Unfoldings of saddle-nodes and their Dulac time

P. Mardešić, D. Marín, M. Saavedra and J. Villadelprat

Institut de Mathématiques de Bourgogne, UFR Sciences et Techniques, Université de Bourgogne, B.P. 47870, 21078 Dijon, France

Departament de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas Universidad de Concepción, Barrio Universitario, Concepción, Casilla 160-C, Chile

Departament d'Enginyeria Informàtica i Matemàtiques, ETSE, Universitat Rovira i Virgili, 43007 Tarragona, Spain

Abstract. In this paper we study unfoldings of saddle-nodes and their Dulac time. By unfolding a saddle-node, saddles and nodes appear. In the first result (Theorem A) we give a uniform asymptotic expansion of the trajectories arriving at the node. Uniformity is with respect to all parameters including the unfolding parameter bringing the node to a saddle-node and a parameter belonging to a space of functions. In the second part, we apply this first result for proving a regularity result (Theorem B) on the Dulac time (time of Dulac map) of an unfolding of a saddle-node. This result is a building block in the study of bifurcations of critical periods in a neighbourhood of a polycycle. Finally, we apply Theorems A and B to the study of critical periods of the Loud family of quadratic centers and we prove that no bifurcation occurs for certain values of the parameters (Theorem C).

1 Introduction and main results

This paper is dedicated to the study of saddle-nodes and their unfoldings in the real plane. Our initial motivation comes from the study of bifurcations of critical periods of quadratic centers, but we think that our results are of more general interest. From the point of view of the study of the period function, the most interesting stratum of quadratic centers is given by the Loud family

$$\begin{cases} \dot{u} = -v + uv, \\ \dot{v} = u + Du^2 + Fv^2, \end{cases} \tag{1}$$

which is Darboux integrable (see Appendix B for a precise definition). Compactifying $\mathbb{R}^2$ to the Poincaré disc, the boundary of the period annulus of the center has two connected components, the center itself and a polycycle. We call them the inner and outer boundary of the period annulus, respectively. In [6], we described one part of the bifurcations of local critical periods from the outer boundary in this family, but many claims remained conjectural, see Figure 1. In particular, the study of the segment $\{D \in (-1, 0), F = 1\}$ requires a theoretical result about the local time function of such a family in a neighbourhood of a saddle-node appearing at infinity. There, the center is bounded in the Poincaré disc by a symmetric polycycle

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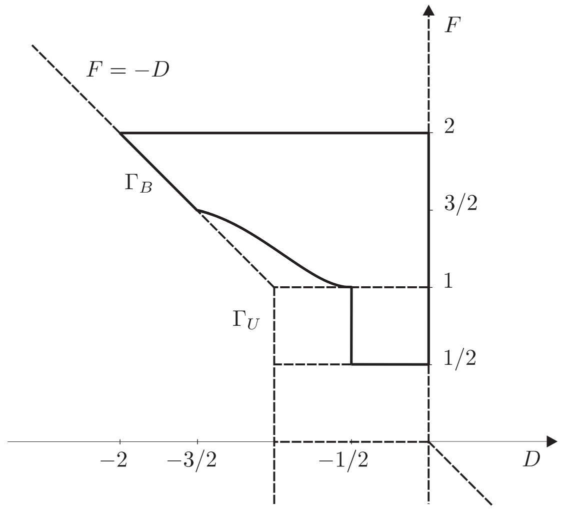


Figure 1: Bifurcation diagram of the period function of (1) at the outer boundary according to Theorem A in [6]. More precisely, $\mathbb{R}^2 \setminus \{\Gamma_B \cup \Gamma_U\}$ corresponds to local regular values, whereas Γ_B are local bifurcation values. The results in that paper did not allow us to determine the character of the parameters in the dotted curve Γ_U.

and, crossing the line $F = 1$, an unfolding of a saddle-node with poles along the line at infinity occurs, see Figure 2. In Theorem C, we prove that no bifurcation of critical periods occurs for the values of the parameter in $\{D \in (-1, 0) \setminus \{-\frac{1}{2}\}, F = 1\}$, corresponding to saddle-nodes. The fundamental tool to obtain this result is an asymptotic expansion of the period function. To this end, taking advantage of the symmetry of the differential system (1), it suffices to study half of the period and then the essential part of the period is given by the Dulac time in a neighbourhood of an unfolding of a saddle-node. By Dulac time we mean the time that a trajectory spends for going between two given transverse sections to the separatrices of a hyperbolic sector (see Figure 4), i.e. the time associated to the Dulac map (see for instance [9] and references therein). By using the Darboux first integral and introducing an auxiliary parameter $\varepsilon = 2(F - 1)$, we will see in the proof of Theorem C that by a local change of coordinates this saddle-node unfolding can be brought to the form

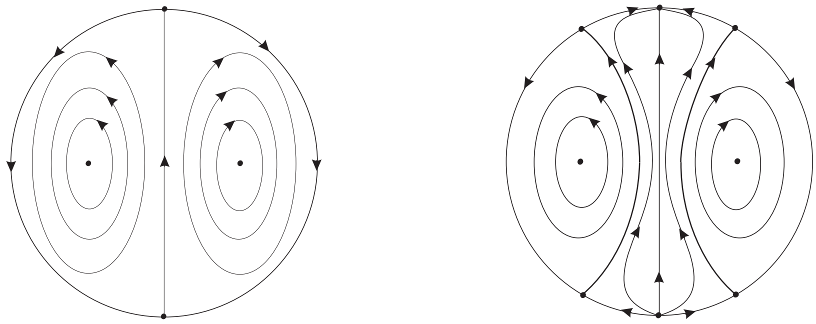
$$\frac{1}{yU(x, y)}\big(x(x^2 - \varepsilon)\partial_x - (2F - x^2)y\partial_y\big), \tag{2}$$

where $y = 0$ corresponds to the line at infinity in (1). More generally, in Theorem B we obtain an asymptotic expansion, uniform with respect to the parameters, of the Dulac time of a saddle-node unfolding of the following type:

$$\frac{1}{yU(x, y)}\big(P_\varepsilon(x)\partial_x - V(x)y\partial_y\big), \tag{3}$$

where P_ε, U and V are analytic functions to be described later. In fact, we will see in Appendix B that any saddle-node unfolding which is locally Darboux integrable is orbitally analytically equivalent to (3). In the applications, the pole along the center manifold $y = 0$ will correspond to the line at infinity.

