{\alpha(\mu)+2j-1}} = \frac{M_j(\bar{\mu})}{x^{\alpha(\bar{\mu})+2j-1}} = 0 \text{ for all } \bar{\mu} \in U_j$$

and then the claim follows by Lemma 2.14. Taking U_j to be the intersection of the previous neighbourhoods we can thus apply Proposition 4.6 and assert that

$$\lim_{z\to+\infty} \frac{\int_0^z f_{j-1}(t;\mu)dt}{z^{\alpha(\mu)+2j-1}} = \frac{a(\mu)\alpha_{j-1}(\mu)}{\alpha(\mu)+2j-1} \text{ uniformly on } U_j.$$

Consequently, by applying Lemma 2.14 once again,

$$\lim_{(z,\mu)\to(+\infty,\bar\mu)} \frac{\int_0^z f_{j-1}(t;\mu)dt}{z^{\alpha(\mu)+2j-1}} = \frac{a(\bar\mu)\alpha_{j-1}(\bar\mu)}{\alpha(\bar\mu)+2j-1} \text{ for all } \bar\mu \in U_j.$$

Then, from (15), the above limit together with (16) show that

$$\lim_{(z,\mu)\to(+\infty,\bar\mu)} \frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = a(\bar\mu)\alpha_{j-1}(\bar\mu) + \frac{a(\bar\mu)\alpha_{j-1}(\bar\mu)}{\alpha(\bar\mu)+2j-1} = a(\bar\mu)\alpha_j(\bar\mu) \neq 0.$$

Therefore $f_j(z;\mu)$ is continuously quantifiable in U_j at $+\infty$ by $\alpha(\mu)+2j$ with limit $a(\mu)\alpha_j(\mu)$. This shows the inductive step and so (a) follows. The proof of (b) follows exactly the same way taking into account that $\alpha_n(\mu)$ is well defined and non-vanishing due to $\alpha(\mu) \notin \{-2n, -2n-1\}$ in a neighbourhood of $\hat\mu$. ■

Now we are in conditions to prove the second main result of this section. In its statement recall that $\mathscr{B}$ is the function defined in (3).

Theorem 2.17. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of continuous functions defined on $[0,\infty)$. Suppose that the family is continuously quantifiable in Λ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)$ and let $\{F_\mu\}_{\mu\in\Lambda}$ defined in (7). Assume also that for some $\hat\mu \in \Lambda$ $\alpha(\hat\mu) < -1$ and take $n \in \mathbb{N}$ such that $-2n-1 \leqslant \alpha(\hat\mu) < -2n+1$. Then, setting $\alpha_j(\mu) := \prod_{i=1}^{j} \frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$ for $j = 1, 2, \ldots, n$, the following assertions hold:*

- (a) *If, for some $1 \leqslant j \leqslant n$, $M_1(\mu) = M_2(\mu) = \ldots = M_{j-1}(\mu) = 0$ for all $\mu \in \Lambda$ and $M_j(\hat\mu) \neq 0$, then $\{F_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat\mu$ at $+\infty$ by $1-2j$ with limit $M_j(\mu)$.*
- (b) *If $M_1(\mu) = M_2(\mu) = \cdots = M_n(\mu) = 0$ for all $\mu \in \Lambda$ and $\alpha(\hat\mu) \notin \{-2n-1, -2n\}$, then $\{F_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat\mu$ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)\alpha_n(\mu)\mathscr{B}(\alpha(\mu)+2n)$.*

Proof. Let us show (a) first. By applying Proposition 2.16 there exists a neighbourhood $\hat U$ of $\hat\mu$ such that $\{f_{j-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat U$ at $+\infty$ by $\alpha(\mu)+2(j-1)$ with limit $a(\mu)\alpha_{j-1}(\mu)$. Then

$$\lim_{(z,\mu)\to(+\infty,\bar\mu)} f_{j-1}(z;\mu)z = \lim_{(z,\mu)\to(+\infty,\bar\mu)} a(\mu)\alpha_{j-1}(\mu)z^{\alpha(\mu)+2j-1} = 0 \text{ for any } \bar\mu \in \hat U$$

due to $j \leqslant n$ and $\alpha(\mu)+2n < 1$. Consequently, since $f_j(z;\mu) = f_{j-1}(z;\mu)z^2 + z\int_0^z f_{j-1}(t;\mu)dt$, by using Lemma 2.15 we get

$$\lim_{(z,\mu)\to(+\infty,\bar\mu)} \frac{f_j(z;\mu)}{z} = \lim_{(z,\mu)\to(+\infty,\bar\mu)} \int_0^z f_{j-1}(t;\mu)dt = M_j(\bar\mu).$$

Accordingly, the family $\{f_j(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat U$ at $+\infty$ by 1 with limit $M_j(\mu)$. Hence, by Lemma 2.4 and Theorem 2.13, $\{F_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat U$ at $+\infty$ by $1-2j$ with limit $M_j(\mu)$. This proves the validity of (a). Let us turn now to the proof of (b). In this case, by Proposition 2.16, $\{f_n(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in a neighbourhood of $\hat\mu$ at $+\infty$ by $\alpha(\mu)+2n$ with limit $a(\mu)\alpha_n(\mu)$. Since $\alpha(\mu)+2n > -1$, by Lemma 2.4 and Theorem 2.13 it follows that $\{F_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat\mu$ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)\alpha_n(\mu)\mathscr{B}(\alpha(\mu)+2n)$. So the result is proved. ■

3 Criticality at the outer boundary of potential centers

This section is devoted to prove the main theoretical results about criticality at the outer boundary. We consider analytic potential differential systems

$$\begin{cases} \dot x = -y, \\ \dot y = V_\mu'(x), \end{cases}$$

depending on a parameter $\mu \in \Lambda$, where Λ is an open subset of $\mathbb{R}^d$. Here V_μ is an analytic function on a certain real interval I_μ that contains $x = 0$. In what follows sometimes we shall use the vector field notation $X_\mu := -y\partial_x + V_\mu'(x)\partial_y$ to refer to the above differential system. We suppose $V_\mu'(0) = 0$ and $V_\mu''(0) > 0$, so that the origin is a non-degenerated center and we shall denote the projection of its period annulus $\mathscr{P}_\mu$ on the x-axis by $\mathcal{I}_\mu = (x_\ell(\mu), x_r(\mu))$. Thus $x_\ell(\mu) < 0 < x_r(\mu)$. The corresponding Hamiltonian function is given by $H_\mu(x, y) = \frac{1}{2}y^2 + V_\mu(x)$, where we fix that $V_\mu(0) = 0$, and we set the energy level of the outer boundary of $\mathscr{P}_\mu$ to be $h_0(\mu)$, so that $V_\mu(\mathcal{I}_\mu) = [0, h_0(\mu))$. Note that $h_0(\mu)$ is a positive number or $+\infty$. In addition we define

$$g_\mu(x) := x\sqrt{\frac{V_\mu(x)}{x^2}},$$

which is clearly a diffeomorphism on $\mathcal{I}_\mu$ since $V_\mu(0) = V_\mu'(0) = 0$ and $V_\mu''(0) > 0$. It is well-known (see [16] for instance) that the period $T_\mu(h)$ of the periodic orbit γ_h inside the energy level $H_\mu = h$ is given by

$$T_\mu(h) = \int_{\gamma_h} \frac{dx}{y} = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'(\sqrt{h}\sin\theta)d\theta, \tag{17}$$

where the definite integral follows by using the polar coordinates that brings the oval $\gamma_h \subset \{\frac{1}{2}y^2 + V_\mu(x) = h\}$ to the circle of radius $\sqrt{h}$. (Here the dependence of γ_h on μ is omitted for shortness.) It is well known that, for each $\mu \in \Lambda$, the function T_μ is an analytic on $(0, h_0(\mu))$ and that can be extended analytically at $h = 0$.

Concerning the dependence of X_μ with respect to the parameter μ, from now on we shall say that the family of potential systems $\{X_\mu\}_{\mu\in\Lambda}$ verifies the hypothesis **(H)** in case that the following holds:

- (H_1) $(x, \mu) \longmapsto V_\mu'''(x)$ is continuous on $\{(x, \mu) \in \mathbb{R} \times \Lambda : x \in I_\mu\}$,
- (H_2) $\mu \longmapsto V_\mu''(0)$ is continuous on Λ,
- (H_3) $\mu \longmapsto x_r(\mu)$ is continuous on Λ or $x_r(\mu) = +\infty$ for all $\mu \in \Lambda$,
- (H_4) $\mu \longmapsto x_\ell(\mu)$ is continuous on Λ or $x_\ell(\mu) = -\infty$ for all $\mu \in \Lambda$,
- (H_5) $\mu \longmapsto h_0(\mu)$ is continuous on Λ or $h_0(\mu) = +\infty$ for all $\mu \in \Lambda$.

Clearly (H_1) and (H_2) imply that $(x, \mu) \longmapsto V_\mu^{(i)}(x)$ is continuous on $\{(x, \mu) \in \mathbb{R} \times \Lambda : x \in I_\mu\}$ for $i = 0, 1, 2$. Indeed, for instance for $i = 2$ this follows from noting that $V_\mu''(x) = \int_0^x V_\mu'''(s)ds - V_\mu''(0)$.

Lemma 3.1. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems satisfying* **(H)**. *Then $(z, \mu) \longmapsto g_\mu^{-1}(z)$ is a continuous map on the open set $\Big\{(z, \mu) \in \mathbb{R} \times \Lambda : z \in \big(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)}\big)\Big\}$.*

Proof. By the assumptions in **(H)**, $\Omega := \{(x, \mu) \in \mathbb{R} \times \Lambda : x \in \mathcal{I}_\mu\}$ is an open subset of $\mathbb{R}^{d+1}$ and the map $G : \Omega \longrightarrow \mathbb{R}^{d+1}$ given by $G(x, \mu) = (g_\mu(x), \mu)$ is continuous. It is also injective because, for each fixed $\mu \in \Lambda$, g_μ is a diffeomorphism from $\big(x_\ell(\mu), x_r(\mu)\big)$ to $\big(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)}\big)$. Then the result follows by the Invariance Domain Theorem (see for instance [4]). ■

Lemma 3.2. *Suppose that $\{X_\mu\}_{\mu\in\Lambda}$ is a family of analytic potential systems satisfying* **(H)**. *Then*

$$\lim_{z\to-\sqrt{h_0(\mu)}} g_\mu^{-1}(z) = x_\ell(\mu) \text{ and } \lim_{z\to\sqrt{h_0(\mu)}} g_\mu^{-1}(z) = x_r(\mu)$$

uniformly in compacts of Λ. Moreover, if the functions h_0, x_ℓ and x_r are finite then $(z, \mu) \longmapsto g_\mu^{-1}(z)$ extends continuously to $(-\sqrt{h_0(\hat{\mu})}, \hat{\mu})$ and $(\sqrt{h_0(\hat{\mu})}, \hat{\mu})$ for all $\mu \in \Lambda$.

Proof. Let us prove the first assertion of the lemma. Consider a given compact subset K of Λ. Let us prove for instance that $\lim_{z\to\sqrt{h_0(\mu)}} g_\mu^{-1}(z) = x_r(\mu)$ uniformly on K. We consider the case when $h_0(\mu) = \infty$ and $x_r(\mu) < \infty$. Set $\delta := \min\{x_r(\mu) : \mu \in K\}$. Then for any $0 < \varepsilon < \delta$ define

$$A_\varepsilon := \max\left\{g_\mu\big(x_r(\mu) - \varepsilon\big) : \mu \in K\right\},$$

which is well defined because K is compact and $\mu \longmapsto g_\mu\big(x_r(\mu)-\varepsilon\big)$ is continuous. Thus $g_\mu(x_r(\mu)-\varepsilon) < z$ for all $z > A_\varepsilon$ and $\mu \in K$, which implies $0 < x_r(\mu) - g_\mu^{-1}(z) < \varepsilon$. This ends the proof in this case. The other cases follows in a similar way with the obvious modifications. Finally the continuity of $(z,\mu) \longrightarrow g_\mu^{-1}(z)$ follows from the first assertion of the lemma together with Lemma 2.14 and the continuity of h_0. ∎

Next two sections are concerned with the criticality at the outer boundary of potential systems verifying **(H)**. Section 3.1 is devoted to prove Theorems A and B, that deal with the case $h_0 = +\infty$, whereas in Section 3.2 we prove Theorems C and D, that tackle the case in which h_0 is finite.

3.1 Outer boundary reached with infinite energy

In this section we shall study the bifurcation of critical periodic orbits in a family of potential systems for which $h_0(\mu) = +\infty$ for all $\mu \in \Lambda$. The first result provides a way to study the criticality at the outer boundary in this situation.

Lemma 3.3. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems satisfying* **(H)** *such that $h_0 \equiv +\infty$ and fix $\hat\mu \in \Lambda$. Then the following holds:*

(a) *Suppose that for all $\mu \in \Lambda$ there exist $\alpha_1(\mu)$ and $\Delta_1(\mu)$ such that*

$$\lim_{h\to+\infty} h^{\alpha_1(\mu)} T'_\mu(h) = \Delta_1(\mu).$$

If there exist two sequences $\{\mu_n^\pm\}_{n\in\mathbb{N}}$ with $\mu_n^\pm \longrightarrow \hat\mu$ such that $\Delta_1(\mu_n^+)\Delta_1(\mu_n^-) < 0$ for all $n \in \mathbb{N}$, then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) \geqslant 1$. If the above limit is uniform on Λ, the map $\mu \longmapsto \alpha_1(\mu)$ is continuous at $\mu = \hat\mu$ and $\Delta_1(\hat\mu) \neq 0$, then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) = 0$.

(b) *If there exist two continuous functions α_1 and α_2 at $\mu = \hat\mu$ such that*

$$\lim_{h\to+\infty} h^{\alpha_2(\mu)}\big(h^{\alpha_1(\mu)} T'_\mu(h)\big)' = \Delta_2(\mu), \text{ uniformly on } \Lambda,$$

and $\Delta_2(\hat\mu) \neq 0$, then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) \leqslant 1$.

Proof. Let us prove the first assertion in (a). The assumption implies that, for all $\delta > 0$ and $\bar h > 0$, there exist $\mu^\pm \in \Lambda$ and $h^\star > 0$ with $\|\mu^\pm - \hat\mu\| < \delta$ and $h^\star > \bar h$ satisfying $T'_{\mu^+}(h)T'_{\mu^-}(h) < 0$ for all $h > h^\star$. Then, on account of the continuity of $\mu \longmapsto T'_\mu(h^\star)$, there exists $\mu^\star$ in the segment that joins μ^+ and μ^- such that $T'_{\mu^\star}(h^\star) = 0$. This shows that $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) \geqslant 1$. Let us turn to the second assertion in (a). For any $(h,\mu) \in (0,\infty)\times\Lambda$ define $f_\mu(h) := h^{\alpha_1(\mu)}T'_\mu(h)$. Then $\{f_\mu\}_{\mu\in\Lambda}$ is a continuous family of continuous functions on $(0,\infty)$ and by (a) in Lemma 2.14 we have $\lim_{(h,\mu)\to(\infty,\hat\mu)} f_\mu(h) = \Delta_1(\hat\mu)$. Then, on account of $\Delta_1(\hat\mu) \neq 0$, there exist a neighbourhood $\mathscr{U}$ of $\hat\mu$ and $h^\star > 0$ such that, for all $\mu \in \mathscr{U}$, $T'_\mu(h) \neq 0$ for all $h \in (h^\star, \infty)$. This shows that $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) = 0$ and completes the proof of (a).

In order to prove (b) we take $\hat f_\mu(h) := h^{\alpha_2(\mu)}\big(h^{\alpha_1(\mu)}T'_\mu(h)\big)'$. Exactly as before, the assumption $\Delta_2(\hat\mu) \neq 0$ implies that $\lim_{(h,\mu)\to(\infty,\hat\mu)} \hat f_\mu(h) \neq 0$. Accordingly there exist a neighbourhood $\mathscr{U}$ of $\hat\mu$ and $h^\star > 0$ such that, for all $\mu \in \mathscr{U}$, $\big(h^{\alpha_1(\mu)}T'_\mu(h)\big)' \neq 0$ for all $h \in (h^\star, \infty)$. Then by applying Bolzano's Theorem it follows that, for all $\mu \in \mathscr{U}$, $T'_\mu(h) = 0$ has at most one root on $(h^\star, \infty)$, multiplicities taking into account. Therefore $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) \leqslant 1$ and so the result is proved. ∎

The previous result is a key tool to prove the main results of this section. Our goal will be then to find sufficient conditions in order that the limits in Lemma 3.3 are uniform with respect to the parameter. In other words, sufficient conditions for $\{T'_\mu\}_{\mu\in\Lambda}$ to be continuously quantifiable at $h = +\infty$. Our next result gives the limit value of the period function as we approach to the outer boundary. It is a non-parametric result and so the dependence on μ is omitted for the sake of shortness.

