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DYNAMICS OF POLYNOMIAL AND RATIONAL DIFFERENTIAL EQUATION WITH EMPHASIS IN THE HAMILTONIAN CENTERS, RATIONAL POTENTIAL, LIMIT CYCLES AND HOMOGENEOUS SYSTEMS

Thesis submitted to Universidad del Bío-Bío in fulfilment of the requirements for the degree of Doctor en Matemática Aplicada

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ABSTRACT

In this thesis firstly we study the global dynamics of planar vector fields in different situations: (I) the vector field is a homogeneous polynomial of degree four; (II) the vector field is given by a symmetric (with respect to the reflection $(x, y) \to (-x, y)$) polynomial Hamiltonian function of degree five; (III) the vector field is given by a separable polynomial Hamiltonian function of degree five; (IV) the vector field is given by a Hamiltonian mechanic (or simply Hamiltonian) with rational potential. Secondly, we study the existence of limit cycles for a perturbation of one Hamiltonian system with a linear type center and present the symmetry of reflection with respect to the first coordinate (case (V)).

Now, we are going to precise the principal problems covered in this thesis. For (I) we present the study of all the possible phase portraits for polynomial homogeneous systems of degree four, i.e., of systems on the type:

$$\begin{aligned} \dot{x} &= P(x,y) &&= P_{10}x^4 + 4P_{11}x^3y + 6P_{12}x^2y^2 + 4P_{13}xy^3 + P_{14}y^4 \\ \dot{y} &= Q(x,y) &&= Q_{10}x^4 + 4Q_{11}x^3y + 6Q_{12}x^2y^2 + 4Q_{13}xy^3 + Q_{14}y^4. \end{aligned}$$

$P_{ij}, Q_{ij} \in \mathbb{R}$. In order to study the global dynamics we first provide the canonical forms for the homogeneous polynomial of degree five. The principal idea is to study the associated function $F = xQ - yP$ and the invariant straight lines, in order to analize the infinite equilibrium points and their stability, we will see that the infinite equilibrium points determine the phase portraits of the previous systems. Considering the canonical forms we characterize all the phase portraits in the Poincaré disk for all quartic homogeneous polynomial differential systems. More precisely, there are exactly 24 different topological phase portraits for the quartic homogeneous polynomial differential systems.

For the problem (II) we consider a Hamiltonian system of degree four with a linear type center at the origin, so we establish a characterization of this type of Hamiltonian system, related to the canonical forms of degree five mentioned above. More precisely, related with this topic, we study the Hamiltonian system whose Hamiltonian function is

$$H(x,y) = \frac{1}{2}(x^2 + y^2) + ax^4y + bx^2y^3 + cy^5,$$

$a, b, c \in \mathbb{R}$, which possesses a reflection symmetry with respect to the y-axis. We give all the global phase portraits and its bifurcation diagram in function of its parameters.

The case (V) is a natural continuation of the case (II) because we have added

to the Hamiltonian system a polynomial perturbation of the form

$$\begin{aligned}\dot{x} = &\ -y - x^4 - 3bx^2y^2 - 5cy^4 + \sum_{i=1}^{7} \varepsilon^i p_i(x,y),\\ \dot{y} = &\ x + 4x^3y + 2bxy^3 + \sum_{i=1}^{7} \varepsilon^i q_i(x,y),\end{aligned}$$

where the perturbation is in the form

$$\begin{aligned}p_i(x,y) = &\ a_1^i x + a_2^i y + a_3^i x^2 + a_4^i xy + a_5^i y^2 + a_6^i x^3 + a_7^i x^2 y +\\ &\ a_8^i xy^2 + a_9^i y^3 + a_{10}^i x^4 + a_{11}^i x^3 y + a_{12}^i x^2y^2 + a_{13}^i xy^3 + a_{14}^i y^4,\\ q_i(x,y) = &\ b_1^i x + b_2^i y + b_3^i x^2 + b_4^i xy + b_5^i y^2 + b_6^i x^3 + b_7^i x^2 y +\\ &\ b_8^i xy^2 + b_9^i y^3 + b_{10}^i x^4 + b_{11}^i x^3 y + b_{12}^i x^2y^2 + b_{13}^i xy^3 + b_{14}^i y^4.\end{aligned}$$

We study the number of limit cycles bifurcating from the origin, as function of the real parameters b, c, a_k^i and b_k^i for $k = 1, \ldots, 14$. We prove using the averaging theory of order 7, that there are quartic polynomial systems close to these Hamiltonian system having 3 limit cycles bifurcating from the origin.

Case (III) consists in the study of separable Hamiltonian system of degree five with a linear type center whose Hamiltonian function is $H(x,y) = H_1(x) + H_2(y)$, where $H_1(x) = \frac{1}{2}x^2 + \frac{a_3}{3}x^3 + \frac{a_4}{4}x^4 + \frac{a_5}{5}x^5$ and $H_2(y) = \frac{1}{2}y^2 + \frac{b_3}{3}y^3 + \frac{b_4}{4}y^4 + \frac{b_5}{5}y^5$. We describe all the possible phase portraits on the Poincaré disk as function of the six real parameters a_3, a_4, a_5, b_3, b_4 and b_5 with $a_5 b_5 \neq 0$.

Finally, in (IV) we study a special type of Hamiltonian functions which are given by a Mechanic Hamiltonian function, whose potential is a rational function, so the Hamiltonian function has the form $H(x,y) = y^2/2 + P(x)/Q(x)$, $P(x), Q(x) \in \mathbb{R}[x]$ are polynomials, in particular H is the sum of the kinetic energy and a rational potential energy. We present a general result for the dynamics associated to this problem, we provide the normal forms by a suitable μ-symplectic change of variables. Then, the global topological classification of the phase portraits of the Hamiltonian systems associated in the Poincaré disk in the cases where degree$(P) = 0, 1, 2$ and degree$(Q) = 0, 1, 2$ are studied as a function of the parameters that define each polynomial. We extend the study to some particular case when degree$(P) = 0$ and degree$(Q) = 3$. Finally, we show some interesting applications.

Key Words: Hamiltonian System; Linear Type Centers; Phase Portraits; Poincaré disk; Limit Cycles; Homogeneous Polynomial Systems; Mechanic Hamiltonian; Rational Potential; Polynomial Vector Fields; Singularities.

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