To prove Theorem B we need to develop some results that in principle have nothing to do with the Dulac time. Since we think that they are interesting on its own we state them separately as Theorem A. For a better understanding of the statement of Theorem A, we briefly outline without technicalities the underlying ideas that lead to the proof of Theorem B. For simplicity in the exposition let us consider (3)



$F \leqslant 1$ $\qquad\qquad$ $F > 1$

Figure 2: Phase portrait of the Loud family (1), with $D \in (-1, 0)$ and $F > 0$ in the Poincaré disc, where the vertical invariant straight line is $x = 1$.

with $\varepsilon = 0$ and suppose that the origin is a saddle-node with a hyperbolic sector in the first quadrant as shown in the phase portrait on the left in Figure 3. We will see in Section 3 that its Dulac time T between $\{y = 1\}$ and $\{x = 1\}$ can be written as a convergent series $T = \sum_{n \geqslant 1} T_n$, where each term T_n is in turn the Dulac time (between the same transverse sections) of

$$\frac{1}{y^n U_n(x)} \big(P_0(x)\partial_x - V(x)y\partial_y\big) \tag{4}$$

for some analytic function U_n. Note that $T_n(s) = \int_s^1 \frac{U_n(x)y^n(x;s)}{P_0(x)}\, dx$, where $y(x; s) = \exp(-\int_s^x \frac{V(\xi)}{P_0(\xi)}\, d\xi)$ is the trajectory of (4) passing through the point $(s, 1)$. A computation, see Section 3, shows that $y = T_n(x)$ is a *trajectory* of the vector field

$$P_0(x)\partial_x + \big(nV(x)y - U_n(x)\big)\partial_y \tag{5}$$

arriving backward in time to the saddle-node at $\left(0, \frac{U_n(0)}{nV(0)}\right)$ through the parabolic sector in $x \geqslant 0$, see the phase portrait on the right in Figure 3. This is the key point and explains why beginning with the problem of computing the *Dulac time* associated to a *hyperbolic sector*, we end up studying the *trajectories* arriving through a *parabolic sector*. Of course when $\varepsilon \neq 0$ the saddle-node bifurcation occurs and we turn to study the Dulac time of a hyperbolic sector in a saddle by considering the trajectories arriving through a parabolic sector to a node. This is delicate because the uniformity with respect to parameters, in particular $\varepsilon \approx 0$, is essential for the applications. The framework in both problems is the same, an unfolding of saddle-node, but we pay attention to different objects. Thus, although our initial motivation was the "temporal result" stated in Theorem B concerning the Dulac time of an unfolding of a saddle-node, as a byproduct of its proof we obtain the "orbital result" given in Theorem A concerning the trajectories arriving through a parabolic sector in an unfolding of a saddle-node. Next, we present the unfolding of saddle-node that we consider in Theorem A, see the family (6) below, but let us advance that our approach to prove Theorem B, writing $T = \sum_{n \geqslant 1} T_n$, forces that the real parameter λ has to be unbounded and that U must be a functional parameter inside a Banach space. They will play, respectively, the role of n and U_n in (5) when we prove Theorem B by applying Theorem A and, once again, the uniformity with respect to these parameters will be crucial.

In order to present our main results properly, we fix $\mu \in \mathbb{N}$ and we consider the following unfolding of a saddle-node

$$X = P_\varepsilon(x)\partial_x + \big(\lambda V_a(x)y - U(x)\big)\partial_y, \tag{6}$$

parametrized by $(\varepsilon, a, \lambda, U)$, with $\varepsilon \approx 0$, a in an open subset A of $\mathbb{R}^\alpha$, $\lambda > 0$, $U \in \mathscr{U}$ and

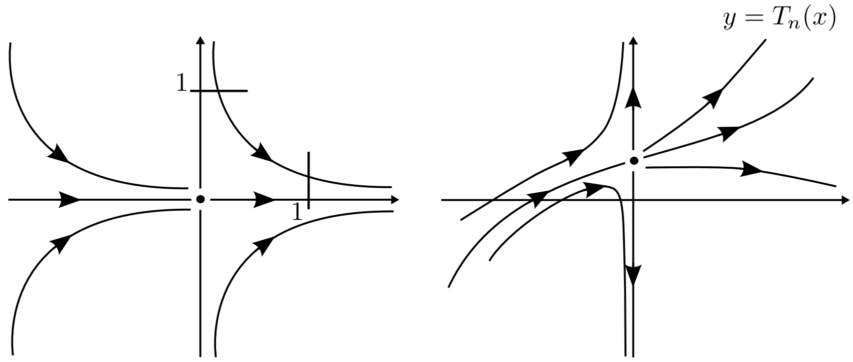


Figure 3: On the left, transverse sections associated to the Dulac time T_n for system (4) and, on the right, trajectory $y = T_n(x)$ arriving backward in time to the saddle-node for system (5).

- $P_\varepsilon(x) = P(x;\varepsilon)$ is an analytic function in (x,ε), for $|x| \leqslant r$, such that $P_0(x)$ has a zero of order $\mu + 1 \geqslant 2$ at $x = 0$;
- $V_a(x)$ is analytic in (x,a), for $|x| \leqslant r$, with $V_a(0) = 1$, for all $a \in A$;
- $\mathscr{U}$ is the space of series $U(x) = \sum_{j\geqslant 0} u_j x^j \in \mathbb{R}\{x\}$, with convergence radius greater than $r > 0$.

By rescaling, we assume that $r = 1$ and $V_a(x) > 0$, for $|x| \leqslant 1$, for all $a \in A$. We endow $\mathscr{U}$ with the norm $\|U\| := \sum_{j\geqslant 0} |u_j|$ and with this norm it becomes a Banach space. We denote $\mathscr{U}_1 := \{U \in \mathscr{U} \,:\, \|U\| \leqslant 1\}$.

By Weierstrass preparation theorem and rescaling, we can assume that $P_\varepsilon(x)$ is a polynomial of degree $\mu + 1$ in x, with $P_0(x) = x^{\mu+1}$. The reason for not including the parameter λ into a is that it will vary in an unbounded interval. As we explained before, this will play a key role in the proof of Theorem B. Moreover, the limit case $\lambda = \infty$ corresponds to a singular deformation or slow-fast system, which is also of independent interest. It is also worth mentioning that, although the unfolding (6) is not necessarily locally Darboux integrable, it is always local Liouville integrable (see Remark 4.4 in Appendix B).