Theorem 3.4. *Let X be an analytic potential differential system with $h_0 = +\infty$ and such that $(g^{-1})''$ is monotonous near the endpoints of the interval $(-\sqrt{h_0}, \sqrt{h_0})$. Then the following statements hold:*

(i) *The limits* $\lim_{z\to-\sqrt{h_0}}(g^{-1})'(z) =: a_\ell$ *and* $\lim_{z\to\sqrt{h_0}}(g^{-1})'(z) =: a_r$ *exist and* $a_\ell, a_r \in [0,+\infty]$. *Moreover* $T(h)$ *tends to* $(a_\ell + a_r)\frac{\pi}{\sqrt{2}}$ *as* $h \to +\infty$.

(ii) *The limits* $\lim_{z\to-\sqrt{h_0}}(g^{-1})''(z) =: b_\ell$ *and* $\lim_{z\to\sqrt{h_0}}(g^{-1})''(z) =: b_r$ *exist. Moreover* $\sqrt{h}T'(h)$ *tends to* $(b_\ell + b_r)\frac{\sqrt{2}}{2}$ *as* $h \to +\infty$ *except for the cases* $\{b_\ell = +\infty, b_r = -\infty\}$ *and* $\{b_\ell = -\infty, b_r = +\infty\}$.

Proof. For the sake of brevity we only prove (i) since (ii) follows similarly. From the expression for the period function in (17) we get $T(s^2) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta$. The monotonicity of $(g^{-1})''$ near the endpoints of $(-\sqrt{h_0}, \sqrt{h_0})$ implies the same property for $(g^{-1})'$. Therefore a_ℓ (respectively, a_r) either exists or it is infinity. In addition, due to $g' > 0$, we have $a_\ell, a_r \in [0,+\infty]$. We claim that $\lim_{s\to\infty}\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta = a_r\frac{\pi}{\sqrt{2}}$. Let us consider first the case $a_r < +\infty$. Due to $\lim_{z\to\sqrt{h_0}}(g^{-1})'(z) = a_r < \infty$ there exist $M > a_r$ such that $(g^{-1})'(x) < M$ for all $x \geqslant 0$. Given $\varepsilon > 0$, define $\varepsilon' = \varepsilon/\sqrt{2}$ and let $\bar{x} > 0$ be such that $\left|(g^{-1})'(x) - a_r\right| < \frac{\varepsilon'}{\pi}$ for all $x > \bar{x}$. Finally, let s_0 be such that $s_0\sin\left(\frac{\varepsilon'}{4M}\right) > \bar{x}$. Then if $s > s_0$ we have

$$
\begin{aligned}
\left|\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta - \frac{a_r\pi}{\sqrt{2}}\right| \leqslant{}& \left|\sqrt{2}\int_0^{\frac{\varepsilon'}{4M}}\left((g^{-1})'(s\sin\theta) - a_r\right)d\theta\right| \\
&+ \left|\sqrt{2}\int_{\frac{\varepsilon'}{4M}}^{\frac{\pi}{2}}\left((g^{-1})'(s\sin\theta) - a_r\right)d\theta\right| \\
\leqslant{}& \sqrt{2}\left(2M\frac{\varepsilon'}{4M} + \frac{\varepsilon'}{\pi}\frac{\pi}{2}\right) = \sqrt{2}\varepsilon' = \varepsilon.
\end{aligned}
$$

Let us consider now the case $a_r = +\infty$. Given any $K > 0$, let $\bar{x} > 0$ be such that $(g^{-1})'(x) > K$ for all $x > \bar{x}$. As before, let s_0 be such that $s_0\sin\left(\frac{\pi}{4}\right) > \bar{x}$. Then, if $s > s_0$ we get that

$$
\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta \geqslant \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta \geqslant K\sqrt{2}\frac{\pi}{4} > K.
$$

Thus $\lim_{s\to\infty}\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta = +\infty$. Exactly the same way can be proved that

$$
\lim_{s\to\infty}\sqrt{2}\int_{-\frac{\pi}{2}}^{0}(g^{-1})'(s\sin\theta)d\theta = a_\ell\frac{\pi}{\sqrt{2}},
$$

so the result follows. ■

Lemma 3.5. *Let* $\{X_\mu\}_{\mu\in\Lambda}$ *be a family of analytic potential systems satisfying* **(H)** *and such that* $h_0 \equiv +\infty$. *Take* $\hat{\mu} \in \Lambda$ *and suppose that* $\{g_\mu\}_{\mu\in\Lambda}$ *is continuously quantifiable in* $\hat{\mu}$ *at* $x_r(\mu)$ *(respectively,* $x_\ell(\mu)$*) by* $\beta(\mu)$ *with limit* b. *Consider a continuous family* $\{f_\mu\}_{\mu\in\Lambda}$ *of continuous functions such that* $\{f_\mu \circ g_\mu\}_{\mu\in\Lambda}$ *is continuously quantifiable in* $\hat{\mu}$ *at* $x_r(\mu)$ *(respectively,* $x_\ell(\mu)$*) by* $\alpha(\mu)$ *with limit* a. *Then* $\{f_\mu\}_{\mu\in\Lambda}$ *is continuously quantifiable in* $\hat{\mu}$ *at* $+\infty$ *(respectively,* $-\infty$*) by* $\frac{\alpha(\mu)}{\beta(\mu)}$ *with limit* $ab^{-\alpha(\hat{\mu})/\beta(\hat{\mu})}$.

Proof. Let us consider the case $x_r(\hat{\mu}) < \infty$ first. On account of Lemma 3.2, $\lim_{x\to+\infty} g_\mu^{-1}(x) = x_r(\mu)$ uniformly in μ. Thus, by applying Lemma 2.14, we have that $\lim_{(x,\mu)\to(+\infty,\hat{\mu})} g_\mu^{-1}(x) = x_r(\hat{\mu})$. Therefore

$$
\begin{aligned}
\lim_{(z,\mu)\to(+\infty,\hat{\mu})} f_\mu(z)z^{-\frac{\alpha(\mu)}{\beta(\mu)}} &= \lim_{(x,\mu)\to(x_r(\hat{\mu}),\hat{\mu})} f_\mu(g_\mu(x))(g_\mu(x))^{\frac{-\alpha(\mu)}{\beta(\mu)}} \\
&= \lim_{(x,\mu)\to(x_r(\hat{\mu}),\hat{\mu})} f_\mu(g_\mu(x))(g_\mu(x))^{-\frac{\alpha(\mu)}{\beta(\mu)}}(x_r(\hat{\mu}) - x)^{\alpha(\mu)}(x_r(\hat{\mu}) - x)^{\beta(\mu)\left(-\frac{\alpha(\mu)}{\beta(\mu)}\right)} \\
&= \lim_{(x,\mu)\to(x_r(\hat{\mu}),\hat{\mu})} f_\mu(g_\mu(x))(x_r(\hat{\mu}) - x)^{\alpha(\mu)}\left(g_\mu(x)(x_r(\hat{\mu}) - x)^{\beta(\mu)}\right)^{-\frac{\alpha(\mu)}{\beta(\mu)}} \\
&= ab^{-\alpha(\hat{\mu})/\beta(\hat{\mu})} \neq 0,
\end{aligned}
$$

where in the last equality we took the assumptions on $f_\mu \circ g_\mu$ and g_μ into account. One can easily show the same in case that $x_r(\hat{\mu}) = \infty$ and so for the sake of brevity we do not include the proof. ■

It is easy to show that if $h_0 \equiv +\infty$ and $\{g_\mu\}$ is continuously quantifiable in $\hat{\mu} \in \Lambda$ at $x_r(\mu)$ by $\beta(\mu)$, then $\beta(\mu) > 0$ for all $\mu \approx \hat{\mu}$. This is also true for the quantifier of $x_\ell(\mu)$. For this reason the quotient $\alpha(\mu)/\beta(\mu)$ for $\mu \approx \hat{\mu}$ is well defined in Lemma 3.5.

Next, by applying the tools developed in Section 2, we prove a criterion for a parameter $\hat{\mu} \in \Lambda$ to be a local regular value of the period function at the outer boundary.

Theorem A. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems satisfying* **(H)** *and such that $h_0 \equiv +\infty$. Assume that the even part of $z\big(g_\mu^{-1}\big)''(z)$ is continuously quantifiable in Λ at $+\infty$ by $\gamma(\mu)$ and, for each $i \in \mathbb{N}$, let $M_i(\mu)$ be the i-th momentum of the even part of $z\big(g_\mu^{-1}\big)''(z)$, whenever it is defined. Then the following hold:*

(a) *If $\gamma(\hat{\mu}) > -1$, then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of the period annulus.*

(b) *If $\gamma(\hat{\mu}) < -1$, let $n \in \mathbb{N}$ be such that $-1-2n \leqslant \gamma(\hat{\mu}) < 1-2n$. Then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of the period annulus in case that*

   (b1) *either $M_j(\hat{\mu}) \neq 0$ for some $j \in \{1,2,\ldots,n\}$ and $M_1 \equiv M_2 \equiv \ldots \equiv M_{j-1} \equiv 0$,*

   (b2) *or $\gamma(\hat{\mu}) \notin \{-1-2n, -2n\}$ and $M_1 \equiv M_2 \equiv \ldots \equiv M_n \equiv 0$.*

Finally the even part of $z\big(g_\mu^{-1}\big)''(z)$ is continuously quantifiable at $+\infty$ by $\gamma(\mu) = 1+\max\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu), \big(\frac{\alpha_r}{\beta_r}\big)(\mu)\big\}$ in case that the following is verified:

(i) *$\{g_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $x_\ell(\mu)$ by $\beta_\ell(\mu)$ and at $x_r(\mu)$ by $\beta_r(\mu)$ with limits $b_\ell(\mu)$ and $b_r(\mu)$, respectively,*

(ii) *$\left\{\frac{-g_\mu''}{(g_\mu')^3}\right\}_{\mu\in\Lambda}$ is continuously quantifiable at $x_\ell(\mu)$ by $\alpha_\ell(\mu)$ and at $x_r(\mu)$ by $\alpha_r(\mu)$ with limits $a_\ell(\mu)$ and $a_r(\mu)$, respectively,*

(iii) *and either $\frac{\alpha_\ell}{\beta_\ell}(\mu) \neq \frac{\alpha_r}{\beta_r}(\mu)$ or, otherwise, $\big(a_\ell b_\ell^{-\frac{\alpha_\ell}{\beta_\ell}} + a_r b_r^{-\frac{\alpha_r}{\beta_r}}\big)(\mu) \neq 0$*

Proof. Let us show first that if $\hat{\mu} \in \Lambda$ verifies (a) or (b) then it is a local regular value. With this aim in view note that, from the expression in (17), the derivative of the period function can be written as

$$\frac{d}{ds}T_\mu(s^2) = 2sT_\mu'(s^2) = \frac{\sqrt{2}}{s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(s\sin\theta)s\sin\theta d\theta = \frac{2\sqrt{2}}{s}\int_0^{\frac{\pi}{2}} f_\mu(s\sin\theta)d\theta,$$

where we define f_μ to be the even part of $z\big(g_\mu^{-1}\big)''(z)$. By hypothesis, $\{f_\mu\}$ is continuously quantifiable at $+\infty$ by $\gamma(\mu)$ with, let us say, limit $d(\mu)$. The assertion in the cases (a) and $(b2)$ follows by applying Theorems 2.13 and 2.17, respectively. Indeed, in case (a) Theorem 2.13 shows that

$$\lim_{(s;\mu)\to(+\infty,\hat{\mu})} s^{-\gamma(\mu)}\int_0^{\frac{\pi}{2}} f_\mu(s\sin\theta)d\theta = d(\hat{\mu})\mathscr{B}(\gamma(\hat{\mu})) \neq 0,$$

and in case $(b2)$, setting $\gamma_j(\mu) := \prod_{i=1}^{j}\frac{\gamma(\mu)+2i}{\gamma(\mu)+2i-1}$ for $j = 1,2,\ldots,n$, Theorem 2.17 shows that

$$\lim_{(s;\mu)\to(+\infty,\hat{\mu})} s^{-\gamma(\mu)}\int_0^{\frac{\pi}{2}} f_\mu(s\sin\theta)d\theta = d(\hat{\mu})\gamma_n(\hat{\mu})\mathscr{B}(\gamma(\hat{\mu})+2n) \neq 0.$$

In both cases this implies that $s^{2-\gamma(\mu)}T_\mu'(s^2)$ tends to a non-zero number as $(s;\mu) \to (+\infty,\hat{\mu})$. Therefore Lemma 3.3 shows that $\hat{\mu}$ is a local regular value. To prove the assertion in case $(b1)$ note that, from (a) in Theorem 2.17,

$$\lim_{(s;\mu)\to(+\infty,\hat{\mu})} s^{2j-1}\int_0^{\frac{\pi}{2}} f_\mu(s\sin\theta)d\theta = M_j(\hat{\mu}) \neq 0$$

and, consequently, $s^{2j+1}T'_\mu(s^2)$ tends to a non-zero number as $(s;\mu)\to(+\infty,\hat\mu)$. Again Lemma 3.3 shows that $\hat\mu$ is a local regular value and the first part of the result follows.

Let us prove the second part. In this regard note that, by Lemma 3.5 and since $(g_\mu^{-1})''\circ g_\mu=\frac{-g''_\mu}{(g'_\mu)^3}$, the combination of (i) and (ii) implies that $\{(g_\mu^{-1})''\}_{\mu\in\Lambda}$ is continuously quantifiable at $+\infty$ by $\big(\frac{\alpha_r}{\beta_r}\big)(\mu)$, with limit $c_r:=a_r(b_r)^{-\left(\frac{\alpha_r}{\beta_r}\right)}$, and at $-\infty$ by $\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu)$, with limit $c_\ell:=a_\ell(b_\ell)^{-\left(\frac{\alpha_\ell}{\beta_\ell}\right)}$. (Here we omit the dependence on μ for the sake of brevity.) Thus, taking (iii) also into account, we can assert that the even part of $z\big(g_\mu^{-1}\big)''(z)$ is continuously quantifiable at $+\infty$ by $\gamma(\mu)=1+\max\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu),\big(\frac{\alpha_r}{\beta_r}\big)(\mu)\big\}$ with limit $d(\mu)/2$, where

$$d(\mu):=\begin{cases} c_\ell(\mu) & \text{if } \big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu)>\big(\frac{\alpha_r}{\beta_r}\big)(\mu),\\ \big(c_\ell+c_r\big)(\mu) & \text{if } \big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu)=\big(\frac{\alpha_r}{\beta_r}\big)(\mu),\\ c_r(\mu) & \text{if } \big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu)<\big(\frac{\alpha_r}{\beta_r}\big)(\mu).\end{cases}\tag{18}$$

This completes the proof of the result. ∎

Remark 3.6. The proof of Theorem A shows that $T'_\mu(h)=h^{\alpha_1(\mu)}\big(\Delta_1(\mu)+f_1(h;\mu)\big)$, with $f_1(h;\mu)$ tending to zero as $(h,\mu)\longrightarrow(+\infty,\hat\mu)$, where

$$\begin{cases} \alpha_1(\mu)=\frac{\gamma(\mu)-2}{2} \quad \text{and} \quad \Delta_1(\mu)=\sqrt{2}d(\mu)\mathscr{B}(\gamma(\mu)), & \text{in case } (a),\\ \alpha_1(\mu)=-\frac{2j+1}{2} \quad \text{and} \quad \Delta_1(\mu)=\sqrt{2}M_j(\mu), & \text{in case } (b1),\\ \alpha_1(\mu)=\frac{\gamma(\mu)-2}{2} \quad \text{and} \quad \Delta_1(\mu)=\sqrt{2}d(\mu)\gamma_n(\mu)\mathscr{B}(\gamma(\mu)+2n), & \text{in case } (b2).\end{cases}$$

In other words, it gives the quantifier of the derivative of the period function when $\Delta_1(\hat\mu)\neq 0$. □

The previous remark, together with (a) in Lemma 3.3, provides a tool to conclude that a certain parameter $\hat\mu$ is a local bifurcation value, and it will be used in Section 4 to study a specific family of potential systems. We finish this section by proving a criterion to bound the number of critical periodic orbits that can bifurcate from the outer boundary of the period annulus.

