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A NOTE ON THE CRITICAL PERIODS OF POTENTIAL SYSTEMS

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In this paper, we consider the planar differential system associated with the potential Hamiltonian $H(x,y) = (1/2)y^2 + V(x)$ where $V(x) = (1/2)x^2 + (a/4)x^4 + (b/6)x^6$ with $b \neq 0$. This family of differential systems always has a center at the origin and, eventually, three other period annuli. We prove that the corresponding period functions can have at most one critical period altogether. More precisely, we show that this critical period is simple and that it corresponds to the period function associated to the center at the origin. To prove the result we use that the period function verifies a Picard–Fuchs equation. Finally we describe the bifurcation diagram of the period function of the center at the origin.

Keywords: Center; period function; critical period; bifurcation.

1. Introduction and Statement of the Result

The study of the period function of centers of planar differential systems has attracted the attention of many authors during the past years. A great effort has been made in order to study the isochronicity and monotonicity problems (see for instance [Cima *et al.*, 1999; Mardešić *et al.*, 1997] and [Chicone, 1987; Zhao, 2002], respectively). In contrast, there are few papers dealing with the qualitative behavior of the period function in a parametrized family of centers. In this setting only the quadratic centers and some families of polynomial potential systems are partially studied. The case in which the potential is given by $V(x) = (1/2)x^2 + (a/3)x^3 + (b/4)x^4$ is solved thanks to the results of Chow and Sanders [1986] and Gavrilov [1993]. We think that the most natural case to study next is the general case of a potential system with two nonlinear monomials, i.e.

$$V(x) = \frac{1}{2}\,x^2 + \frac{a}{m+1}\,x^{m+1} + \frac{b}{n+1}\,x^{n+1},$$

with $m, n \in \mathbb{N}$ and $a^2 + b^2 \neq 0$. Note that this family becomes one-parametric by means of a homothetic transformation (see the final remarks at the end of the paper). This paper is devoted to the case $m = 3$ and $n = 5$. Taking advantage of the symmetry of the potential and using tools similar to those in Gavrilov's paper we are able to describe completely the bifurcation diagram of the period function of this family. We think (see Remark 2.1) that there is little hope of success in applying the same tools to other pairs of monomials.

More concretely, let $V(x) = (1/2)x^2 + (a/4)x^4 + (b/6)x^6$ with $a, b \in \mathbb{R}$ and $b \neq 0$ and consider the