Notice that the singularity $(x,y) = (0, U(0)/\lambda)$ is a saddle-node of $X|_{\varepsilon=0}$, whose (real) parabolic sector is contained in the half plane $x \geqslant 0$. We will assume that $P_\varepsilon(x)$ has a real root for $\varepsilon \approx 0$. In what follows, ϑ_ε will denote the biggest real root of $P_\varepsilon(x)$. As it will be clear in a moment, our results refer to this root, and the reason for choosing this one among the others is because we can approach it from the right inside a parabolic sector that does not shrink as ε tends to zero. In the study of bifurcations, having uniformity on the parameters is crucial and, with respect to ε, this only makes sense by approaching from the right to ϑ_ε. Moreover, this is the only relevant situation in the study of the period function near the outer boundary of the period annulus. In the sequel, we will assume

(H0) $P'_\varepsilon(\vartheta_\varepsilon) > 0$, for $\varepsilon \not\approx 0$, so that the singular point $(x,y) = \left(\vartheta_\varepsilon, \frac{U(\vartheta_\varepsilon)}{\lambda V_a(\vartheta_\varepsilon)}\right)$ is a node of X.

The polynomial $P_\varepsilon(x)$ need not be irreducible. We identify the two branches that contain the root $x = \vartheta_\varepsilon$, for $\varepsilon \geqslant 0$ and $\varepsilon \leqslant 0$, and we apply Puiseux theorem to each one, obtaining $\rho_\pm \in \mathbb{N}$ and analytic functions $\sigma_\pm(z) \in \mathbb{R}\{z\}$, such that

$$\vartheta_\varepsilon = \begin{cases} \sigma_-\left((-\varepsilon)^{1/\rho_-}\right), & \text{if } \varepsilon \leqslant 0, \\ \sigma_+\left((+\varepsilon)^{1/\rho_+}\right), & \text{if } \varepsilon \geqslant 0. \end{cases} \tag{7}$$

Note that $\sigma_\pm(0) = 0$, because ϑ_ε tends to zero as $\varepsilon \to 0$. This gives the continuity of the function ϑ_ε. Note that this function in general is not analytic at $\varepsilon = 0$, even though σ_- and σ_+ are. In our first result, Theorem A, we treat the unfolding (6), as $\varepsilon \to 0^+$, or $\varepsilon \to 0^-$. Since the substitution $\varepsilon \longmapsto -\varepsilon$ interchanges both situations, we will restrict to the case $\varepsilon \geqslant 0$ and, in what follows, when there is no risk of confusion, we will omit the subscript $+$, for the sake of shortness.

Besides the natural assumption (H0), we need to impose two technical conditions on $P_\varepsilon(x) = P(x;\varepsilon)$. In order to state them precisely, we introduce the function

$$\mathcal{Q}(s,\varepsilon) := \frac{P\big(s+\sigma(\varepsilon);\varepsilon^\rho\big)}{s}, \tag{8}$$

which is analytic at $(s,\varepsilon) = (0,0)$ and polynomial in s. Moreover, $\mathcal{Q}(s,0) = s^\mu$ and on account of (H0), $\mathcal{Q}(0,\varepsilon) = \chi\,\varepsilon^\nu + \ldots$, with $\chi > 0$, for some $\nu \in \mathbb{N}$. Taking this notation into account, the aforementioned assumptions are the following:

(H1) The Newton's diagram of $\mathcal{Q}(s,\varepsilon)$ has only one compact side (connecting the endpoints $(\mu,0)$ and $(0,\nu)$), i.e. $\mathcal{Q}$ admits a Taylor's expansion of the form

$$\mathcal{Q}(s,\varepsilon) = \sum_{\frac{i}{\mu}+\frac{j}{\nu}\geqslant 1} q_{ij}s^i\varepsilon^j.$$

(H2) The principal (μ,ν)-quasi-homogeneous part of $\mathcal{Q}(s,\varepsilon)$ is positive definite on the first quadrant, i.e.

$$\sum_{\frac{i}{\mu}+\frac{j}{\nu}=1} q_{ij}\sin^i\theta\cos^j\theta > 0, \text{ for all } \theta \in \left[0,\frac{\pi}{2}\right].$$

Notice that (H2) implies (H0) because $P'_\varepsilon(\vartheta_\varepsilon) = \mathcal{Q}(0,\varepsilon)$. On the other hand, if $\gcd(\mu,\nu) = 1$, then (H1) implies (H2). However, the last implication does not hold in general, as the following example shows.

Example 1.1. If $P_\varepsilon(x) = x((x-\varepsilon)^2+\varepsilon^4)$, then $\vartheta_\varepsilon \equiv 0$, $\mathcal{Q}(x,\varepsilon) = (s-\varepsilon)^2+\varepsilon^4 = s^2-2s\,\varepsilon+\varepsilon^2+\varepsilon^4$ and $\mu = \nu = 2$. One can easily show that P_ε satisfies (H0) and (H1), but it does not satisfy (H2). □

Let $y(x) = y(x;x_0,y_0,\varepsilon,a,\lambda,U)$ be the trajectory of (6), i.e. the solution of the linear differential equation

$$P_\varepsilon(x)y'(x) = \lambda V_a(x)y(x) - U(x), \tag{9}$$

with initial condition $y(x_0) = y_0$. We have $y(x) = D(x)\frac{y_0}{D(x_0)} + y_L(x)$, where

$$D(x) = D(x;\varepsilon,a,\lambda) := \exp\left(\lambda\int_1^x \frac{V_a(s)}{P_\varepsilon(s)}ds\right)$$

and

$$y_L(x;x_0,\varepsilon,a,\lambda,U) := D(x;\varepsilon,a,\lambda)\int_x^{x_0}\frac{U(s)}{P_\varepsilon(s)}\frac{ds}{D(s;\varepsilon,a,\lambda)}.$$

Here $D(x)$ is a fundamental solution of the homogeneous equation and it coincides with the Dulac map of the saddle point $(x,y) = (\vartheta_\varepsilon,0)$ of the vector field $P_\varepsilon(x)\partial_x - \lambda V_a(x)y\partial_y$, for $x \geqslant \vartheta_\varepsilon$. Moreover, $y_L(x)$ is the particular solution with initial condition $y_0 = 0$ and it depends linearly on $U \in \mathscr{U}$. We are now in position to state our first main result, which describes how the trajectories of (6) arrive at the node $(x,y) = \left(\vartheta_\varepsilon, \frac{U(\vartheta_\varepsilon)}{\lambda V_a(\vartheta_\varepsilon}\right)$ given by hypothesis (H0). For convenience, in its statement we use the differential operator

$$\Theta_\lambda = \frac{1}{\lambda}s\partial_s. \tag{10}$$

Theorem A. *Let us consider the saddle-node unfolding given in* (6)*, with* $\varepsilon \geqslant 0$*. Assume that* $P_\varepsilon(x)$ *satisfies the hypothesis (H1) and (H2). Then, there exist functions* $c_j(\varepsilon,\lambda,a,U)$*,* $j \in \mathbb{Z}^+$*, satisfying that, for each* $\ell, k \in \mathbb{Z}^+$*,* $\lambda_0 > 0$ *and every compact set* $K_a \subset A$*, there exists* $\varepsilon_0 > 0$*, such that* $c_0,\ldots,c_\ell$ *are analytic on* $\mathcal{A} := [0,\varepsilon_0]\times K_a\times[\lambda_0,\infty)$ *and are uniformly bounded linear operators on* $\mathscr{U}$ *and the following assertions hold:*