Theorem B. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems satisfying* **(H)** *and such that $h_0\equiv+\infty$. Assume that there exists a continuous function $\upsilon:\Lambda\longrightarrow\mathbb{R}$ such that the even part of*

$$f_\mu(z):=(g_\mu^{-1})'''(z)z^2-\upsilon(\mu)(g_\mu^{-1})''(z)z,$$

is continuously quantifiable in Λ at $+\infty$ by $\xi(\mu)$. For each $i\in\mathbb{N}$, let $M_i(\mu)$ be the i-th momentum at of the even part of f_μ, whenever it is defined. Then the following hold:

(a) If $\xi(\hat\mu)>-1$, then $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant 1$.

(b) If $\xi(\hat\mu)<-1$, let $n\in\mathbb{N}$ be such that $-1-2n\leqslant\xi(\hat\mu)<1-2n$. Then $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant 1$ *if*

  (b1) either, $M_j(\hat\mu)\neq 0$ for some $j\in\{1,2,\ldots,n\}$ and $M_1\equiv M_2\equiv\ldots\equiv M_{j-1}\equiv 0$, for all μ in a neighbourhood of $\hat\mu$,

  (b2) or, $\xi(\hat\mu)\notin\{-1-2n,-2n\}$ and $M_1\equiv M_2\equiv\ldots\equiv M_n\equiv 0$ for all μ in a neighbourhood of $\hat\mu$

Finally the even part of f_μ is continuously quantifiable at $+\infty$ by $\xi(\mu)=\max\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu),\big(\frac{\alpha_r}{\beta_r}\big)(\mu)\big\}$ in case that the following is verified:

(i) $\{g_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $x_\ell(\mu)$ by $\beta_\ell(\mu)$ and at $x_r(\mu)$ by $\beta_r(\mu)$ with limits $b_\ell(\mu)$ and $b_r(\mu)$, respectively,

(ii) $\{f_\mu\circ g_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $x_\ell(\mu)$ by $\alpha_\ell(\mu)$ and at $x_r(\mu)$ by $\alpha_r(\mu)$ with limits $a_\ell(\mu)$ and $a_r(\mu)$, respectively,

(iii) and either $\frac{\alpha_\ell}{\beta_\ell}(\mu)\neq\frac{\alpha_r}{\beta_r}(\mu)$ or, otherwise, $a_r(b_\ell)^{-\frac{\alpha_\ell}{\beta_\ell}}(\mu)+a_\ell(b_r)^{-\frac{\alpha_r}{\beta_r}}(\mu)\neq 0$.

Proof. An easy computation using the expression in (17) shows that

$$\left(s^{-\upsilon(\mu)}\frac{d}{ds}T_\mu(s^2)\right)' = \sqrt{2}s^{-\upsilon(\mu)-2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\mu(s\sin\theta)d\theta = 2\sqrt{2}s^{-\upsilon(\mu)-2}\int_0^{\frac{\pi}{2}} \hat{f}_\mu(s\sin\theta)d\theta,$$

where f_μ is the function defined in the statement and $\hat{f}_\mu$ its even part. If $\xi(\hat{\mu}) > -1$ then Theorem 2.13 shows that

$$\lim_{(s,\mu)\to(+\infty,\hat{\mu})} s^{-\xi(\mu)}\int_0^{\frac{\pi}{2}} \hat{f}_\mu(s\sin\theta)d\theta =: L \neq 0.$$

Consequently $s^{\upsilon(\mu)-\xi(\mu)+2}\big(s^{-\upsilon(\mu)}\frac{d}{ds}T_\mu(s^2)\big)'$ tends to $2\sqrt{2}L$ as $(s,\mu)\to(+\infty,\hat{\mu})$. On account of Bolzano's Theorem, this implies that there exists $M>0$ and a neighbourhood $\mathscr{U}$ of $\hat{\mu}$ such that if $\mu\in\mathscr{U}$ then T_μ' has at most one zero for $s>M$. Hence the criticality at the outer boundary of $X_{\hat{\mu}}$ with respect to the deformation X_μ is at most one. By using (b) in Theorem 2.17 instead, exactly the same proof applies in case that $\xi(\hat{\mu})\in(-1-2n,1-2n)\setminus\{-2n\}$ and $M_1\equiv M_2\equiv\ldots\equiv M_n\equiv 0$, i.e., $(b2)$ is verified. Finally, if $(b1)$ holds, then by applying (a) in Theorem 2.17 we conclude that

$$\lim_{(s,\mu)\to(+\infty,\hat{\mu})} s^{\upsilon(\mu)+2j+1}\left(s^{-\upsilon(\mu)}\frac{d}{ds}T_\mu(s^2)\right)' = 2\sqrt{2}M_j(\hat{\mu}) \neq 0,$$

which exactly as before implies that the criticality at the outer boundary of $X_{\hat{\mu}}$ with respect to the deformation X_μ is at most one. This proves the first part of the result. In order to show the second part note that, by Lemma 3.5, the assumptions (i) and (ii) imply that f_μ is continuously quantifiable at $-\infty$ by $\frac{\alpha_\ell}{\beta_\ell}$ and at $+\infty$ by $\frac{\alpha_r}{\beta_r}$, with limits $a_\ell(b_\ell)^{-\frac{\alpha_\ell}{\beta_\ell}}$ and $a_r(b_r)^{-\frac{\alpha_r}{\beta_r}}$, respectively. (Here we omit again the dependence on μ for the sake of brevity.) Finally, by the assumption in (iii), we have that the even part of f_μ is continuously quantifiable at $+\infty$ by $\xi(\mu)=\max\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu),\big(\frac{\alpha_r}{\beta_r}\big)(\mu)\big\}$. So the result is true. ■

Remark 3.7. The proof of Theorem A is based on the quantification of T_μ'. More concretely, it gives sufficient conditions in order that $T_\mu'(h)=\Delta_1(\mu)h^{\alpha_1(\mu)}+h^{\alpha_1(\mu)}f_1(h;\mu)$, with $\Delta_1(\hat{\mu})\neq 0$ and the remainder $f_1(h;\mu)$ tending to 0 as $h\longrightarrow+\infty$, uniformly on $\mu\approx\hat{\mu}$. The explicit value of the quantifier α_1 is given in Remark 3.6. In case that $\Delta_1(\hat{\mu})=0$ we must go further in the asymptotic development to get

$$T_\mu'(h)=\Delta_1(\mu)h^{\alpha_1(\mu)}+\Delta_2(\mu)h^{\alpha_2(\mu)}+h^{\alpha_2(\mu)}f_2(h;\mu),\ \text{with } \alpha_1(\hat{\mu})>\alpha_2(\hat{\mu}).$$

If the new remainder has "good properties" with respect to the *division-derivation process*, then

$$\lim_{h\to+\infty} h^{\alpha_1(\mu)-\alpha_2(\mu)+1}\big(h^{-\alpha_1(\mu)}T_\mu'(h)\big)'=\big(\alpha_2(\mu)-\alpha_1(\mu)\big)\Delta_2(\mu),\ \text{uniformly on } \mu\approx\hat{\mu}.$$

From this point of view, the proof of Theorem B is based on the quantification of a combination of the first and the second derivative of the period function, more concretely, $hT_\mu''(h)-\alpha_1(\mu)T_\mu'(h)$. Thus, in order to apply Theorem B, a good choice is to take the function υ in its statement as the quantifier α_1 of T_μ'. □

It is to be noted that, for any given $n\in\mathbb{N}$, it is possible to obtain a criterion for $\mathrm{Crit}\big((\Pi_{\hat{\mu}},X_{\hat{\mu}}),X_\mu\big)\leqslant n$ by using Theorems 2.13 and 2.17 exactly as we do in Theorem B for $n=1$.

3.2 Outer boundary reached with finite energy

In this section we shall study the bifurcation of critical periodic orbits in a family of potential systems for which the energy level $h_0(\mu)$ is finite for all $\mu\in\Lambda$. Our first result is the counterpart of Lemma 3.3 for this situation and, since its proof is very similar, we omit it for brevity.

Lemma 3.8. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems satisfying* **(H)** *such that $h_0(\mu)$ is finite and fix $\hat{\mu}\in\Lambda$. Then the following holds:*

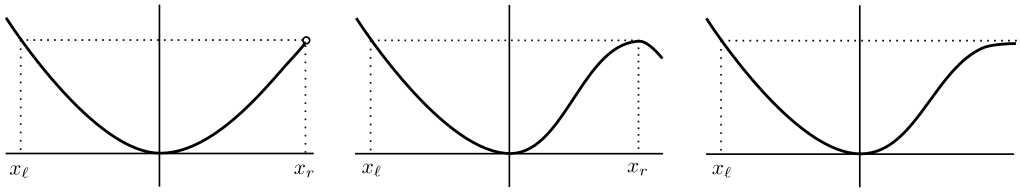

Figure 3: Graph of V for admissible potential systems with finite energy and only one non-regular endpoint, cf. (a) in Definition 3.9.

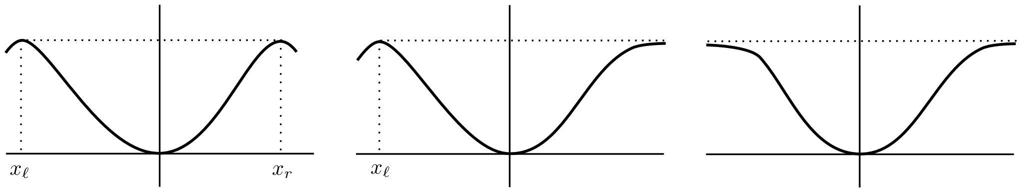

Figure 4: Graph of V for admissible potential systems with finite energy and two non-regular endpoints, cf. (b) in Definition 3.9.

(a) *Suppose that for all $\mu \in \Lambda$ there exist $\Delta_1(\mu)$ such that*

$$\lim_{h \to h_0(\mu)} T'_\mu(h) = \Delta_1(\mu).$$

If there exist two sequences $\{\mu_n^\pm\}_{n\in\mathbb{N}}$ with $\mu_n^\pm \longrightarrow \hat\mu$ such that $\Delta_1(\mu_n^+)\Delta_1(\mu_n^-) < 0$ for all $n \in \mathbb{N}$, then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) \geqslant 1$. If the above limit is uniform on Λ and $\Delta_1(\hat\mu) \neq 0$, then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) = 0$.

(b) *If $\lim_{h\to h_0(\mu)} T'_\mu(h) = \infty$ uniformly on Λ, then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) = 0$ for all $\hat\mu \in \Lambda$.*

Definition 3.9. Let $X = -y\partial_x + V'(x)\partial_y$ be an analytic potential system with a non-degenerated center at the origin and let (x_ℓ, x_r) be the projection on the x-axis of its period annulus. We say that x_ℓ (respectively, x_r) is *regular* if V is analytic at x_ℓ (respectively, x_r) and $V'(x_\ell) \neq 0$ (respectively, $V'(x_r) \neq 0$). Otherwise we say that the endpoint is *non-regular*. Moreover, we say that the potential system is *admissible* if it verifies one of the following conditions:

(a) either x_ℓ or x_r is regular.

(b) $\lim_{x\to x_\ell} V'(x) = \lim_{x\to x_r} V'(x) = 0$.

□

We point out that x_ℓ and x_r cannot be regular simultaneously, otherwise the projection of the period annulus is larger than the interval (x_ℓ, x_r). In what follows, without lost of generality, we shall assume that x_r is non-regular. Figures 3 and 4 display the graph of V for all the possible cases giving rise to an admissible potential system under this assumption.

Lemma 3.10. *Suppose that $X = -y\partial_x + V'(x)\partial_y$ is an admissible analytic potential system with two non-regular endpoints and such that $(g^{-1})''$ is monotonous near the endpoints of $(-\sqrt{h_0}, \sqrt{h_0})$. Then $(g^{-1})''(z)$ tends to $+\infty$ (respectively, $-\infty$) as $z \nearrow \sqrt{h_0}$ (respectively, $z \searrow -\sqrt{h_0}$).*

Proof. By hypothesis, $\lim_{x\to x_\ell} V'(x) = \lim_{x\to x_r} V'(x) = 0$. Since $g(x) = \mathrm{sgn}(x)\sqrt{V(x)}$, this implies that $\lim_{z\to\pm\sqrt{h_0}} (g^{-1})'(z) = +\infty$. Then, due to the fact that the interval $(-\sqrt{h_0}, \sqrt{h_0})$ is bounded, there exist two sequences $a_n \nearrow \sqrt{h_0}$ and $b_n \searrow -\sqrt{h_0}$ such that $(g^{-1})''(a_n)$ and $(g^{-1})''(b_n)$ tend, respectively, to $+\infty$ and $-\infty$ as $n \longrightarrow \infty$. Now the result follows on account of the monotonicity of $(g^{-1})''$ near the endpoints of the interval $(-\sqrt{h_0}, \sqrt{h_0})$. ■

Proposition 3.11. *Let $F:[0,\sigma)\longrightarrow\mathbb{R}$ be a continuous function that is monotonous near $x=\sigma$. Then, for any $n\in\mathbb{N}$,*

$$\lim_{s\to 1^-}\int_0^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta=\int_0^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta,$$

where the improper integral on the right either converges or it tends to infinity.

Proof. Let us prove first the result in case that $L:=\int_0^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta$ is a convergent integral. Clearly the limit of $F(z)$ as $z\nearrow\sigma$ exists due to the monotonicity of F near $z=\sigma$. If this limit is finite then the result is straightforward. Hence let us suppose, for instance, that $\lim_{z\to\sigma}F(z)=+\infty$. Thus F is a positive increasing function on $(\sigma-\kappa,\sigma)$ for some $\kappa>0$. Consider any $\varepsilon>0$ and let η and δ_1 be small enough positive numbers such that $s\sigma\sin\theta>\sigma-\kappa$ for all $\theta\in(\frac{\pi}{2}-\eta,\frac{\pi}{2})$ and $s\in(1-\delta_1,1)$. Then, for these values, $0<F(s\sigma\sin\theta)<F(\sigma\sin\theta)$ and consequently

$$0<\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta<\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta<\frac{\varepsilon}{4}\text{ for all } s\in(1-\delta_1,1),$$

where the last inequality follows due to the fact that $\int_0^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta$ is a convergent integral and taking η smaller if necessary. On the other hand, since $s\longmapsto\int_0^{\frac{\pi}{2}-\eta} F(s\sigma\sin\theta)\sin^n\theta d\theta$ is continuous at $s=1$, there exists $\delta_2>0$ such that

$$\left|\int_0^{\frac{\pi}{2}-\eta}\big(F(\sigma\sin\theta)-F(s\sigma\sin\theta)\big)\sin^n\theta d\theta\right|<\frac{\varepsilon}{2}\text{ for all } s\in(1-\delta_2,1).$$

Accordingly if $s\in(1-\delta,1)$ with $\delta:=\min\{\delta_1,\delta_2\}$, then

$$\begin{aligned}\left|L-\int_0^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta\right|&\leqslant\left|\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta\right|+\left|\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta\right|\\&\quad+\left|\int_0^{\frac{\pi}{2}-\eta}\big(F(\sigma\sin\theta)-F(s\sigma\sin\theta)\big)\sin^n\theta d\theta\right|<\varepsilon,\end{aligned}$$

and the result follows.

Now let us prove the result in case that $\int_0^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta$ does not converge. This implies, due to the monotonicity of $F(z)$ at $z=\sigma$, that $\int_0^{\frac{\pi}{2}-\eta} F(\sigma\sin\theta)\sin^n\theta d\theta$ tends to infinity as $\eta\searrow 0$. Suppose, for instance, that it tends to $+\infty$. Hence $F(z)$ tends to $+\infty$ as $z\nearrow\sigma$. Take $\bar z\in(0,\sigma)$ such that F is positive on $(\bar z,\sigma)$. Let η_1 and δ_1 be positive numbers such that $s\sigma\sin\theta>\bar z$ for all $\theta\in(\frac{\pi}{2}-\eta_1,\frac{\pi}{2})$ and $s\in(1-\delta_1,1)$.

$$\int_0^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta\geqslant\int_0^{\frac{\pi}{2}-\eta_1} F(s\sigma\sin\theta)\sin^n\theta d\theta\text{ for all } s\in(1-\delta_1,1).\tag{19}$$

Consider at this point any $M>0$. Then, due to $\int_0^{\frac{\pi}{2}} F(\sigma\sin\theta)\sin^n\theta d\theta=+\infty$, there exists $\eta_2\in(0,\eta_1)$ small enough such that

$$\int_0^{\frac{\pi}{2}-\eta_2} F(\sigma\sin\theta)\sin^n\theta d\theta>M.$$

Define $S(s):=\int_0^{\frac{\pi}{2}-\eta_2} F(s\sigma\sin\theta)\sin^n\theta d\theta$, which is a continuous function on $[0,1]$. Therefore, on account of $S(1)>M$, there exists $\delta_2\in(0,\delta_1)$ such that $S(s)>M$ for all $s\in(1-\delta2,1)$. Hence, since $F(s\sigma\sin\theta)>0$ for all $\theta\in(\frac{\pi}{2}-\eta_1,\frac{\pi}{2})$ and $s\in(1-\delta_1,1)$, from (19) we can assert that

$$\int_0^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta\geqslant\int_0^{\frac{\pi}{2}-\eta_2} F(s\sigma\sin\theta)\sin^n\theta d\theta=S(s)>M\text{ for all } s\in(1-\delta_2,1),$$

where in the first inequality we take $0<\delta_2<\delta_1$ and $0<\eta_2<\eta_1$ also into account. This shows that $\lim_{s\to 1^-}\int_0^{\frac{\pi}{2}} F(s\sigma\sin\theta)\sin^n\theta d\theta=+\infty$, as desired, and completes the proof of the result. ■

Next result gives the limit value of the period function and its derivative as we approach the outer boundary. Since it is non-parametric, the dependence on μ is omitted for the sake of brevity.

Corollary 3.12. *Let X be an admissible analytic potential system with $h_0 < +\infty$ and such that $(g^{-1})''$ is monotonous near the endpoints of the interval $(-\sqrt{h_0}, \sqrt{h_0})$. Then either $\lim_{h\nearrow h_0} T(h) = +\infty$ or*

$$\lim_{h\nearrow h_0} T(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h_0}\sin\theta)d\theta$$

and the integral is convergent. Similarly, either $\lim_{h\nearrow h_o} T'(h) = \pm\infty$ or

$$\lim_{h\nearrow h_0} T'(h) = \frac{1}{\sqrt{2h_0}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})''(\sqrt{h_0}\sin\theta)\sin\theta d\theta,$$

and the integral is convergent.

Proof. Clearly the monotonicity assumption on $(g^{-1})''$ implies that $(g^{-1})'$ is monotonous near the endpoints of $(-\sqrt{h_0}, \sqrt{h_0})$ as well. Let us prove the assertion concerning the first limit. Denote $f(z) := \sqrt{2}(g^{-1})'(z)$. Then, from (17), we can write

$$T(h_0 s^2) = I_+(s) + I_-(s), \text{ where } I_\pm(s) := \int_0^{\frac{\pi}{2}} f(\pm s\sqrt{h_0}\sin\theta)d\theta.$$

By applying Proposition 3.11 we have that $I_\pm(s)$ tends to $I_\pm(1)$ as $s \nearrow 1$, with $I_\pm(1)$ being a positive number or $+\infty$ since $(g^{-1})'$ is a positive function. This proves the first assertion.

Let turn now to the second assertion. In this case, setting $\hat{f}(z) := \sqrt{2h_0}(g^{-1})''(z)$, we write

$$\frac{d}{ds}T(h_0 s^2) = 2h_0 s T'(h_0 s^2) = R_+(s) - R_-(s), \text{ where } R_\pm(s) := \int_0^{\frac{\pi}{2}} \hat{f}(\pm s\sqrt{h_0}\sin\theta)\sin\theta d\theta.$$

Again, by Proposition 3.11, $R_\pm(s)$ tends to $R_\pm(1)$ as $s \nearrow 1$, with $R_\pm(1)$ being a real number or ∞. Accordingly the result follows except in case that $R_-(1)$ and $R_+(1)$ are both ∞. However, due to the admissibility assumption (see Definition 3.9), this can only occur if V' tends to zero as we approach to the endpoints of (x_ℓ, x_r). Hence, by Lemma 3.10, $\hat{f}(z)$ tends to $+\infty$ (respectively, $-\infty$) as $z \nearrow \sqrt{h_0}$ (respectively, $z \searrow -\sqrt{h_0}$) and, consequently, $R_-(1)$ and $R_+(1)$ are both $+\infty$. This completes the proof of the result. ∎

Once we have stablished the limit of $T'(h)$ as h tends to $h_0(\mu)$, our next goal is to give sufficient conditions to ensure that this limit is uniform with respect to μ. With this aim in view we prove the following result.

Lemma 3.13. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying* **(H)** *and such that h_0 and x_ℓ are finite. Assume that $x_\ell(\mu)$ is regular. Then the map $(z,\mu) \longmapsto (g_\mu^{-1})''(z)$ is continuous on $\big\{(z,\mu) \in \mathbb{R}\times\Lambda : z \in [-\sqrt{h_0(\mu)}, 0]\big\}$.*

Proof. Since $V_\mu(x) = g_\mu(x)^2$ and $x_\ell(\mu)$ is regular, $g'_\mu\big(x_\ell(\mu)\big) \neq 0$. On the other hand, by implicit derivation, $(g_\mu^{-1})'' = \frac{-g''_\mu}{(g'_\mu)^3}\circ g_\mu^{-1}$. Note also that $(x,\mu) \longmapsto \frac{-g''_\mu}{(g'_\mu)^3}(x)$ is continuous on $\{(z,\mu) \in \mathbb{R}\times\Lambda : z \in [-\sqrt{h_0(\mu)}, 0]\}$ thanks to hypothesis **(H)**. By Lemma 3.1, $(x,\mu) \longmapsto g_\mu^{-1}(x)$ is continuous on $\{(z,\mu) \in \mathbb{R}\times\Lambda : z \in (-\sqrt{h_0(\mu)}, 0]\}$ and it extends continuously at $(-\sqrt{h_0(\mu)}, \mu)$ by Lemma 3.2. The result follows then by composition. ∎

Definition 3.14. Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of continuous functions defined on $I_\mu = \big(a(\mu), b(\mu)\big)$. Suppose that each endpoint of I_μ is either a continuous function on Λ or identically ∞. We say that the family $\{f_\mu\}_{\mu\in\Lambda}$ is *uniformly monotonous* in $\hat{\mu} \in \Lambda$ at $a(\mu)$ (respectively, at $b(\mu)$) if there exist a neighbourhood U of $\hat{\mu}$ and $\bar{z} \in \mathbb{R}$ such that, for all $\mu \in U$, $\bar{z} \in I_\mu$ and $x \longmapsto f_\mu(x)$ is monotonous on $(a(\mu), \bar{z})$ (respectively, on $(\bar{z}, b(\mu))$). □

Definition 3.15. Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems. We say that a given parameter $\hat{\mu}\in\Lambda$ satisfies condition **(C)** if the following holds:

(C_1) The family $\left\{\frac{g_\mu''}{(g_\mu')^3}\right\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat{\mu}$ at the non-regular endpoints of $\mathcal{I}_\mu$.

(C_2) The families $\{g_\mu-\sqrt{h_0(\mu)}\}_{\mu\in\Lambda}$, $\{g_\mu'\}_{\mu\in\Lambda}$ and $\{g_\mu''\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at the non-regular endpoints of $\mathcal{I}_\mu$.

Let $\alpha_r(\mu)$ be the quantifier of $\{g_\mu-\sqrt{h_0(\mu)}\}_{\mu\in\Lambda}$ at $x_r(\mu)$, which recall that it is non-regular by convention (see Figures 3 and 4). If $x_\ell(\mu)$ is non-regular too, then we denote the corresponding quantifier at $x_\ell(\mu)$ by $\alpha_\ell(\mu)$. With this notation we define

$$M(\mu):=\begin{cases}-\frac{3}{2}\alpha_r(\mu) & \text{if } x_\ell(\mu) \text{ is regular,}\\ \max\{-\frac{3}{2}\alpha_\ell(\mu),-\frac{3}{2}\alpha_r(\mu)\} & \text{if } x_\ell(\mu) \text{ is non-regular,}\end{cases}$$

and

$$m(\mu):=\begin{cases}-\frac{3}{2}\alpha_r(\mu) & \text{if } x_\ell(\mu) \text{ is regular,}\\ \min\{-\frac{3}{2}\alpha_\ell(\mu),-\frac{3}{2}\alpha_r(\mu)\} & \text{if } x_\ell(\mu) \text{ is non-regular.}\end{cases}$$

□

The functions $M(\mu)$ and $m(\mu)$ are positive. Indeed, since $g_\mu(x)-\sqrt{h_0(\mu)}\longrightarrow 0$ as x tends to $x_r(\mu)$, it follows that $\alpha_r(\mu)<0$, and exactly the same occurs for $x_\ell(\mu)$.

Lemma 3.16. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems and suppose that $\hat{\mu}\in\Lambda$ verifies (C_2). Then the family $\left\{\frac{g_\mu''g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x_r(\mu)$ by $-\frac{3}{2}\alpha_r(\mu)$. Moreover, if $x_\ell(\mu)$ is non-regular too, then the family is continuously quantifiable in $\hat{\mu}$ at x_ℓ by $-\frac{3}{2}\alpha_\ell(\mu)$.*

Proof. By Hôpital's Rule it is easy to see that if $x_r(\mu)$ is finite then the quantifiers of $\{g_\mu'\}_{\mu\in\Lambda}$ and $\{g_\mu''\}_{\mu\in\Lambda}$ are $\alpha_r(\mu)+1$ and $\alpha_r(\mu)+2$ respectively. If $x_r(\mu)$ is infinite then the quantifiers are $\alpha_r(\mu)-1$ and $\alpha_r(\mu)-2$. The result follows then by product of limits. The proof for the left endpoint follows in the same way. ■

Proposition 3.17. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying* **(H)** *such that $h_0(\mu)$ is finite and $\mathcal{I}_\mu$ is bounded. Consider $\hat{\mu}\in\Lambda$ satisfying* **(C)**. *Then,*

(a) If $M(\hat{\mu})<1$ then

$$\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})}T_\mu'(h)=\frac{1}{\sqrt{2h_0(\hat{\mu})}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g_{\hat{\mu}}^{-1})''\big(\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin\theta d\theta$$

and the integral is convergent.

(b) If $M(\hat{\mu})>1$ and $m(\hat{\mu})\neq 1$ then $\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})}T_\mu'(h)=\pm\infty$.

Proof. Let us first prove (a). Setting $H(z;\mu):=\sqrt{2h_0(\mu)}(g_\mu^{-1})''(z)$ for the sake of brevity, the derivation of the expression of the period function in (17) yields to

$$\frac{d}{ds}T_\mu\big(h_0(\mu)s^2\big)=2h_0(\mu)sT_\mu'\big(h_0(\mu)s^2\big)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}H\big(\sqrt{h_0(\mu)}s\sin\theta;\mu\big)\sin\theta d\theta. \tag{20}$$

We split the interval of integration into $(-\frac{\pi}{2},0)$ and $(0,\frac{\pi}{2})$. We shall prove that

$$\lim_{(s,\mu)\to(1,\hat{\mu})}\int_0^{\frac{\pi}{2}}H\big(s\sqrt{h_0(\mu)}\sin\theta;\mu\big)\sin\theta d\theta=\int_0^{\frac{\pi}{2}}H\big(\sqrt{h_0(\hat{\mu})}\sin\theta;\hat{\mu}\big)\sin\theta d\theta=:L \tag{21}$$

and that L is a convergent integral. Since the potential systems are admissible, two different situations are considered: either x_ℓ is regular or both x_ℓ and x_r are non-regular. We point out that in the first case the assertion is immediate on $(-\frac{\pi}{2},0)$. Indeed, in this situation the potential family is analytic on $x_\ell(\mu)$ and $V_\mu'(x_\ell(\mu))\neq 0$ for all $\mu\in\Lambda$. Consequently, by Lemma 3.13 the function $(z,\mu)\longmapsto H(z;\mu)$ is continuous on $\{(x,\mu)\in\mathbb{R}\times\Lambda : x\in[-\sqrt{h_0(\mu)},0]\}$. On the other hand, if both x_ℓ and x_r are non-regular, the proof of the assertion on $(-\frac{\pi}{2},0)$ follows in the same way as the assertion on $(0,\frac{\pi}{2})$. Accordingly the result will follow once we prove (21). With this aim in view we claim that, for a given $\varepsilon>0$, there exist positive η, δ and r small enough such that

$$\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}}\left|H(s\sqrt{h_0(\mu)}\sin\theta;\mu)\right|\sin\theta d\theta<\varepsilon \text{ for all } \mu\in B_r(\hat\mu) \text{ and } s\in(1-\delta,1). \tag{22}$$

Here, and in what follows, $B_r(\hat\mu):=\{\mu\in\Lambda;\|\mu-\hat\mu\|\leqslant r\}$. To show this let us note first that $\alpha_r(\mu)$ in condition **(C)** is negative. Indeed, condition (C_2) implies that the limit

$$\lim_{(x,\mu)\to(x_r(\hat\mu),\hat\mu)}\big(g_\mu(x)-\sqrt{h_0(\mu)}\big)\big(x_r(\mu)-x\big)^{-\alpha_r(\mu)}$$

is finite a different from zero. Due to $g_\mu(x)\nearrow\sqrt{h_0(\mu)}$ as x tends to $x_r(\mu)$ we have then $\alpha_r(\hat\mu)<0$. The continuity of $\mu\longmapsto\alpha_r(\mu)$, see Remark 2.12, allows us to suppose $\alpha_r(\mu)<0$ for all $\mu\in B_r(\hat\mu)$. On the other hand, by condition (C_2) and Lemma 3.16, the family $\left\{\frac{g_\mu''g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat\mu$ at $x_r(\mu)$ by $\beta(\mu):=-\frac{3}{2}\alpha_r(\mu)>0$. Moreover, by hypothesis $M(\hat\mu)<1$ so we have $0<\beta(\mu)<1$ for all $\mu\in B_r(\hat\mu)$ considering r smaller if necessary. Therefore, due to the continuity of $\mu\longmapsto x_r(\mu)$, there exist positive C, ξ and r such that, for all $\mu\in B_r(\hat\mu)$,

$$\left|\frac{g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2}\right|\frac{1}{\sqrt{h_0(\mu)-V_\mu(x)}}<\frac{C}{(x_r(\mu)-x)^{\beta(\mu)}} \text{ for all } x\in\big(x_r(\mu)-\xi,x_r(\mu)\big).$$

Therefore

$$\int_{x_r(\mu)-\xi}^{x_r(\mu)}\left|\frac{g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2}\right|\frac{dx}{\sqrt{h_0(\mu)-V_\mu(x)}}<C\frac{\xi^{1-\beta(\mu)}}{1-\beta(\mu)}$$

and so, taking ξ and r smaller if necessary, we can assert that

$$\int_{x_r(\mu)-\xi}^{x_r(\mu)}\left|\frac{g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2}\right|\frac{dx}{\sqrt{h_0(\mu)-V_\mu(x)}}<\varepsilon \text{ for all } \mu\in B_r(\hat\mu).$$

If we perform the change of variable $x=(g_\mu^{-1})(\sqrt{h_0(\mu)}\sin\theta)$ in the integral above, the inequality easily implies that

$$\int_{\frac{\pi}{2}-\hat\eta}^{\frac{\pi}{2}}\left|H(\sqrt{h_0(\mu)}\sin\theta;\mu)\right|\sin\theta d\theta<\varepsilon \text{ for all } \mu\in B_r(\hat\mu), \tag{23}$$

where $\hat\eta:=\frac{\pi}{2}-\max\left\{\arcsin\left(\frac{g_\mu(x_r(\mu)-\xi)}{\sqrt{h_0(\mu)}}\right);\mu\in B_r(\hat\mu)\right\}>0$.