(a) for every compact set $K_x \subset (0,1]$, the particular solution y_L of (9) *is of the form*

$$y_L(s+\vartheta_\varepsilon; x_0, \varepsilon, a, \lambda, U) = \sum_{j=0}^{\ell} c_j(\varepsilon^{1/\rho}, a, \lambda, U)s^j + s^\ell h_\ell(s; x_0, \varepsilon, a, \lambda, U),$$

where $\Theta_\lambda^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $K_x \times \mathcal{A} \times \mathscr{U}_1$, for $r = 0, 1, \ldots, k$;

(b) the fundamental homogeneous solution of (9) *is of the form $D(s+\vartheta_\varepsilon; \varepsilon, a, \lambda) = s^\ell h_\ell(s; \varepsilon, a, \lambda)$, where $\Theta_\lambda^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $\mathcal{A}$, for $r = 0, 1, \ldots, k$.*

Theorem A can be compared to the results [10] and [11] of Rousseau and Teyssier but important differences exist. Both studies deal with unfoldings of saddle-nodes. Rousseau and Teyssier deal with the complex foliation, whereas our study is essentially real. They construct what they call *squid sectors* on which, by a holomorphic change of coordinates, the vector field can be brought to a model and give moduli of analytic classification in terms of comparison of these normalizing coordinates and the period functions on asymptotic cycles giving the temporal part of the moduli. Their model equation is like our equation (6), but with $U = 0$. The real interval $[\vartheta_\varepsilon, x_0]$, which we study, would belong to one of their squid sectors, having the node in its boundary. Our study gives the asymptotic expansion of the solutions at the boundary of such a squid sector with a good uniform control together with all derivatives of the remainder term. It is algorithmic. We think that it cannot be obtained from the results in [11]. Note that requiring the uniform flatness property of a remainder term has proved its efficiency in studying the cyclicity (creation of cycles) and their bifurcations from hyperbolic polycyles, see e.g. [8]. It is also the right condition for studying the bifurcation of critical periods from monodromic polycycles.

A specific situation which will be useful for further applications is the following:

$$P_\varepsilon(x) = x(x^\mu - \varepsilon) \text{ and } U(x) = x^m \bar{U}(x), \text{ with } m \in \mathbb{Z}^+. \tag{11}$$

In this case we have

$$\vartheta_\varepsilon = \begin{cases} 0, & \text{if } \varepsilon \leqslant 0, \\ \varepsilon^{1/\mu}, & \text{if } \varepsilon \geqslant 0. \end{cases} \tag{12}$$

Our next main result follows almost directly by applying twice Theorem A. We point out that it deals with both cases $\varepsilon \geqslant 0$ and $\varepsilon \leqslant 0$.

Corollary A. *Consider the saddle-node unfolding given in* (6)*, taking the functions given in* (11) *and setting ϑ_ε, as in* (12)*. Then there exist functions $c_j(\varepsilon, \lambda, a, U)$, $j \in \mathbb{Z}^+$, satisfying that, for each $\ell, k \in \mathbb{Z}^+$, $\lambda_0 > 0$ and every compact set $K_a \subset A$, there exists $\varepsilon_0 > 0$ such that $c_0, \ldots, c_\ell$ are continuous on $\mathcal{A} := [-\varepsilon_0, \varepsilon_0] \times K_a \times [\lambda_0, \infty)$ and are uniformly bounded linear operators on $\mathscr{U}$, with $c_j(\varepsilon, a, \lambda, U) = 0$, for $\varepsilon \leqslant 0$ and $j = 0, 1, \ldots, m-1$; and such that the following assertions hold:*

(a) for every compact set $K_x \subset (0,1]$, the particular solution y_L of (9) *is of the form*

$$y_L(s+\vartheta_\varepsilon; x_0, \varepsilon, a, \lambda, U) = \sum_{j=0}^{\ell} c_j(\varepsilon, a, \lambda, U)s^j + s^\ell h_\ell(s; x_0, \varepsilon, a, \lambda, U),$$

where $\Theta_\lambda^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $K_x \times \mathcal{A} \times \mathscr{U}_1$, for $r = 0, 1, \ldots, k$;

(b) the fundamental homogeneous solution of (9) *is of the form $D(s+\vartheta_\varepsilon; \varepsilon, a, \lambda) = s^\ell h_\ell(s; \varepsilon, a, \lambda)$, where $\Theta_\lambda^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $\mathcal{A}$, for $r = 0, 1, \ldots, k$.*

It is worth to notice that the case $\mu = 1$, $\varepsilon = 0$, $\lambda = 1$, $V_a \equiv 1$, $m = 1$ and $\bar{U} \equiv 1$ in the above corollary corresponds to the classical Euler equation $x^2\partial_x + (y+x)\partial_y$, having an irregular singular point at the origin whose center manifold has a divergent asymptotic expansion $y(x) = -\sum_{n\geq 1}(n-1)!\, x^n$ at $x = 0$.

Before stating properly the second main result of the paper let us recall that the general setting is the study of the period function of a family of polynomial centers in the plane. It is well-known that, by blowing-ups, any singularity of a vector field reduces to simple singularities or saddle-nodes, see for instance [13]. Hence, the period function around any monodromic polycycle can always be expressed as the sum of Dulac times of saddle or saddle-node singularities, composed by their corresponding Dulac maps. Therefore, local Dulac time of saddles or saddle-nodes in the finite plane or at infinity can be thought of as the basic building blocks in the study of the period function near the outer boundary of a period annulus. In [5] and [7], we deal with orbitally linearizable and resonant saddles, respectively. In this paper we consider the remaining case: saddle-node singularities. Since the Dulac time and its derivative of a singularity in the finite plane tends to infinity, the interesting situation occurs when there are vertices of the polycycle bounding the period annulus that belong to the divisor at infinity obtained by desingularization. We study here the Dulac time of an unfolding of a saddle-node at infinity. Generically, the hyperbolic sector of the saddle-node belonging to the polycycle bounding the period annulus is deformed to a hyperbolic sector of a saddle point. This saddle point either remains at infinity or comes to the finite plane in the unfolding. In the second situation there is a superposition of two different geometric phenomena. For this reason we study the first case. The simplest way to assure this situation is by requiring that the line at infinity is the center manifold. This requirement, together with the considerations about the Darboux integrability explained before (see Proposition 4.5), motivates us to consider in the temporal setting the saddle-node unfolding (3). Next we rewrite it for the reader's convenience, making explicit the dependence on an auxiliary parameter $a \in A \subset \mathbb{R}^\alpha$:

$$\frac{1}{yU_a(x,y)}\left(P_\varepsilon(x)\partial_x - V_a(x)y\partial_y\right). \tag{13}$$

Without loss of generality, we assume that $U_a(x,y) \not\equiv 0$ has an absolutely convergent Taylor series at $(x,y) = (0,0)$ on $|x|, |y| \leqslant 1$, and that $V_a(x)$ is an analytic function on $|x| \leqslant 1$, with $V_a(0) > 0$, for all $a \in A$. Notice that under assumption (H0), the point $(\vartheta_\varepsilon, 0)$, where ϑ_ε is the biggest root of $P_\varepsilon(x)$, is now a *saddle* of the differential system (13), for $\varepsilon \approx 0$. In these local coordinates, the period annulus is in the quadrant $y \geqslant 0$ and $x \geqslant \vartheta_\varepsilon$. In the statement of our next result, $\mathcal{T}(s;\varepsilon,a)$ is the Dulac time of the saddle-node unfolding (13) between the transverse sections $\{y = 1\}$ and $\{x = 1\}$, i.e. it is the time that the trajectory starting at $(s + \vartheta_\varepsilon, 1)$ spends to arrive to $\{x = 1\}$. We also use $\Theta = \Theta_1$, see (10), for shortness.

Theorem B. *Let us consider the Dulac time $\mathcal{T}(s;\varepsilon,a)$ of the saddle-node unfolding* (13)*, with $\varepsilon \geqslant 0$. Assume that $P_\varepsilon(x)$ satisfies conditions (H1) and (H2). Then there exist functions $c_j(\varepsilon,a)$, $j \in \mathbb{Z}^+$, satisfying that, for each $\ell, k \in \mathbb{Z}^+$ and every compact set $K_a \subset A$, there exists $\varepsilon_0 > 0$ such that $c_0, \ldots, c_\ell$ are analytic on $[0,\varepsilon_0] \times K_a$; and the Dulac time can be written as*

$$\mathcal{T}(s;\varepsilon,a) = \sum_{j=0}^{\ell} c_j(\varepsilon^{1/\rho},a)s^j + s^\ell h_\ell(s;\varepsilon,a),$$

with $\Theta^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $[0,\varepsilon_0] \times K_a$, for $r = 0, 1, \ldots, k$.

As we already did in Corollary A, we particularize the unfolding (13) considered in Theorem B by taking

$$P_\varepsilon(x) = x(x^\mu - \varepsilon) \text{ and } U_a(x,y) = x^m \bar{U}_a(x,y), \tag{14}$$

where $m \in \mathbb{Z}^+$ and $a \in A$. As before, we stress that our next result deals with both cases, $\varepsilon \geqslant 0$ and $\varepsilon \leqslant 0$.

Corollary B. *Let us consider the Dulac time $\mathcal{T}(s;\varepsilon,a)$ of the saddle-node unfolding* (13) *taking the functions in* (14) *and setting ϑ_ε as in* (12)*. Then there exist functions $c_j(\varepsilon,a)$, $j \in \mathbb{Z}^+$, satisfying that for*

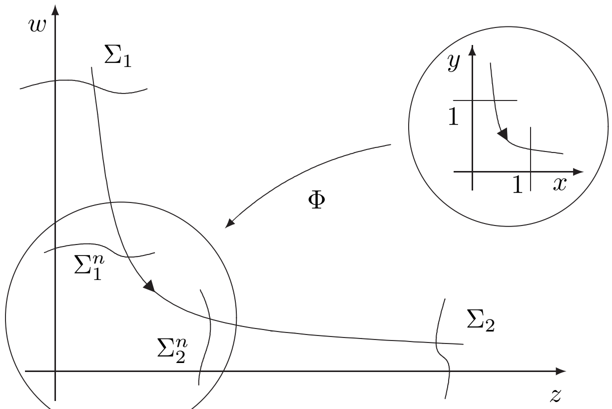


Figure 4: Local Dulac time and normalized transverse sections.

each $\ell, k \in \mathbb{Z}^+$ *and every compact set* $K_a \subset A$*, there exists* $\varepsilon_0 > 0$ *such that* $c_0, \ldots, c_\ell$ *are continuous on* $[-\varepsilon_0, \varepsilon_0] \times K_a$*; and the Dulac time can be written as*

$$\mathcal{T}(s; \varepsilon, a) = \sum_{j=0}^{\ell} c_j(\varepsilon, a) s^j + s^\ell h_\ell(s; \varepsilon, a)$$

with $\Theta^r h_\ell(s) \to 0$*, as* $s \to 0^+$*, uniformly on* $[-\varepsilon_0, \varepsilon_0] \times K_a$*, for* $r = 0, 1, \ldots, k$*. Moreover,* $c_j(\varepsilon, a) = 0$*, for* $\varepsilon \leqslant 0$ *and* $j = 0, 1, \ldots, m-1$*.*

Before introducing Theorem B we mentioned that the *local* Dulac time is the basic building block in the study of the period function near the polycycle at the outer boundary of the period annulus. Let us now clarify the role of Theorem B and Corollary B in this study and state our last main result. After blowing-up the singularities, we can decompose the period function near the polycycle as a sum of Dulac times between *arbitrary* transverse sections Σ_1 and Σ_2 as it is shown in Figure 4. In order to study each Dulac time, we use a diffeomorphism that brings the unfolding of the singularity to its normal form; a saddle or a saddle-node. In this paper, we study the saddle-node unfolding, given in (13). We use the normalizing diffeomorphism Φ to introduce two auxiliary *normalized* transverse sections $\Sigma_1^n := \Phi(\{y = 1\})$ and $\Sigma_2^n := \Phi(\{x = 1\})$. The function $\mathcal{T}$ in Theorem B and Corollary B is precisely this local Dulac time between Σ_1^n and Σ_2^n. In order to have a general result on the Dulac time between arbitrary transverse sections, one must add to the local Dulac time the two times necessary to go from given transverse sections to the normalized ones. For applications it is convenient to express these times in the coordinate on the source transversal and this leads to a composition problem. The symmetry of the differential system (1) makes this composition problem easier than the general situation and we are able to solve it with the tools developed in the present paper. More precisely, Corollaries A and B, together with a result obtained in [5], enable us to answer the initial question motivating this paper. We can thus prove the following result, see also Figure 1, where for a precise definition of local regular value we refer the reader to [6].