Recall at this point that, by condition (C_1) and taking $r>0$ smaller if necessary, there exists $\bar x\in\mathbb{R}$ such that, for all $\mu\in B_r(\hat\mu)$, it holds $\bar x\in(x_r(\mu)-\xi,x_r(\mu))$ and $\frac{g_\mu''}{(g_\mu')^3}$ is monotonous on $(\bar x,x_r(\mu))$. Since g_μ is a diffeomorphism from $\big(x_\ell(\mu),x_r(\mu)\big)$ to $\big(-\sqrt{h_0(\mu)},\sqrt{h_0(\mu)}\big)$ and $(g_\mu^{-1})''=\frac{-g_\mu''}{(g_\mu')^3}\circ g_\mu^{-1}$, if we set $\bar z:=\max\{g_\mu(\bar x);\mu\in B_r(\hat\mu)\}$, then for all $\mu\in B_r(\hat\mu)$ the function $(g_\mu^{-1})''$ is monotonous on $(\hat z,\sqrt{h_0(\mu)})$. Accordingly, for all $\mu\in B_r(\hat\mu)$, $z\longmapsto|H(z;\mu)|$ is monotonous on $(\hat z,\sqrt{h_0(\mu)})$. Let us take now $\eta\in(0,\hat\eta)$ and $\delta>0$ small enough in order that $\sqrt{h_0(\mu)}s\sin\theta>\hat z$ for all $s\in(1-\delta,1)$, $\theta\in(\frac{\pi}{2}-\eta,\frac{\pi}{2})$ and $\mu\in B_r(\hat\mu)$. If $|H(\,\cdot\,;\mu)|$ is increasing then, for these values, $|H(s\sqrt{h_0(\mu)}\sin\theta;\mu)|<|H(\sqrt{h_0(\mu)}\sin\theta;\mu)|$ and consequently, taking (23) also into account,

$$\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}}\left|H(s\sqrt{h_0(\mu)}\sin\theta;\mu)\right|\sin\theta d\theta\leqslant\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}}\left|H(\sqrt{h_0(\mu)}\sin\theta;\mu)\right|\sin\theta d\theta<\varepsilon$$

for all $s \in (1-\delta, 1)$ and $\mu \in B_r(\hat{\mu})$. Hence the claim follows in this case. Suppose finally that $|H(\,\cdot\,;\mu)|$ is decreasing. Then, for the same values as before, $|H(s\sqrt{h_0(\mu)}\sin\theta;\mu)| < |H\big((1-\delta)\sqrt{h_0(\mu)}\sin\theta;\mu\big)|$, which yields

$$\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \left|H(s\sqrt{h_0(\mu)}\sin\theta;\mu)\right| \sin\theta d\theta \leqslant \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \left|H\big((1-\delta)\sqrt{h_0(\mu)}\sin\theta;\mu\big)\right| \sin\theta d\theta.$$

It is clear that the integral on the right tends to zero as $\eta \longrightarrow 0^+$ uniformly for $\mu \in B_r(\hat{\mu})$ because, by Lemma 3.1 and hypothesis **(H)**, the function $(\theta,\mu) \longmapsto |H\big((1-\delta)\sqrt{h_0(\mu)}\sin\theta;\mu\big)|$ is continuous on $[\frac{\pi}{2}-\eta, \frac{\pi}{2}] \times B_r(\hat{\mu})$. Thus the inequality in (22) is true for $\eta > 0$ small enough and so the claim follows also in this case.

We are now in position to show (21). The fact that L is a convergent integral follows easily by using that, due to the assumption in (C_2) and Lemma 3.16, $\frac{g''_{\hat{\mu}} g_{\hat{\mu}}}{(g'_{\hat{\mu}})^2\sqrt{h_0(\hat{\mu})-V_{\hat{\mu}}}}$ is quantifiable at $x_r(\hat{\mu})$ by $0 < \beta(\hat{\mu}) < 1$. On the other hand,

$$\left|\int_0^{\frac{\pi}{2}} H\big(s\sqrt{h_0(\mu)}\sin\theta;\mu\big)\sin\theta d\theta - L\right| \leqslant \left|\int_0^{\frac{\pi}{2}} \Big(H\big(s\sqrt{h_0(\mu)}\sin\theta;\mu\big) - H\big(s\sqrt{h_0(\hat{\mu})}\sin\theta;\hat{\mu}\big)\Big)\sin\theta d\theta\right| + \left|\int_0^{\frac{\pi}{2}} H\big(s\sqrt{h_0(\hat{\mu})}\sin\theta;\hat{\mu}\big)\sin\theta d\theta - L\right|.$$

Let us denote the first and second summands above by S_1 and S_2, respectively, and consider any $\varepsilon > 0$. Then, by Proposition 3.11, there exists $\delta_2 > 0$ such that $S_2 < \varepsilon/2$ for all $s \in (1-\delta_2, 1)$. In addition, taking any $\eta \in (0, \frac{\pi}{2})$, we get

$$S_1 \leqslant \left|\int_0^{\frac{\pi}{2}-\eta} \Big(H\big(s\sqrt{h_0(\mu)}\sin\theta;\mu\big) - H\big(s\sqrt{h_0(\hat{\mu})}\sin\theta;\hat{\mu}\big)\Big)\sin\theta d\theta\right| + \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \left|H\big(s\sqrt{h_0(\mu)}\sin\theta;\mu\big)\right|\sin\theta d\theta + \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \left|H\big(s\sqrt{h_0(\hat{\mu})}\sin\theta;\hat{\mu}\big)\right|\sin\theta d\theta.$$

Let us denote by S_{11}, S_{12} and S_{13} the first, second and third summands above, respectively. By applying the claim in (22) twice, there exist positive η, δ_1 and r small enough such that $S_{12} + S_{13} < \varepsilon/4$ for all $\mu \in B_r(\hat{\mu})$ and $s \in (1-\delta_1, 1)$. Finally, since the function $(\theta, s, \mu) \longmapsto H\big(s\sqrt{h_0(\mu)}\sin\theta;\mu\big)$ is continuous on $[0, \frac{\pi}{2}-\eta] \times [0,1] \times B_r(\hat{\mu})$, thanks to Lemma 3.1, by making δ_1 and r smaller if necessary, we get that $S_{11} < \varepsilon/4$ for all $\mu \in B_r(\hat{\mu})$ and $s \in (1-\delta_1, 1)$. Hence $S_1 + S_2 < \varepsilon$ for all $\mu \in B_r(\hat{\mu})$ and $s \in (1-\delta, 1)$ with $\delta := \min\{\delta_1, \delta_2\}$. This shows (21) and completes the proof of (a).

Let us prove (b). In this case two different situations can occur: either $m(\hat{\mu}) < 1 < M(\hat{\mu})$ or $M(\hat{\mu}) \geqslant m(\hat{\mu}) > 1$. Let us start proving the result in the first situation. In this case x_ℓ and x_r are both non-regular. Let us fix that $m(\hat{\mu}) = -\frac{3}{2}\alpha_\ell(\hat{\mu})$ and $M(\hat{\mu}) = -\frac{3}{2}\alpha_r(\hat{\mu})$ (the other situation follows exactly in the same way). Lemma 3.16 shows that $M(\hat{\mu})$ and $m(\hat{\mu})$ are the respective quantifiers of family $\left\{\frac{g''_\mu g_\mu}{(g'_\mu)^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$. We split the integration interval of (20) into $(-\frac{\pi}{2}, 0)$ and $(0, \frac{\pi}{2})$ giving rise to two integrals that we denote respectively by $L^-(s;\mu)$ and $L^+(s;\mu)$. On account of $m(\hat{\mu}) < 1$ the same proof as in (a) shows that $L^-(s;\mu)$ converges as $(s,\mu) \to (1,\hat{\mu})$. We claim at this point that $L^+(s;\mu)$ tends to infinity as $s \nearrow 1$ uniformly in a neighbourhood of $\hat{\mu}$. Note that once we show this the result will follow taking into account that $h_0(\mu)$ is a continuous function. In order to show the claim we first note that, on account of condition (C_1), g''_μ is non-vanishing near $x_r(\mu)$. Suppose, for instance, that it is negative. Note that, on account of the assumption in (C_2) and Lemma 3.16, there exist $\bar{x} \in \mathbb{R}$ and $\bar{r} > 0$ verifying

$$\frac{-g''_\mu(x) g_\mu(x)}{(g'_\mu(x))^2\sqrt{h_0(\mu)-V_\mu(x)}} > \frac{C}{(x_r(\mu)-x)^{-\frac{3}{2}\alpha_r(\mu)}} \text{ for all } \mu \in B_{\bar{r}}(\hat{\mu}) \text{ and } x \in \big(\bar{x}, x_r(\mu)\big),$$

where we can take $C > 0$ because $g''_{\hat{\mu}}$ is negative near $x_r(\hat{\mu})$. Moreover, by Lemma 3.2, there exist $\delta > 0$ and

$r\in(0,\bar r)$ such that $\bar x< g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big)<x_r(\mu)$ for all $s\in(1-\delta,1)$ and $\mu\in B_r(\hat\mu)$. Consequently,

$$
\begin{aligned}
L_1^+(s;\mu):=&\int_{\hat x}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)}\frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2-V_\mu(x)}}>\int_{\hat x}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)}\frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)-V_\mu(x)}}\\
>&\int_{\hat x}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)}C(x_r(\mu)-x)^{\frac32\alpha_r(\mu)}dx\\
=&\frac{C}{-(\frac32\alpha_r(\mu)+1)}\Big((x_r(\mu)-g_\mu^{-1}(\sqrt{h_0(\mu)}s))^{\frac32\alpha_r(\mu)+1}-(x_r(\mu)-\hat x)^{\frac32\alpha_r(\mu)+1}\Big).
\end{aligned}
$$

Since $M(\hat\mu)>1$ then $\lim_{\mu\to\hat\mu}\frac32\alpha_r(\mu)+1=\frac32\alpha_r(\hat\mu)+1=1-M(\hat\mu)<0$. Moreover Lemma 3.2 shows $g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big)\longrightarrow x_r(\mu)$ as $s\nearrow 1$ uniformly on $B_r(\hat\mu)$. Therefore the above inequalities show that $L_1^+(s;\mu)$ tends to $+\infty$ as $s\nearrow 1$ uniformly on $B_r(\hat\mu)$ as desired. This shows the claim and so the result follows in the case $M(\hat\mu)>1>m(\hat\mu)$.

Finally let us consider the case when $M(\hat\mu)\geqslant m(\hat\mu)>1$. If x_ℓ is regular then $L^-(s;\mu)$ converges to a number as $(s,\mu)\to(1,\hat\mu)$ and the same procedure before shows that $L^+(s;\mu)$ tends to infinity as s tends to 1 uniformly on $B_r(\hat\mu)$. So the result holds in this case. On the other hand, in case that both x_ℓ and x_r are non-regular, with the same argue we can prove that both $L^-(s;\mu)$ and $L^+(s;\mu)$ tend to infinity as s tends to 1 uniformly on $B_r(\hat\mu)$. Moreover, on account of Lemma 3.10, both integrals tends to $+\infty$. Then, the result follows in this case by additivity. ∎

The next one is the last ingredient for the proof of the main results in the present section.

Proposition 3.18. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying* **(H)** *such that $h_0(\mu)$ is finite and $\mathcal{I}_\mu$ is unbounded. Consider $\hat\mu\in\Lambda$ satisfying condition* **(C)**. *Then $T'(h)$ tends to $\pm\infty$ as $(h,\mu)\longrightarrow(h_0(\hat\mu),\hat\mu)$.*

Proof. The derivative of the expression of the period function in (17) gives

$$
\frac{d}{ds}T_\mu\big(h_0(\mu)s^2\big)=2sh_0(\mu)T_\mu'\big(h_0(\mu)s^2\big)=\sqrt{2h_0(\mu)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g_\mu^{-1})''(\sqrt{h_0(\mu)}s\sin\theta)\sin\theta d\theta.
$$

We split the integration interval into $(-\frac{\pi}{2},0)$ and $(0,\frac{\pi}{2})$, namely $L^-(s;\mu)$ and $L^+(s;\mu)$ respectively. Due to the hypothesis of the endpoints of $\mathcal{I}_\mu$ three different cases can be considered: either $x_\ell(\mu)$ is regular and $x_r\equiv+\infty$, or $x_\ell(\mu)\not\equiv-\infty$ non-regular and $x_r\equiv+\infty$, or $x_\ell\equiv-\infty$ and $x_r\equiv+\infty$. Notice that in the three cases $x_r\equiv+\infty$ so the proof for $L^+(s;\mu)$ will be the same.

Let us consider first that $x_\ell(\mu)$ is regular and $x_r\equiv+\infty$. In this case is clear by Lemma 3.13 that $L^-(s;\mu)$ tends to a number when $(s,\mu)\longrightarrow(1,\hat\mu)$. Then let us focus to show that $L^+(s;\mu)$ tends to infinity uniformly on a neighbourhood of $\hat\mu$. By making the change of variable $x=g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\sin\theta\big)$, we obtain

$$
L^+(s;\mu)=\int_0^{\frac{\pi}{2}}(g_\mu^{-1})''(\sqrt{h_0(\mu)}s\sin\theta)\sin\theta d\theta=\frac{1}{\sqrt{h_0(\mu)}s}\int_0^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)}\frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2-V_\mu(x)}}.
$$

Note that, on account of condition (C_1), g_μ'' must be non-vanishing near $x_r(\mu)$. Suppose, for instance, that it is negative. We claim that $L^+(s;\mu)$ tends to $+\infty$ as $s\nearrow 1$ uniformly on some neighbourhood of $\hat\mu$ (respectively, if g_μ'' is positive near $x_r(\mu)$ then $L^+(s;\mu)$ tends to $-\infty$ uniformly). It is clear due to the continuity of $h_0(\mu)$ that the result will follow in this case once we prove this. With this aim in view note that, on account of the assumption in (C_2), $\alpha_r(\mu)$ is positive. Indeed, we have that the limit

$$
\lim_{(x,\mu)\to(+\infty,\hat\mu)}\frac{g_\mu(x)-\sqrt{h_0(\mu)}}{x^{\alpha_r(\mu)}}
$$

is finite and different from zero. Due to $g_\mu(x)\nearrow\sqrt{h_0(\mu)}$ as x tends to $+\infty$ we have then $\alpha_r(\hat\mu)<0$. The continuity of the map $\mu\longmapsto\alpha_r(\mu)$, see Remark 2.12, allows us to consider $\alpha_r(\mu)<0$ for all $\mu\approx\hat\mu$. On account of Lemma 3.16 we have that the family $\left\{\frac{g_\mu'' g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at infinity by $\beta(\mu) := -\frac{3}{2}\alpha_r(\mu) > 0$. Therefore, since $\lim_{\mu\to\hat{\mu}} x_r(\mu) = +\infty$, there exists $\bar{x}\in\mathbb{R}$ and $\bar{r}>0$ verifying

$$\frac{-g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2\sqrt{h_0(\mu)-V_\mu(x)}} > Cx^{\beta(\mu)} \text{ for all } \mu\in B_{\bar{r}}(\hat{\mu}) \text{ and } x\in\big(\bar{x}, x_r(\mu)\big),$$

where we can take $C>0$ because we assumed $g_{\hat{\mu}}''$ to be negative near $x_r\equiv+\infty$. Moreover, by Lemma 3.2, there exists $\delta>0$ and $r\in(0,\bar{r})$ such that $\bar{x} < g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big) < +\infty$ for all $s\in(1-\delta,1)$ and $\mu\in B_r(\hat{\mu})$. Then

$$\begin{aligned} L_1^+(s;\mu) &:= \int_{\hat{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2-V_\mu(x)}} > \int_{\hat{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)-V_\mu(x)}} \\ &> \int_{\hat{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} Cx^{\beta(\mu)}dx = \frac{C}{\beta(\mu)+1}\left(g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big)^{\beta(\mu)+1} - \hat{x}^{\beta(\mu)+1}\right) \end{aligned}$$

Since $\beta(\mu)>0$ and by Lemma 3.2 we have $g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big)\longrightarrow+\infty$ as $s\nearrow 1$ uniformly on $B_r(\hat{\mu})$, the above inequalities show that $L_1^+(s;\mu)$ tends to $+\infty$ as $s\nearrow 1$ uniformly on $B_r(\hat{\mu})$. On the other hand

$$L_2^+(s;\mu) := \int_0^{\hat{x}} \frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2-V_\mu(x)}}$$

is continuous on $K:=[1-\delta,1]\times B_r(\hat{\mu})$ because, by construction, $\bar{x} < \min\{g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big), (s,\mu)\in K\}$. Accordingly $L_2^+(s;\mu)$ tends to a number as $(s,\mu)\longrightarrow(1,\hat{\mu})$. Therefore, due to $L^+(s;\mu) = \frac{L_1^+(s;\mu)+L_2^+(s;\mu)}{\sqrt{h_0(\mu)}s}$, the claim is true and the result follows in this case.

Now let us consider x_ℓ to be non-regular and finite, and $x_r\equiv+\infty$. In this case $L^-(s;\mu)$ tends to a number as $(s,\mu)\to(1,\hat{\mu})$. We refer the reader to the proof of Proposition 3.17 for the details in this case. On the other hand, we have $L^+(s;\mu)$ tends to infinity uniformly on a neighbourhood of $\hat{\mu}$ as we proved before. Consequently $T_\mu'(h)$ tends to infinity as h approach $h_0(\mu)$ uniformly on a neighbourhood of $\hat{\mu}$.

Finally let us consider $x_\ell\equiv-\infty$ and $x_r\equiv+\infty$. The same proof for $x_r\equiv+\infty$ proves that $L^-(s;\mu)$ tends to infinity uniformly on a neighbourhood of $\hat{\mu}$ in case that $x_\ell\equiv-\infty$. Moreover, Lemma 3.10 shows that both L^- and L^+ tend to $+\infty$. Then, in this case we have that $T_\mu'(h)$ tends to $+\infty$ as $h\nearrow h_0(\mu)$ uniformly on a neighbourhood of $\hat{\mu}$. This shows the validity of the result in this case and completes the proof. ■

Now we are in position to prove a criterion for a parameter to be a local regular value of the period function at the outer boundary of the period annulus.