Theorem C. *Denoting* $a = (D, F)$*, let* $\{X_a, a \in \mathbb{R}^2\}$ *be the family of differential systems in* (1) *and consider the period function of the center at the origin. Then the parameters* $a \in \big\{D \in (-1, 0) \setminus \{-\frac{1}{2}\}, F = 1\big\}$ *are local regular values of the period function at the outer boundary.*

We point out that, by a result in [6], the exceptional parameter $(D, F) = (-\frac{1}{2}, 1)$ is a local bifurcation value, as it can be seen in Figure 1.

The paper is organised as follows. In Section 2 we obtain the results dealing with the "orbital setting" explained before, including the proofs of Theorem A and Corollary A. Sections 3 and 4 are devoted, respectively, to the proof of Theorems B and C. In Appendix A we prove a L'Hôpital's rule with uniformity in the parameters which is fundamental in the proof of Theorem A. Finally, in Appendix B we discuss the relations between the notions of Darboux and Liouville local integrability in regard to the different unfoldings that we consider in the paper. We thank the anonymous referee, whose comments helped improve the presentation of the results.

2 Orbital results

This section is dedicated to the proof of Theorem A and Corollary A, but first some preliminary and auxiliary results must be proved. To this end, we fix once for all $\lambda_0 > 0$ and compact subsets $K_a \subset A$ and $K_x \subset (0, 1]$. Unless explicitly stated, we shall assume that $\varepsilon \geqslant 0$ and in the sequel we shall use

$$\mathrlap{\,|}\varepsilon = \mathrlap{\,|}\varepsilon(\varepsilon) := \varepsilon^{1/\rho},$$

where $\rho = \rho_+ \in \mathbb{N}$ is the inverse of the Puiseux exponent given in (7). Recall that a trajectory $y = y(x)$ of the unfolding, given in (6), verifies the linear differential equation $P_\varepsilon(x)y'(x) = \lambda V_a(x)y(x) - U(x)$. Accordingly,

$$P_\varepsilon(s + \vartheta_\varepsilon)y'(s + \vartheta_\varepsilon) = \lambda V_a(s + \vartheta_\varepsilon)y(s + \vartheta_\varepsilon) - U(s + \vartheta_\varepsilon).$$

On account of $\vartheta_\varepsilon = \sigma(\mathrlap{\,|}\varepsilon)$, from the definition in (8), we get $\frac{P_\varepsilon(s+\vartheta_\varepsilon)}{s} = \mathcal{Q}(s, \mathrlap{\,|}\varepsilon)$. Thus, since $\Theta_\lambda = \frac{1}{\lambda} s\partial_s$, setting

$$\mathcal{T}(s, \mathrlap{\,|}\varepsilon) := y(s + \sigma(\mathrlap{\,|}\varepsilon)),\ \mathcal{V}(s, \mathrlap{\,|}\varepsilon) := V_a(s + \sigma(\mathrlap{\,|}\varepsilon)) \text{ and } \mathcal{U}(s, \mathrlap{\,|}\varepsilon) := \frac{1}{\lambda}U(s + \sigma(\mathrlap{\,|}\varepsilon)),$$

the above linear differential equation writes as $\mathcal{Q}\Theta_\lambda\mathcal{T} = \mathcal{V}\mathcal{T} - \mathcal{U}$. The idea to obtain the asymptotic expansion is to search for a formal solution $\mathcal{T}(s) = c_0 + c_1 s + c_2 s^2 + \ldots$ satisfying

$$\frac{1}{\lambda}\mathcal{Q}(s, \mathrlap{\,|}\varepsilon)s(c_1 + 2c_2 s + \ldots) = \mathcal{V}(s)(c_0 + c_1 s + c_2 s^2 + \ldots) - \mathcal{U}(s).$$

Since $\mathcal{Q}(s, \mathrlap{\,|}\varepsilon)s\big|_{s=0} = 0$, evaluating in $s = 0$, we get $c_0 = \frac{\mathcal{U}(0)}{\mathcal{V}(0)}$. Next step gives

$$c_1 = \lim_{s\to 0}\left[\frac{1}{s}\left(\frac{\mathcal{U}(s)}{\mathcal{V}(s)} - \frac{\mathcal{U}(0)}{\mathcal{V}(0)}\right)\frac{\mathcal{V}(s)}{\mathcal{V}(s) - \frac{1}{\lambda}\mathcal{Q}(s)}\right].$$

We formalize this inductive procedure as follows.

Definition 2.1. Consider the linear finite difference operator, acting on functions $f(s)$ analytic at $s = 0$, given by

$$(\nabla f)(s) := \begin{cases} \frac{f(s)-f(0)}{s} & \text{for } s > 0, \\ f'(0) & \text{for } s = 0. \end{cases}$$

Setting $F_0 = \mathcal{U}$, we define inductively $F_{\ell+1} = \mathcal{V}_\ell\nabla(F_\ell/\mathcal{V}_\ell)$, where $\mathcal{V}_\ell(s) := \mathcal{V}(s) - \frac{\ell}{\lambda}\mathcal{Q}(s; \mathrlap{\,|}\varepsilon)$. Finally, define

$$c_\ell := \frac{F_\ell}{\mathcal{V}_\ell}\bigg|_{s=0} \text{ and } \Sigma_\ell(s) := \sum_{j=0}^{\ell} c_j s^j.$$

Note that, for each ℓ, $c_\ell = c_\ell(\mathrlap{\,|}\varepsilon, \lambda, a, U)$ is a well defined function on $[-\varepsilon_\ell, \varepsilon_\ell] \times [\lambda_0, \infty) \times K_a \times \mathscr{U}$, for some $\varepsilon_\ell > 0$, which may go to zero, as $\ell \to +\infty$. In the previous definitions, ℓ belongs to $\mathbb{Z}^+$, for convenience we define $\Sigma_{-1} := 0$. □

Notice that the functions F_ℓ, $\ell \in \mathbb{N}$, are obtained from $F_0 = \mathcal{U}$, by iterating a sort of finite differences operators, but conjugated by multiplication by $\mathcal{V}_\ell$.