Theorem C. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying* **(H)** *such that $h_0(\mu)$ is finite and consider $\hat{\mu}\in\Lambda$ satisfying* **(C)**. *Then $\hat{\mu}$ is a local regular value of the period function at the outer boundary if one of the following conditions is verified:*

(*a*) *$\mathcal{I}_\mu$ is bounded, $M(\hat{\mu})<1$ and* $\Delta_1(\hat{\mu}) := \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''\big(\sqrt{h_0(\mu)}\sin\theta\big)\sin\theta d\theta \neq 0.$

(*b*) *$\mathcal{I}_\mu$ is bounded, $M(\hat{\mu})>1$ and $m(\hat{\mu})\neq 1$.*

(*c*) *$\mathcal{I}_\mu$ is unbounded.*

Proof. The assertion in (a) follows from Proposition 3.17, which shows that

$$\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})} T_\mu'(h) = \frac{1}{\sqrt{2h_0(\hat{\mu})}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''\big(\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin\theta d\theta = \frac{\Delta_1(\hat{\mu})}{\sqrt{2h_0(\hat{\mu})}} \neq 0,$$

and then by applying Lemma 3.8. Assertion in (b) follows also from Proposition 3.17. Indeed, this result shows that $\lim_{(h,\mu)\to(h_0(\hat\mu),\hat\mu)} T'_\mu(h) = \pm\infty$. On account of Lemma 2.14 we have that $\lim_{h\to h_0(\mu)} T'_{\hat\mu}(h) = \pm\infty$ uniformly on compact neighbourhood of $\hat\mu$. Then the result follows on account of Lemma 3.8. Finally assertion in (c) follows from Proposition 3.18 using again Lemma 3.8. ∎

The previous result guarantees that $\hat\mu$ is a local regular value of the period function at the outer boundary except for the case in which $\mathcal{I}_\mu$ is bounded and $M(\hat\mu) < 1$ but $\Delta_1(\hat\mu) = 0$. Next result can be applied to bound the criticality in this situation. Since the proof is very similar to the one of Theorem C, for the sake of brevity we do not include it here.

Theorem D. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying* **(H)** *with $h_0(\mu)$ finite and $\mathcal{I}_\mu$ bounded. Suppose that $\hat\mu \in \Lambda$ satisfies the following:*

(i) The family $\left\{\frac{3(g''_\mu)^2 - g'''_\mu g'_\mu}{(g'_\mu)^5}\right\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat\mu$ at the non-regular endpoints of $\mathcal{I}_\mu$.

(ii) The families $\{g_\mu - \sqrt{h_0(\mu)}\}_{\mu\in\Lambda}$, $\{g'_\mu\}_{\mu\in\Lambda}$, $\{g''_\mu\}_{\mu\in\Lambda}$ and $\{g'''_\mu\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat\mu$ at the non-regular endpoints of $\mathcal{I}_\mu$.

Then $\mathrm{Crit}\big((\Pi_{\hat\mu}, X_{\hat\mu}), X_\mu\big) \leqslant 1$ in the following situations:

(a) $M(\hat\mu) < \frac{3}{5}$ and $\Delta_2(\hat\mu) := \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'''\big(\sqrt{h_0(\mu)}\sin\theta\big)\sin^2\theta d\theta \neq 0$.

(b) $M(\hat\mu) \in \left(\frac{3}{5}, 1\right) \setminus \{\frac{3}{4}\}$ and $m(\hat\mu) \notin \{\frac{3}{5}, \frac{3}{4}\}$.

4 Application

This section is devoted to the application of the previous tools to an specific family of potential centers. As we explained in Section 1, we shall study the bifurcation problem at the outer boundary of system (1), which recall that it is given by

$$\begin{cases} \dot x = -y, \\ \dot y = (x+1)^p - (x+1)^q, \end{cases}$$

defined for $x > -1$ and $\mu := (q,p) \in \Lambda = \{(q,p) \in \mathbb{R}^2 : p > q\}$. The corresponding potential function is

$$V_\mu(x) := \int_1^{x+1} (u^p - u^q)du, \tag{24}$$

which satisfies $V_\mu(0) = V'_\mu(0) = 0$ and $V''_\mu(0) > 0$ for all $\mu \in \Lambda$. Clearly the centers are determined by the local minima of $V_\mu(x)$. In this case, for all $\mu \in \Lambda$, the origin is the only center of system (1). Let us define the following three subsets of Λ,

$$\begin{aligned} \Lambda_1 &:= \Lambda \cap \{\mu \in \mathbb{R}^2 : -1 < q < p\}, \\ \Lambda_2 &:= \Lambda \cap \{\mu \in \mathbb{R}^2 : q \leqslant -1 \leqslant p\}, \\ \Lambda_3 &:= \Lambda \cap \{\mu \in \mathbb{R}^2 : q < p < -1\}, \end{aligned}$$

which form a partition of Λ. The projection of the period annulus $\mathscr{P}_\mu$ on the x-axis is $\mathcal{I}_\mu = (-1, \rho(\mu))$ if $\mu \in \Lambda_1$, $\mathcal{I}_\mu = (-1, +\infty)$ if $\mu \in \Lambda_2$ and $\mathcal{I}_\mu = (\rho(\mu), +\infty)$ if $\mu \in \Lambda_3$, where

$$\rho(\mu) := \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}} - 1.$$

Notice that $\rho(\mu)$ is a continuous function in Λ_1 and Λ_3. Both regions correspond to parameters such that the energy level of the outer boundary is finite, more concretely, $h_0(\mu) = \frac{p-q}{(p+1)(q+1)}$, which is clearly continuous. The energy level is $+\infty$ for the parameters in Λ_2.

We consider each region separately and the proof of Theorem E follows from Propositions 4.3, 4.4 and 4.5, which are proved in Sections 4.1, 4.2 and 4.3, respectively.

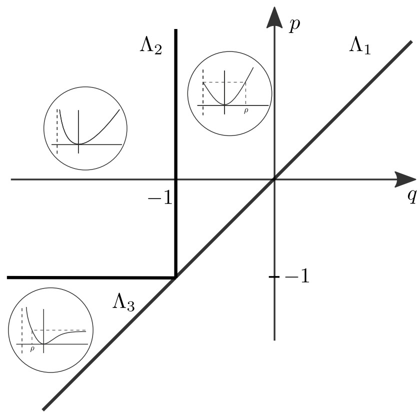

Figure 5: Graph of V_μ for each parameter region.

4.1 Criticality for parameters inside Λ_1

As we already mentioned, $\mathcal{I}_\mu = (-1, \rho(\mu))$ and $h_0(\mu) = \frac{p-q}{(p+1)(q+1)}$ for all $\mu \in \Lambda_1$. Hence, condition **(H)** is satisfied on Λ_1.

Lemma 4.1. *Let X_μ be the potential vector field defined in (1). Then the following statements hold*

(*a*) *X_μ is admissible for all $\mu \in \Lambda_1$ and $x_r(\mu)$ is regular.*

(*b*) *If $\hat{\mu} \in \{(q,p) \in \Lambda_1 : q(2q+1) \neq 0\}$ then $\hat{\mu}$ satisfies condition* **(C)** *and $M(\mu) = \frac{3}{2}(q+1)$.*

(*c*) *If $\hat{\mu} \in \{(q,p) \in \Lambda_1 : q(2q+1)(3q+2) \neq 0\}$ then $\{\frac{3(g''_\mu)^2 - g'''_\mu g'_\mu}{(g'_\mu)^5}\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat{\mu}$ at x_ℓ and $\{g'''_\mu\}_{\mu\in\Lambda}$ is continuous quantifiable in $\hat{\mu}$ at x_ℓ.*

Proof. For proving the first assertion of the lemma let us show that condition (a) of Definition 3.9 is satisfied for $\mu \in \Lambda_1$. Indeed, V_μ is analytic at $x_r(\mu) = \rho(\mu)$ and $V'_\mu(x_r(\mu)) \neq 0$ so $x_r(\mu)$ is regular.

To prove (b) fix $\hat{\mu} = (\hat{q}, \hat{p}) \in \Lambda_1$ with $\hat{q} \neq 0$ and $\hat{q} \neq -1/2$. We shall prove first condition (C_1). That is, the family $\{g''_\mu/(g'_\mu)^3\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat{\mu}$ at $x_\ell = -1$. With this aim in view we shall show that $(g''_\mu/(g'_\mu)^3)' = \frac{g'_\mu g'''_\mu - 3g''_\mu}{(g'_\mu)^4}$ does not accumulate zeroes near $x_\ell = -1$ for $\mu \approx \hat{\mu}$. Since $g'_\mu(x)$ is smooth in $\mathcal{I}_\mu$ it is enough to show that the function $g'_\mu g'''_\mu - 3g''_\mu$ does not accumulate zeroes at $x_\ell = -1$ for $\mu \approx \hat{\mu}$. By definition,

$$g'_\mu g'''_\mu - 3g''_\mu = \frac{3V''_\mu(V'_\mu)^2 + 6(V''_\mu)^2 V_\mu - 2V'''_\mu V'_\mu V_\mu}{8V_\mu^2}.$$

Again, in this case due to the regularity of V_μ in $\mathcal{I}_\mu$, it is enough to prove that the function on the numerator does not accumulate zeroes. Let us denote by P_μ the numerator of the previous expression. Then some computations show

$$P_\mu(x-1) = \frac{a_0(\mu) + a_1(\mu)x^{2(p-q)} + a_2(\mu)x^{1+3p-2q} + a_3(\mu)x^{p-q} + a_4(\mu)x^{1+2p-q} + a_5(\mu)x^{1+p} + a_6(\mu)x^{1+q}}{x^{2-2q}}$$

where $a_i(\mu)$ are continuous rational functions on $\mu = (q,p)$ in Λ_1 that we omitted for the sake of shortness. Since $\mu \in \Lambda_1$ we have $p+1 > q+1 > 0$ so all the exponents on the numerator are positive. Notice that the function $x^{2-2q}P_\mu(x-1)$ is continuous on the variables (x,μ). Therefore we have that

$$\lim_{(x,\mu)\to(0,\hat{\mu})} x^{2-2q}P_\mu(x-1) = a_0(\hat{q},\hat{p}).$$

An easy computation shows that $a_0(\hat{q},\hat{p}) = \frac{2(\hat{p}-\hat{q})\hat{q}(1+2\hat{q})}{(\hat{p}+1)(\hat{q}+1)}$, which is different from zero in the region under consideration. Consequently the function $P_\mu(x)$ does not vanish near $x_\ell = -1$ for all $\mu \approx \hat{\mu}$ and therefore the family $\{(g_\mu)''/(g'_\mu)^3\}$ is uniformly monotonous on $x_\ell = -1$ at $\hat{\mu}$. This proves (C$_1$). Notice that the change of sign in the coefficient $a_0(q,p)$ when $q \approx -\frac{1}{2}$ implies there is no uniformity on the monotonicity in $\hat{q} = -\frac{1}{2}$.

Let us check that $\hat{\mu}$ verifies (C$_2$). On account of the expression in (24) we have that

$$V_\mu(x) = \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + h_0(\mu)$$

and then

$$\lim_{(x,\mu)\to(-1,\hat{\mu})} (h_0(\mu) - V_\mu(x))(x+1)^{-(q+1)} = \lim_{(x,\mu)\to(-1,\hat{\mu})} \frac{1}{q+1} - \frac{(x+1)^{p-q}}{p+1} = \frac{1}{\hat{q}+1} \neq 0.$$

Due to $h_0(\mu) - V_\mu = (\sqrt{h_0(\mu)} - g_\mu)(\sqrt{h_0(\mu)} + g_\mu)$ and $g_\mu(x)$ tends to $\sqrt{h_0(\mu)}$ as x tends to -1 we have then that $\{\sqrt{h_0(\mu)} - g_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $\hat{\mu}$ in $x_\ell = -1$ by $\alpha_\ell(\mu) = -(q+1)$. Moreover, on account of expression in (24), we can easily see that

$$\lim_{(x,\mu)\to(-1,\hat{\mu})} V'_\mu(x)(x+1)^{-q} = \lim_{(x,\mu)\to(-1,\hat{\mu})} 1 - (x+1)^{p-q} = 1,$$

$$\lim_{(x,\mu)\to(-1,\hat{\mu})} V''_\mu(x)(x+1)^{1-q} = \lim_{(x,\mu)\to(-1,\hat{\mu})} q - p(x+1)^{p-q} = \hat{q} \neq 0.$$

Consequently the families $\{V'_\mu\}_{\mu\in\Lambda}$ and $\{V''_\mu\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at $x_\ell = -1$ by $-q$ and $1-q$, respectively. Taking this into account, and using that $g'_\mu = \frac{V'_\mu}{2(V_\mu)^{1/2}}$ and $g''_\mu = \frac{1}{4}\frac{2V''_\mu V_\mu - (V'_\mu)^2}{(V_\mu)^{3/2}}$, one can easily show that the families $\{g'_\mu\}_{\mu\in\Lambda}$ and $\{g''_\mu\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at $x_\ell = -1$ by $-q$ and $1-q$, respectively. This shows that condition (C$_2$) is verified. Finally, since x_r is regular, by definition $M(\mu) = -\frac{3}{2}\alpha_\ell(\mu) = \frac{3}{2}(q+1)$.

Let us prove (c). The assertion concerning the uniform monotonicity of the family $\{\frac{3(g''_\mu)^2 - g'''_\mu g'_\mu}{(g'_\mu)^5}\}_{\mu\in\Lambda}$ follows similarly as the proof we have shown for proving (C$_1$) in (b). In this case we use that

$$\frac{3(g''_\mu)^2 - g'''_\mu g'_\mu}{(g'_\mu)^5} = -\frac{4V_\mu^{\frac{1}{2}}(3V''_\mu(V'_\mu)^2 - 6(V''_\mu)^2 V_\mu + 2V'''_\mu V'_\mu V_\mu)}{(V'_\mu)^5}$$

so we shall proof that the derivative of that function does not accumulate zeroes at $x_\ell = -1$. For the sake of simplicity we omit the computations and we have that

$$\left(\frac{3(g''_\mu)^2 - g'''_\mu g'_\mu}{(g'_\mu)^5}\right)'(x) = \frac{-4\sqrt{V_\mu(x)}(x+1)^{3q-3}}{(p+1)^2(q+1)^2 V'_\mu(x)^5} Q_\mu(x-1)$$

where $Q_\mu(z)$ is a continuous function such that $\lim_{(z,\mu)\to(0,\hat{\mu})} Q_\mu(z) = -24(\hat{p}-\hat{q})^2\hat{q}(\hat{q}+\frac{2}{3})(\hat{q}+\frac{1}{2}) \neq 0$. Therefore, and taking into account the regularity of V_μ and V'_μ we have that the derivative of the family under consideration does not accumulate zeroes at $x_\ell = -1$. Consequently, the family is uniformly monotonous in $\hat{\mu}$ at x_ℓ.