Remark 2.2. Let $g(s)$ be an analytic function at $s = 0$. Then, for each $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m\}$, we have that $\nabla^k\big(s^m g(s)\big) = s^{m-k} g(s)$. $\square$

Lemma 2.3. *$\mathcal{Q}\Theta_\lambda \Sigma_\ell = \mathcal{V}\Sigma_\ell - \mathcal{U} + s^{\ell+1} F_{\ell+1}$, for each $\ell \in \mathbb{N} \cup \{-1, 0\}$.*

Proof. We proceed by induction on ℓ. For $\ell = -1$, $\Sigma_\ell \equiv 0$ and the assertion holds. Assume now that the claim is true for $\ell - 1$. Then

$$\begin{aligned}\mathcal{Q}\Theta_\lambda \Sigma_\ell &= \mathcal{Q}\Theta_\lambda \Sigma_{\ell-1} + \mathcal{Q}\Theta_\lambda (c_\ell s^\ell) = \mathcal{V}\Sigma_{\ell-1} - \mathcal{U} + s^\ell F_\ell + \ell \mathcal{Q} c_\ell s^\ell = \mathcal{V}\Sigma_\ell - \mathcal{U} + s^\ell \big(F_\ell - c_\ell \mathcal{V}_\ell\big)\\ &= \mathcal{V}\Sigma_\ell - \mathcal{U} + s^{\ell+1} F_{\ell+1},\end{aligned}$$

because $F_\ell - c_\ell \mathcal{V}_\ell = s F_{\ell+1}$, by definition. $\blacksquare$

Definition 2.4. For each $k \in \mathbb{Z}^+$ and $d \in \{0, 1\}$, we say that a real function $F(s, \rlap{|}\varepsilon; a, \lambda, U)$ belongs to the set $\mathcal{F}_k^d$, if it can be written as

$$F(s, \rlap{|}\varepsilon; a, \lambda, U) = \frac{f(s, \rlap{|}\varepsilon; a, \lambda, U)}{\mathcal{Q}(s; \rlap{|}\varepsilon)^k},$$

where $\mathcal{Q}$ verifies hypothesis (H1) and (H2) and f is a function such that

(*a*) $f(s, \rlap{|}\varepsilon; a, \lambda, U)$ is analytic at $(s, \rlap{|}\varepsilon) = (0, 0)$, for fixed a, λ, U and it is homogeneous of degree d in U, more precisely, for $d = 0$, it does not depend on U and, for $d = 1$, it depends linearly on U,

(*b*) $f(s, \rlap{|}\varepsilon; a, \lambda, U) = \sum_{i,j \geqslant 0} f_{ij}(a, \lambda, U) s^i \rlap{|}\varepsilon^j$ with $f_{ij}(a, \lambda, U) \equiv 0$, for $\frac{i}{\mu} + \frac{j}{\nu} < k$, and

(*c*) there exists a neighbourhood W of $(s, \rlap{|}\varepsilon) = (0, 0)$ in $\mathbb{C}^2$ such that the complex-analytic extension of f in $(s, \rlap{|}\varepsilon)$ satisfies

$$\sup\{|f(s, \rlap{|}\varepsilon; a, \lambda, U)| : (s, \rlap{|}\varepsilon) \in W, (a, \lambda, U) \in [\lambda_0, +\infty) \times K_a \times \mathscr{U}_1\} < +\infty.$$

When we write $\frac{f}{\mathcal{Q}^k} \in \mathcal{F}_k^d$, we will assume implicitly that f satisfies conditions (a), (b) and (c). $\square$

Lemma 2.5. *The following properties hold:*

(a) $\mathcal{F}_k^d$ is stable by addition, $\mathcal{F}_k^0 \mathcal{F}_m^d \subset \mathcal{F}_{k+m}^d$ and $\mathcal{F}_k^d \subset \mathcal{F}_{k+1}^d$;

(b) $\nabla \mathcal{F}_0^d \subset \mathcal{F}_0^d$;

(c) $\Theta_\lambda \mathcal{F}_k^d \subset \mathcal{F}_{k+1}^d$;

(d) $\frac{s^\mu}{\mathcal{Q}} \in \mathcal{F}_1^0$, $\mathcal{V}_\ell, \frac{1}{\mathcal{V}_\ell} \in \mathcal{F}_0^0$ and $\mathcal{U} : (s, \rlap{|}\varepsilon; a, \lambda, U) \longmapsto \frac{1}{\lambda} U(s + \vartheta_\varepsilon) \in \mathcal{F}_0^1$;

(e) If $F \in \mathcal{F}_k^d$, then there exists a neighbourhood W of $(s, \rlap{|}\varepsilon) = (0, 0)$ in $\mathbb{R}^+ \times \mathbb{R}^+$ such that F is bounded on $W \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$.

Proof. Assertion (a) is straightforward. To prove assertion (b), note first that since ∇ is a linear operator it preserves the homogeneous degree d on U. On the other hand, the condition on the Newton's diagram for belonging to $\mathcal{F}_0^d$ (i.e. for $k = 0$) is empty. Let f be an element of $\mathcal{F}_0^d$. There exists $r_0 > 0$, such that the function $f(s, \rlap{|}\varepsilon; a, \lambda, U)$ is defined, for every $s \in \mathbb{C}$, with $|s| \leqslant r_0$. By applying Cauchy's integral formula to the function $s \longmapsto f(s, \rlap{|}\varepsilon; a, \lambda, U)$, which is analytic at $s = 0$, we get

$$\nabla f(s, \rlap{|}\varepsilon; a, \lambda, U) = \frac{1}{2\pi i} \int_{|\zeta| = r_0} \frac{f(\zeta, \rlap{|}\varepsilon; a, \lambda, U)}{(\zeta - s)\zeta} d\zeta.$$

If $|s| \leq r_0/2$, then the denominator in the integrand is bounded away from zero and the boundedness of the complex analytic extension of f implies the boundedness of that of ∇f.
(c) Suppose that $\frac{f}{\mathcal{Q}^k} \in \mathcal{F}_k^d$. We will prove first that $\frac{\Theta_\lambda f}{\mathcal{Q}^k} \in \mathcal{F}_k^d$. To see this, note first that the derivative Θ_λ is a linear operator and it does not affect the condition about the Newton's diagram of f. On the other hand, by Cauchy's differentiation formula, we have that

$$\Theta_\lambda f(s, \ddagger; a, \lambda, U) = \frac{1}{2i\pi} \int_{|\xi|=r_0} \frac{s\, f(\zeta, \ddagger; a, \lambda, U)}{\lambda(\zeta - s)^2} d\zeta$$