Finally let us prove that $\{g'''_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at x_ℓ. On account of the expression of V_μ in (24) we can easily see that $\{V'''_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at x_ℓ by $2-q$ with limit $\hat{q}(\hat{q}-1)$. Then, using that $\{V'_\mu\}_{\mu\in\Lambda}$ and $\{V''_\mu\}_{\mu\in\Lambda}$ are continuously quantifiable and their quantifiers together with the equality

$$g'''_\mu = \frac{3(V'_\mu)^3 - 6V''_\mu V'_\mu V_\mu + 4V'''_\mu V_\mu^2}{8V_\mu^{\frac{5}{2}}}$$

we have that $\{g'''_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at x_ℓ by $2-q$ as we desired. ■

Proposition 4.2. *Consider the period function T_μ of the center at the origin of system (1) with $(q,p) \in \Lambda_1$. Then the following holds:*

$$(i)\ \lim_{h\to h_0(\mu)} T_\mu(h) = \begin{cases} \sqrt{2\pi}\frac{\sqrt{q+1}}{p-q}\left(\frac{p+1}{q+1}\right)^{\frac{1-q}{2(p-q)}} \frac{\Gamma\left(\frac{1-q}{2(p-q)}\right)}{\Gamma\left(\frac{1+p-q}{2(p-q)}\right)} & \text{if } -1<q<1, \\ +\infty & \text{if } q \geqslant 1. \end{cases}$$

$$(ii)\ \lim_{h\to h_0(\mu)} T'_\mu(h) = \begin{cases} -\sqrt{2\pi}\frac{(p+1)^{\frac{3}{2}}(p+2q+1)}{2(p-q)^2\rho(\mu)^{\frac{3p+1}{2}}} \frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)} & \text{if } -1<q<-\frac{1}{3}, \\ -\infty & \text{if } -\frac{1}{3}\leqslant q<0, \\ +\infty & \text{if } q>0. \end{cases}$$

Proof. Since $\mu \in \Lambda_1$ we have that $h_0(\mu)$ is finite. Taking $(g_\mu(x))^2 = V_\mu(x) = \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + h_0(\mu)$ into account and deriving implicitly it easily follows that $(g_\mu^{-1})'''$ is non-vanishing near the endpoints of $(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)})$. Consequently $(g_\mu^{-1})''$ is monotonous near the endpoints of $(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)})$. Since on the other hand X_μ is admissible thanks to Lemma 4.1, we can apply Corollary 3.12 to conclude that

$$\lim_{h\to h_0(\mu)} T_\mu(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'(\sqrt{h_0(\mu)}\sin\theta)d\theta$$

and

$$\lim_{h\to h_0(\mu)} T'_\mu(h) = \frac{1}{\sqrt{2h_0(\mu)}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}\sin\theta)\sin\theta d\theta,$$

where the improper integral on the right either converges or it tends to infinity. In the first case, if we perform the change of variable $x = g_\mu^{-1}(\sqrt{h_0(\mu)}\sin\theta)$ we have

$$\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'(\sqrt{h_0(\mu)}\sin\theta)d\theta = \sqrt{2}\int_{-1}^{\rho(\mu)} \frac{dx}{\sqrt{h_0(\mu)-V_\mu(x)}}.$$

Then (i) follows by the first assertion on Lemma 4.9 in the Appendix. In the second case, with the same change of variable we have that

$$\frac{1}{\sqrt{2h_0(\mu)}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}\sin\theta)\sin\theta d\theta = \frac{\sqrt{2}}{2h_0(\mu)}\int_{-1}^{\rho(\mu)} \frac{-g''_\mu(x)g_\mu(x)}{g'_\mu(x)^2\sqrt{h_0(\mu)-V_\mu(x)}}.$$

Using that $g_\mu^2 = V_\mu$ it follows that

$$\frac{1}{\sqrt{2h_0(\mu)}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}\sin\theta)\sin\theta d\theta = \frac{\sqrt{2}}{h_0(\mu)}\int_{-1}^{\rho(\mu)} \frac{\frac{1}{2}-\frac{V_\mu(x)V''_\mu(x)}{V'_\mu(x)^2}}{\sqrt{h_0(\mu)-V_\mu(x)}}.$$

Then (ii) follows by the second assertion on Lemma 4.9 in the Appendix. ■

Next result proves Theorem E for the parameters inside Λ_1.

Proposition 4.3. *If $\hat{\mu} = (\hat{q},\hat{p}) \in \{\mu\in\Lambda_1 : q(p+2q+1)(2q+1)(3q+1)\neq 0\}$ then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of system (1). Moreover,*

(a) *If $\hat{\mu} \in \{\mu\in\Lambda_1 : q(p+2q+1)>0, (2q+1)(3q+1)\neq 0\}$ then the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.*

(b) *If $\hat{\mu} \in \{\mu\in\Lambda_1 : q(p+2q+1)<0, (2q+1)(3q+1)\neq 0\}$ then the period function of $X_{\hat{\mu}}$ is decreasing near the outer boundary.*

On the other hand, if $\hat{\mu} \in \{\mu\in\Lambda_1 : q(p+2q+1)=0\}$ then $\hat{\mu}$ is a local bifurcation value of the period function at the outer boundary of system (1). Moreover, if $\hat{\mu} = (\hat{q},-2\hat{q}-1)$ with $\hat{q}\in(-\frac{3}{5},-\frac{1}{3})\setminus\{-\frac{1}{2}\}$, then Crit$((\Pi_{\hat{\mu}},X_{\hat{\mu}}),X_\mu)=1$.

Proof. Consider $\hat{\mu} = (\hat{q}, \hat{p}) \in \{\mu \in \Lambda_1 : q(p+2q+1)(2q+1)(3q+1) \neq 0\}$. On account of Lemma 4.1 we have that the potential family is admissible and that $\hat{\mu}$ satisfies condition **(C)**. Moreover, $M(\hat{\mu}) = \frac{3}{2}(\hat{q}+1)$.

If $\hat{q} > -\frac{1}{3}$ then $M(\hat{\mu}) > 1$ and, by applying Theorem C, $\hat{\mu}$ is a local regular value of the period function at the outer boundary. Moreover Proposition 4.2 shows that if $\hat{q} < 0$ (respectively, $\hat{q} > 0$) then the period function tends to $-\infty$ (respectively, $+\infty$) as $h \longrightarrow h_0(\mu)$. This proves (a) and (b) for $\hat{q} > -\frac{1}{3}$ and also that, by Lemma 3.8, $\{\mu \in \Lambda_1 : q = 0\}$ consists of local bifurcation value of the period function at the outer boundary. On the other hand, if $\hat{q} < -\frac{1}{3}$ then $M(\hat{\mu}) < 1$. In addition, Proposition 4.2 shows that function $\Delta_1(\mu)$ defined in Theorem C is

$$\Delta_1(\mu) = -\sqrt{2\pi}\frac{(p+1)^{\frac{3}{2}}(p+2q+1)}{2(p-q)^2\rho(\mu)^{\frac{3p+1}{2}}}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}.$$

Due to $\hat{q}(\hat{p}+2\hat{q}+1)(2\hat{q}+1)(3\hat{q}+1) \neq 0$, we have $\Delta_1(\hat{\mu}) \neq 0$ so Theorem C guarantees that $\hat{\mu}$ is a local regular value of the period function at the outer boundary. This proves the assertion about the regularity. Moreover, if $\hat{p}+2\hat{q}+1 < 0$ and $\hat{q} < -\frac{1}{3}$, then $\Delta_1(\hat{\mu}) > 0$ whereas if $\hat{p}+2\hat{q}+1 > 0$, then $\Delta_1(\hat{\mu}) < 0$. This proves the assertion concerning the monotonicity of the period function near the outer boundary if $\hat{q} < -\frac{1}{3}$. Due to the change of sign of Δ_1, Lemma 3.8 shows that for all $\hat{\mu} \in \{\mu \in \Lambda_1 : p+2q+1 = 0\}$ we have $\mathrm{Crit}((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu) \geqslant 1$, so they are local bifurcation values of the period function at the outer boundary. Finally, if $\hat{q} \in (-\frac{3}{5}, -\frac{1}{3}) \setminus \{-\frac{1}{2}\}$, then $M(\hat{\mu}) \in (\frac{3}{5}, 1) \setminus \{\frac{3}{4}\}$, so together with Lemma 4.1 we have that $\mathrm{Crit}((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu) = 1$ by Theorem D. ∎

4.2 Criticality for parameters inside Λ_2

Recall that $h_0(\mu) = +\infty$ and $\mathcal{I}_\mu = (-1, +\infty)$ for all $\mu \in \Lambda_2$. We note also that condition **(H)** is not satisfied for $\hat{\mu} \in \{\mu \in \Lambda_2 : (q+1)(p+1) = 0\}$. Indeed, in every neighbourhood $\mathscr{U}$ of $\hat{\mu}$ there exist $\mu_1, \mu_2 \in \mathscr{U}$ such that $h_0(\mu_1)$ is finite and $h_0(\mu_2)$ is infinite. Hence the techniques developed in this paper do not apply for these parameters. The proof of Theorem E on Λ_2 follows from next result.

Proposition 4.4. *Consider $\hat{\mu} = (\hat{q}, \hat{p}) \in \{\mu \in \Lambda_2 : (q+1)(p+1) \neq 0\}$. If $\hat{p} \neq 1$ then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of system* (1)*. Moreover,*

(a) If $\hat{p} < 1$ then the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.

(b) If $\hat{p} > 1$ then the period function of $X_{\hat{\mu}}$ is decreasing near the outer boundary.

On the other hand, if $\hat{p} = 1$ then $\hat{\mu}$ is a local bifurcation value of the period function at the outer boundary of system (1)*. Moreover, if $\hat{p} = 1$ and $\hat{q} < -3$ then* $\mathrm{Crit}((\Pi_\mu, X_{\hat{\mu}}), X_\mu) = 1$.

Proof. First we shall apply Theorem A in order to prove that any $\hat{\mu} = (\hat{q}, \hat{p}) \in \{\mu \in \Lambda_2 : (q+1)(p+1) \neq 0\}$ with $\hat{p} \neq 1$ is a local regular value of the period function at the outer boundary. Since $g_\mu^2 = V_\mu$,

$$\frac{-g_\mu''}{(g_\mu')^3} = 2\frac{(V_\mu')^2 - 2V_\mu V_\mu''}{(V_\mu')^3}.$$

On account of expression in (24), $\{V_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x = \infty$ by $p+1$ with limit $\frac{1}{\hat{p}+1}$. In the same way, $\{V_\mu'\}_{\mu\in\Lambda}$ and $\{V_\mu''\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at $x = \infty$ by p with limit 1 and by $p-1$ with limit $\hat{p}$ respectively. Using this together with the equality above we have that

$$\lim_{(x,\mu)\to(\infty,\hat{\mu})} \frac{-g_\mu''(x)}{g_\mu'(x)^3}\frac{1}{x^{-p}} = \frac{2(1-\hat{p})}{1+\hat{p}}$$

which is different from zero if $\hat{p} \neq 1$. Then, if $\hat{p} \neq 1$, the family $\{-g_\mu''/(g_\mu')^3\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x_r = +\infty$ by $\alpha_r(\mu) = -p$ with limit $a_r = 2(1-\hat{p})/(1+\hat{p})$. Similarly one can prove that $\{-g_\mu''/(g_\mu')^3\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x_\ell = -1$ by $\alpha_\ell(\mu) = q$ with limit $a_\ell = 2(\hat{q}-1)/(\hat{q}+1)$.

On the other hand, taking into account the expression of g_μ, one can prove that $\{g_\mu(x)\}_{\mu\in\Lambda}$ is also continuously quantifiable in $\hat\mu$ at both endpoints of $\mathcal{I}_\mu$, at x_ℓ by $\beta_\ell(\mu) = -\frac{q+1}{2}$ with limit $b_\ell = (-(\hat q+1))^{-1/2}$ and at x_r by $\beta_r(\mu) = \frac{p+1}{2}$ with limit $b_r = (\hat p+1)^{-1/2}$.

According with Theorem A, we have then that the even part of $z(g_\mu^{-1})''(z)$ is continuously quantifiable at infinity by $\gamma(\mu) := 1 + \max\{\frac{-2q}{q+1}, \frac{-2p}{p+1}\} = 1 - \frac{2p}{p+1}$. Moreover, $\gamma(\hat\mu) > -1$ for all $\hat\mu$ under consideration. Therefore, if $\hat\mu = (\hat q, \hat p) \in \{(q,p) \in \Lambda_2 : (q+1)(p+1) \neq 0\}$ with $\hat p \neq 1$, by Theorem A, we have that $\hat\mu$ is a local regular value of the period function at the outer boundary. Particularly, Remark 3.6 shows that in this situation

$$\lim_{(h,\mu)\to(+\infty,\hat\mu)} h^{1-\frac{\gamma(\mu)}{2}} T'_\mu(h) = \Delta_1(\hat\mu) \neq 0$$

where $\Delta_1(\mu) = 2\sqrt{2\pi}(1-p)(1+p)^{-\frac{2p+1}{p+1}} \frac{\Gamma\left(\frac{1}{p+1}\right)}{\Gamma\left(\frac12 + \frac{1}{p+1}\right)}$. Notice that $\Delta_1(\hat\mu) > 0$ if $\hat p < 1$ and $\Delta_1(\hat\mu) < 0$ if $\hat p > 1$. This proves the assertion concerning the monotonicity of the period function near the outer boundary. Moreover, Lemma 3.3 shows in this case that $\mathrm{Crit}((\Pi_\mu, X_{\hat\mu}), X_\mu) \geqslant 1$ if $\hat\mu = (\hat q, 1)$, so we have that $\hat\mu$ is a local bifurcation value of the period annulus at the outer boundary.

Finally let us prove that $\mathrm{Crit}((\Pi_\mu, X_{\hat\mu}), X_\mu) \leqslant 1$ for $\hat\mu = (\hat q, 1)$ with $\hat q < -3$. To this end we shall apply Theorem B taking $\upsilon(\mu) := -\frac{2p}{p+1}$. (This choice is based on Remark 3.7.) Define

$$f_\mu(z) := (g_\mu^{-1})'''(z)z^2 - \upsilon(\mu)(g_\mu^{-1})''(z)z,$$

and then one can verify that

$$f_\mu \circ g_\mu = \frac{2\sqrt{V_\mu}\left((2V_\mu V''_\mu - (V'_\mu)^2)(\upsilon(\mu)(V'_\mu)^2 + 6V_\mu V''_\mu) - 4V_\mu^2 V'_\mu V'''_\mu\right)}{(V'_\mu)^5}.$$

On account of expression in (24), some long and tedious computations show that

$$\lim_{(x,\mu)\to(\infty,\hat\mu)} \frac{(f_\mu\circ g_\mu)(x)}{x^{\frac{-1-3p}{2}}} = \frac{4\hat p(1+3\hat p)(\hat p - \hat q)}{(1+\hat p)^{5/2}(1+\hat q)} = \frac{2\sqrt2(1-\hat q)}{1+\hat q} \neq 0,$$

and

$$\lim_{(x,\mu)\to(-1,\hat\mu)} (f_\mu\circ g_\mu)(x)(x+1)^{\frac{q-1}{2}} = \frac{4(\hat p - \hat q)(\hat q - 1)}{(1+\hat p)(-1-\hat q)^{5/2}} = \frac{-2(1-\hat q)^2}{(-1-\hat q)^{5/2}} \neq 0,$$

where we omit the explicit expression of $f_\mu\circ g_\mu$ for shortness. Therefore, the family $\{(f_\mu\circ g_\mu)(x)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat\mu$ at $x_\ell = -1$ by $\alpha_\ell(\mu) = \frac{q-1}{2}$ and at $x_r = \infty$ by $\alpha_r(\mu) = -\frac{1+3p}{2}$. Accordingly $\xi(\mu) := \max\{\left(\frac{\alpha_\ell}{\beta_\ell}\right)(\mu), \left(\frac{\alpha_r}{\beta_r}\right)(\mu)\} = \max\{\frac{-1-3p}{p+1}, \frac{1-q}{q+1}\} = \frac{1-q}{q+1}$ due to $\hat q < -3$. In addition $\xi(\hat\mu) \in (-2,-1)$. Moreover, by Definition 2.3, we have that the first momentum of the even part of f_μ is

$$\begin{aligned} M_1(\mu) &= \int_{-\infty}^{\infty} f_\mu(z)dz = \int_{-1}^{\infty} f_\mu(g_\mu(x))g'_\mu(x)dx \\ &= \left(x(2+\upsilon(\mu)) - \frac{2(1+\upsilon(\mu))V_\mu(x)}{V'_\mu(x)} - \frac{4V_\mu(x)^2 V''_\mu(x)}{V'_\mu(x)^3} \right)\Bigg|_{-1}^{\infty} = 0 \end{aligned}$$

for all $\mu \in \{(q,p) \in \Lambda_2 : (q+1)(p+1) \neq 0\}$. So, by applying case $(b2)$ of Theorem B we conclude that $\hat\mu$ has criticality at most one. That proves $\mathrm{Crit}((\Pi_\mu, X_{\hat\mu}), X_\mu) = 1$ for $\hat\mu = (\hat q, 1)$ with $\hat q < -3$ as we desired. ■

Theorem B can not be applied to study the criticality of the local bifurcation parameters $\hat\mu = (\hat q, 1)$ with $\hat q \in (-3,-1)$ because $\xi(\hat\mu) = -2$ and $M_1(\hat\mu) = 0$. In this case the result does not hold even in the non-parametric setting, cf. Remark 2.8.

4.3 Criticality for parameters inside Λ_3

For parameters inside Λ_3 we have $\mathcal{I}_\mu = (\rho(\mu), +\infty)$, with $\rho(\mu) = \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}} - 1$, and $h_0(\mu) = \frac{p-q}{(p+1)(q+1)}$. We also point out that condition **(H)** is satisfied on Λ_3. The assertion in Theorem E concerning Λ_3 follows from the next result.

Proposition 4.5. *If $\hat{\mu} \in \Lambda_3$ then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of system* (1)*. Moreover the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.*

This result can be proved by using the techniques developed in Section 3.2. However, we omit the proof because it is a corollary of Theorem A in [15], where the authors prove the (global) monotonicity of the period function for this parameter region.

Appendix

In this Appendix we show some technical results that are needed in the previous proofs. The first result is a uniform Hôpital's Rule. The authors in [17] give a uniform version of this classical result in case that the function on the denominator tends to infinity. Here we adapt their proof to the case in which the numerator and denominator tend to zero.

Proposition 4.6 (Uniform Hôpital's Rule). *Let f_μ and g_μ be two real valued functions defined on an interval (a,b) and depending on a parameter $\mu \in \Lambda \subset \mathbb{R}^d$. Suppose that:*

(a) f_μ and g_μ are differentiable on (a,b),

(b) $g'_\mu(x) \neq 0$ for all $x \in (a,b)$ and $\mu \in \Lambda$,

(c) for all $\mu \in \Lambda$, there exists $L_\mu \in \mathbb{R}$ such that $\lim_{x\to a^+} \frac{f'_\mu(x)}{g'_\mu(x)} = L_\mu$ uniformly on $\mu \in \Lambda$,

(d) $\sup\{|L_\mu|\,;\mu \in \Lambda\} < +\infty$,

(e) there exists $c \in (a,b)$ such that, for each $x \in (a,c)$ we have that

$$\lim_{y\to a^+} \frac{f_\mu(y)}{g_\mu(x)} = 0 \text{ and } \lim_{y\to a^+} \frac{g_\mu(y)}{g_\mu(x)} = 0 \text{ uniformly on } \mu \in \Lambda.$$

Then $\lim_{x\to a^+} \frac{f_\mu(x)}{g_\mu(x)} = L_\mu$ uniformly on $\mu \in \Lambda$.

Proof. Consider a given $\varepsilon > 0$. Setting $M := \sup\{|L_\mu|\,;\mu \in \Lambda\}$, which is well defined by the assumption (d), let us take $\varepsilon_1 := \min\{\frac{\varepsilon}{3+M}, 1\}$. From (c) there exists $\delta > 0$ such that, if $c \in (a, a+\delta)$, then $\left|\frac{f'_\mu(c)}{g'_\mu(c)} - L_\mu\right| < \varepsilon_1$ for all $\mu \in \Lambda$. Let us fix at this point any $x \in (a, a+\delta)$. By the Mean Value Theorem, for each $y \in (a,x)$ there exists $c = c(x,y,\mu) \in (y,x) \subset (a, a+\delta)$ such that $\frac{f_\mu(x)-f_\mu(y)}{g_\mu(x)-g_\mu(y)} = \frac{f'_\mu(c)}{g'_\mu(c)}$. Therefore

$$\left|\frac{\frac{f_\mu(x)}{g_\mu(x)} - \frac{f_\mu(y)}{g_\mu(x)}}{1 - \frac{g_\mu(y)}{g_\mu(x)}} - L_\mu\right| = \left|\frac{f'_\mu(c)}{g'_\mu(c)} - L_\mu\right| < \varepsilon_1. \tag{25}$$

On the other hand, the assumption (e) guarantees that there exists $z_x \in (a,x)$ such that

$$\left|\frac{f_\mu(y)}{g_\mu(x)}\right| < \varepsilon_1 \text{ and } \left|\frac{g_\mu(y)}{g_\mu(x)}\right| < \varepsilon_1 \text{ for all } y \in (a, z_x) \text{ and } \mu \in \Lambda. \tag{26}$$

Note then that $\left|(L_\mu \pm \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)}\right| < (|L_\mu| + \varepsilon_1)\varepsilon_1$ and, accordingly,

$$-(|L_\mu| + \varepsilon_1)\varepsilon_1 < (L_\mu \pm \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)} < (|L_\mu| + \varepsilon_1)\varepsilon_1. \tag{27}$$

The second inequality in (26) shows in particular that $1 - \frac{g_\mu(y)}{g_\mu(x)} > 0$ because $\varepsilon_1 < 1$. Hence, from (25),

$$(-\varepsilon_1 + L_\mu)\left(1 - \frac{g_\mu(y)}{g_\mu(x)}\right) + \frac{f_\mu(y)}{g_\mu(x)} < \frac{f_\mu(x)}{g_\mu(x)} < (\varepsilon_1 + L_\mu)\left(1 - \frac{g_\mu(y)}{g_\mu(x)}\right) + \frac{f_\mu(y)}{g_\mu(x)}.$$

Therefore,

$$-\varepsilon_1 - (L_\mu - \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)} + \frac{f_\mu(y)}{g_\mu(x)} < \frac{f_\mu(x)}{g_\mu(x)} - L_\mu < \varepsilon_1 - (L_\mu + \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)} + \frac{f_\mu(y)}{g_\mu(x)}.$$

From this, on account of (27) and the first inequality in (26), we get that

$$-2\varepsilon_1 - (|L_\mu| + \varepsilon_1)\varepsilon_1 < \frac{f_\mu(x)}{g_\mu(x)} - L_\mu < 2\varepsilon_1 + (|L_\mu| + \varepsilon_1)\varepsilon_1.$$

Accordingly, for all $x \in (a, a+\delta)$ and $\mu \in \Lambda$,

$$\left|\frac{f_\mu(x)}{g_\mu(x)} - L_\mu\right| < \varepsilon_1(2 + |L_\mu| + \varepsilon_1) < \varepsilon_1(3 + |L_\mu|) < \varepsilon_1(3 + M) < \varepsilon,$$

and this proves the result. ∎

Next three lemmas deal with the computation of some integrals used in the proof of Proposition 4.2.

Lemma 4.7. *Let α and β be any complex number with strictly positive real part. Then,*

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \int_0^\infty u^{\alpha-1}(1+u)^{-(\alpha+\beta)}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where Γ denotes the Gamma function.

Proof. See for instance (6.2.1) and (6.2.2) of [1]. ∎

Lemma 4.8. *Let α and β real numbers such that $\alpha+\beta+1 \neq 0$. Then,*

$$\int u^\alpha(u+1)^\beta du = \frac{\beta}{\alpha+\beta+1}\int u^\alpha(u+1)^{\beta-1}du + \frac{1}{\alpha+\beta+1}u^{\alpha+1}(1+u)^\beta.$$

Proof. The result follows from

$$\left(\frac{1}{\alpha+\beta+1}u^{\alpha+1}(u+1)^\beta\right)' = u^\alpha(u+1)^\beta - \frac{\beta}{\alpha+\beta+1}u^\alpha(u+1)^{\beta-1}.$$

∎

Lemma 4.9. *Let $\mu \in \{(q,p) \in \mathbb{R}^2 : p > q > -1\}$. Consider $V_\mu(x) = \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + \frac{p-q}{(p+1)(q+1)}$ and $\rho(\mu) = \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}} - 1$. Then,*

$$(i)\quad \int_{-1}^{\rho(\mu)} \frac{dx}{\sqrt{\frac{p-q}{(p+1)(q+1)} - V_\mu(x)}} = \begin{cases} \sqrt{\pi}\,\frac{\sqrt{q+1}}{p-q}\left(\frac{p+1}{q+1}\right)^{\frac{1-q}{2(p-q)}}\frac{\Gamma\left(\frac{1-q}{2(p-q)}\right)}{\Gamma\left(\frac{1+p-q}{2(p-q)}\right)} & \text{if } -1<q<1,\\ +\infty & \text{if } q \geqslant 1.\end{cases}$$

$$(ii)\quad \int_{-1}^{\rho(\mu)} \frac{\frac{1}{2} - \frac{V_\mu''(x)V_\mu(x)}{V_\mu'(x)^2}}{\sqrt{\frac{p-q}{(p+1)(q+1)} - V_\mu(x)}}dx = \begin{cases} \frac{-\sqrt{\pi}(p+1)^{\frac{1}{2}}(p+2q+1)}{2(p-q)(q+1)\rho(\mu)^{\frac{3p+1}{2}}}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)} & \text{if } -1<q<-\frac{1}{3},\\ -\infty & \text{if } -\frac{1}{3} \leqslant q < 0,\\ +\infty & \text{if } q>0.\end{cases}$$

Proof. Let us prove (i). The improper integral under consideration can be written as

$$\int_{-1}^{\rho(\mu)} \frac{dx}{(x+1)^{\frac{q+1}{2}}\sqrt{\frac{1}{q+1} - \frac{(x+1)^{p-q}}{p+1}}}.$$

In case that $q \geqslant 1$ it is clear that the improper integral is $+\infty$. Let us consider $q < 1$ and let us perform the change of variable $x = \left(\frac{(p+1)(1-u)}{q+1}\right)^{\frac{1}{p-q}} - 1$. Therefore, the improper integral becomes

$$\frac{\sqrt{q+1}}{p-q}\left(\frac{p+1}{q+1}\right)^{\frac{1-q}{2(p-q)}} \int_0^1 u^{-\frac{1}{2}}(1-u)^{\frac{1-2p+q}{2(p-q)}}du.$$

Notice that due to $q < 1$ the integral satisfies assumptions in Lemma 4.7 and so the result follows immediately applying this lemma.

Let us prove (ii). If we denote $\nu_1(x) = -\frac{1}{2} + \left(\frac{V_\mu(x)}{V_\mu'(x)}\right)'$ the improper integral under consideration can be expressed as

$$\int_{-1}^{\rho(\mu)} \frac{\nu_1(x)dx}{\sqrt{h_0(\mu) - V_\mu(x)}}.$$

By substituting the expression of the function V_μ, we have that the improper integral is given by

$$\int_{-1}^{\rho(\mu)} \frac{1}{(x+1)^{\frac{3}{2}(q+1)}}\left(\frac{q(p-q)}{2(p+1)\sqrt{q+1}} + G(x;\mu)\right)dx$$

where $G(x;\mu)$ is a continuous function on $\{(x,\mu) : x \in [-1,\rho(\mu)], \mu \in \Lambda_1\}$ with $G(-1;\mu) = 0$ for all $\mu \in \Lambda_1$. Consequently, the improper integral is $\pm\infty$ in case that $-\frac{1}{3} \leqslant q \neq 0$ and the sign of the infinity is given by the sign of q, so the result holds in this cases. On the other hand, if $q < -\frac{1}{3}$ then the integral converges. Let us assume that $-1 < q < -\frac{1}{3}$ and let us compute the integral. Let us denote for the sake of simplicity

$$\Phi(z;\mu) := \frac{z^{q+1}}{p+1}\left(\frac{p+1}{q+1} - z^{p-q}\right),\ l(z;\mu) := \frac{1}{z^p - z^q},\ h(z;\mu) := -\Phi(z;\mu)^{\frac{1}{2}} + \frac{p-q}{(p+1)(q+1)}\Phi(z;\mu)^{-\frac{1}{2}}.$$

With this notation and considering the expression of V_μ, the improper integral under consideration can be written as follows

$$\int_{-1}^{\rho(\mu)} \frac{\nu_1(x)dx}{\sqrt{h_0(\mu) - V_\mu(x)}} = \lim_{R\to\rho(\mu)+1}\left(\frac{1}{2}\int_0^R \Phi(z;\mu)^{-\frac{1}{2}}dz + \int_0^R l'(z;\mu)h(z;\mu)dz\right).$$

Integrating by parts the second integral it holds that

$$\int_{-1}^{\rho(\mu)} \frac{\nu_1(x)dx}{\sqrt{h_0(\mu) - V_\mu(x)}} = \lim_{R\to\rho(\mu)+1}\left(\frac{1}{2}\int_0^R \Phi(z;\mu)^{-\frac{1}{2}}dz + l(z;\mu)h(z;\mu)|_0^R - \int_0^R l(z;\mu)h'(z;\mu)dz\right).$$

Since $l(z;\mu)h'(z;\mu) = \frac{1}{2}\Phi(z;\mu)^{-\frac{1}{2}} + \frac{1}{2}\frac{p-q}{(p+1)(q+1)}\Phi(z;\mu)^{-\frac{3}{2}}$ and due to $\lim_{z\to 0} l(z;\mu)h(z;\mu) = 0$ we have then

$$\int_{-1}^{\rho(\mu)} \frac{\nu_1(x)dx}{\sqrt{h_0(\mu) - V_\mu(x)}} = \lim_{R\to\rho(\mu)+1}\left(l(R;\mu)h(R;\mu) - \frac{1}{2}\frac{p-q}{(p+1)(q+1)}\int_0^R \Phi(z;\mu)^{-\frac{3}{2}}dz\right). \tag{28}$$

Moreover, let us denote $\lambda := \frac{1}{2}\frac{3p+1}{p-q}$ and perform the change of variable $u = f(z)$ with $f(z) = \frac{z^{p-q}}{\frac{p+1}{q+1} - z^{p-q}}$. We have that

$$\int_0^R \Phi(z;\mu)^{-\frac{3}{2}}dz = \frac{(p+1)^{\frac{3}{2}}}{(p-q)\rho(\mu)^{\frac{3p+1}{2}}}\int_0^{f(R)} u^{\frac{1}{2}-\lambda}(u+1)^{\lambda-1}du.$$

Applying Lemma 4.8 to the above integral and taking into account that $\lim_{u\to 0} 2u^{\frac{3}{2}-\lambda}(u+1)^{\lambda-1} = 0$ since $-1 < q < -\frac{1}{3}$, we have that

$$\int_0^R \Phi(z;\mu)^{-\frac{3}{2}}dz = \frac{(p+1)^{\frac{3}{2}}}{(p-q)\rho(\mu)^{\frac{3p+1}{2}}}\left(2f(R)^{\frac{3}{2}-\lambda}(f(R)+1)^{\lambda-1} + 2(\lambda-1)\int_0^{f(R)}\frac{u^{\frac{1}{2}-\lambda}}{(u+1)^{2-\lambda}}du\right). \tag{29}$$

At this point we claim that

$$\lim_{R\to\rho(\mu)+1}\left(l(R;\mu)h(R;\mu)-\frac{f(R)^{-\frac{3q+1}{2(p-q)}}(f(R)+1)^{\frac{p+2q+1}{2(p-q)}}(p+1)^{\frac{1}{2}}}{(q+1)\rho(\mu)^{\frac{3p+1}{2}}}\right)=0.$$

Indeed, if we substitute $f(R)=\frac{R^{p-q}}{\frac{p+1}{q+1}-R^{p-q}}$ then we have

$$\frac{f(R)^{-\frac{3q+1}{2(p-q)}}(f(R)+1)^{\frac{p+2q+1}{2(p-q)}}(p+1)^{\frac{1}{2}}}{(q+1)\rho(\mu)^{\frac{3p+1}{2}}}=(p+1)^{-\frac{1}{2}}R^{-\frac{3q+1}{2}}\left(\frac{p+1}{q+1}-R^{p-q}\right)^{-\frac{1}{2}},$$

and so using the expressions of $l(R;\mu)$ and $h(R;\mu)$ we can obtain that

$$l(R;\mu)h(R;\mu)-(p+1)^{-\frac{1}{2}}R^{-\frac{3q+1}{2}}\left(\frac{p+1}{q+1}-R^{p-q}\right)^{-\frac{1}{2}}=\frac{-\left(\frac{p+1}{q+1}-R^{p-q}\right)^{-\frac{1}{2}}}{\sqrt{p+1}R^{\frac{q+1}{2}}(R^p-R^q)}$$

which clearly tends to 0 as $R\longrightarrow\rho(\mu)+1=\left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}$, so the claim is proved.

Substituting expression in (29) into the equality in (28) and using the claim we have that

$$\int_{-1}^{\rho(\mu)}\frac{\nu_1(x)dx}{\sqrt{h_0(\mu)-V_\mu(x)}}=\lim_{R\to\rho(\mu)+1}\frac{-(p+1)^{\frac{1}{2}}}{(q+1)\rho(\mu)^{\frac{3p+1}{2}}}(\lambda-1)\int_0^{f(R)}\frac{u^{\frac{1}{2}-\lambda}}{(u+1)^{2-\lambda}}du.$$

Finally, since $\lim_{R\to\rho(\mu)+1}f(R)=+\infty$, using Lemma 4.7 with $\alpha=-\frac{3q+1}{2(p-q)}>0$ and $\beta=\frac{1}{2}>0$, and substituting the value of λ we have that

$$\lim_{R\to\rho(\mu)+1}\frac{-(p+1)^{\frac{1}{2}}}{(q+1)\rho(\mu)^{\frac{3p+1}{2}}}(\lambda-1)\int_0^{f(R)}\frac{u^{\frac{1}{2}-\lambda}}{(u+1)^{2-\lambda}}du=\frac{-(p+1)^{\frac{1}{2}}(p+2q+1)}{2(p-q)(q+1)\rho(\mu)^{\frac{3p+1}{2}}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}.$$

Consequently we obtain that the value of the improper integral is given by

$$\int_{-1}^{\rho(\mu)}\frac{\nu_1(x)dx}{\sqrt{h_0(\mu)-V_\mu(x)}}=\frac{-(p+1)^{\frac{1}{2}}(p+2q+1)}{2(p-q)(q+1)\rho(\mu)^{\frac{3p+1}{2}}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}$$

as we desired. ■

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