is bounded on $W \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$, where W is a neighbourhood of $(s, \ddagger) = (0, 0)$ in $\mathbb{C}^2$. In particular, taking $f = \mathcal{Q}$ and $k = 1$, we deduce that $\frac{\Theta_\lambda \mathcal{Q}}{\mathcal{Q}} \in \mathcal{F}_1^0$. Finally, on account of $\Theta_\lambda(\frac{f}{\mathcal{Q}^k}) = \frac{\Theta_\lambda f}{\mathcal{Q}^k} - k \frac{f}{\mathcal{Q}^k} \frac{\Theta_\lambda \mathcal{Q}}{\mathcal{Q}}$, we conclude that $\Theta_\lambda \mathcal{F}_k^d \subset \mathcal{F}_{k+1}^d$, by using the assertions in (a).
(d) Obviously $\frac{s^\mu}{\mathcal{Q}} \in \mathcal{F}_1^0$. Due to $V_a(0) = 1$ and $\mathcal{Q}(0,0) = 0$, it follows that, for every $\ell \in \mathbb{N}$ and $\lambda_0 > 0$, there exists a neighbourhood W of $(s, \ddagger) = (0,0)$ in $\mathbb{C}^2$ such that

$$\frac{1}{2} \leqslant V_a(s + \vartheta_\varepsilon) - \frac{\ell}{\lambda} \mathcal{Q}(s; \ddagger) \leqslant 2$$

on $W \times K_a \times [\lambda_0, \infty)$. This shows that $\mathcal{V}_\ell$ and $\frac{1}{\mathcal{V}_\ell}$ belong to $\mathcal{F}_0^0$. Finally, $\mathcal{U} \in \mathcal{F}_0^1$ because $\frac{U(s+\vartheta_\varepsilon)}{\lambda}$ is clearly linear in U and bounded on $W \times [\lambda_0, \infty) \times \mathscr{U}_1$, where W is a sufficiently small neighbourhood of $(s, \ddagger) = (0,0)$ in $\mathbb{C}^2$ (in this case there is no dependence on a).
(e) If $F = \frac{f}{\mathcal{Q}^k}$ belongs to $\mathcal{F}_k^d$, then $f(r^\nu \sin\theta, r^\mu \cos\theta; \xi) = r^{k\mu\nu} \tilde{f}(r; \tilde{\xi}) \in \mathcal{F}_0^0$, where $\tilde{\xi} := (\tilde{a}, \lambda, U)$ with $\tilde{a} := (a, \theta)$ varying in the compact set $K_a \times [0, \frac{\pi}{2}]$. By Remark 2.2, we have that $\tilde{f}(r; \tilde{\xi}) = \nabla^{k\mu\nu}(r^{k\mu\nu} \tilde{f}(r; \tilde{\xi}))$, which belongs to $\mathcal{F}_0^0$ thanks to the assertion (2), i.e. $\tilde{f}(r; \tilde{\xi})$ is bounded on $V \times K_a \times [0, \frac{\pi}{2}] \times [\lambda_0, \infty) \times \mathscr{U}_1$, where V is some neighbourhood of $r = 0$ in $\mathbb{C}$. On the other hand, hypothesis (H1) and (H2) imply that $\mathcal{Q}(r^\nu \sin\theta, r^\mu \cos\theta) = r^{\mu\nu} \tilde{q}(r, \theta)$ with $\tilde{q}(0, \theta) \geqslant \delta > 0$, for all $\theta \in [0, \frac{\pi}{2}]$. Hence, there exists $r_0 > 0$ such that $\tilde{q}(r, \theta) \geqslant \delta/2$, for all $r \in [0, r_0]$ and $\theta \in [0, \frac{\pi}{2}]$. Accordingly, this shows that $F(r^\nu \sin\theta, r^\mu \cos\theta; \xi) = \frac{\tilde{f}(r, \tilde{\xi})}{\tilde{q}(r, \theta)^k}$ is bounded, when $r \in [0, r_0]$, $\theta \in [0, \frac{\pi}{2}]$ and $(a, \lambda, U) \in K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$, as desired. ■

The reason to require the boundedness of f on $W \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$, where W is a neighbourhood of the origin in $\mathbb{C}^2$, and not just in $\mathbb{R}^2$, is illustrated by the following example.

Example 2.6. The analytic function $f(s; \lambda) = \sin(\lambda^2 s)$ is bounded on $\mathbb{R} \times [\lambda_0, \infty)$. However $\Theta_\lambda f(s; \lambda) = \lambda s \cos(\lambda^2 s)$ is not bounded on $(-s_0, s_0) \times [\lambda_0, \infty)$, for any $s_0 > 0$. Notice that, although there exists a neighbourhood of $\mathbb{R} \times [\lambda_0, \infty)$ in $\mathbb{C} \times [\lambda_0, \infty)$ where the analytic extension of f is bounded, this function is unbounded on $U \times [\lambda_0, \infty)$, for any neighbourhood U of $s = 0$ in $\mathbb{C}$. Thus, f does not belong to $\mathcal{F}_0^0$ according to Definition 2.4. □

Proposition 2.7. *For each $\ell \in \mathbb{Z}^+$, $F_\ell \in \mathcal{F}_0^1$ and there exists $\varepsilon_\ell > 0$ such that $c_\ell(\ddagger, a, \lambda, U)$ is an analytic function in $(\ddagger, a, \lambda) \in [0, \varepsilon_\ell] \times K_a \times [\lambda_0, \infty)$ and a uniformly bounded linear operator on $\mathscr{U}$.*

Proof. To prove that $F_\ell \in \mathcal{F}_0^1$, we proceed by induction on ℓ. The case $\ell = 0$ follows from assertion (d) of Lemma 2.5, because $F_0 = \mathcal{U}$. The inductive step follows easily from the recursive definition $F_{\ell+1} = \mathcal{V}_\ell \nabla(F_\ell \mathcal{V}_\ell^{-1})$, by using assertions (a), (b) and (d) of Lemma 2.5. Using again Lemma 2.5, we deduce that $\frac{F_\ell}{\mathcal{V}_\ell} \in \mathcal{F}_0^1$, which implies that $c_\ell = \frac{F_\ell}{\mathcal{V}_\ell}\big|_{s=0} \in \mathcal{F}_0^1$. By condition (c) in Definition 2.4, we have

$$\sup\{|c_\ell(s, \ddagger; a, \lambda, U)| : (s, \ddagger) \in W, (a, \lambda, U) \in [\lambda_0, +\infty) \times K_a \times \mathscr{U}_1\} < +\infty,$$

for some neighborhood W of $(s, \varepsilon) = (0,0)$. The analyticity of c_ℓ in the remaining parameters $(\ddagger, a, \lambda)$ follows easily from Definition 2.1 by the analyticity of $\frac{1}{\lambda} U(s + \sigma(\ddagger))$ and $V_a(s + \sigma(\ddagger))$. ■

Next, we shall study the remainder term

$$h_\ell(s) := \frac{\mathcal{T}(s) - \Sigma_\ell(s)}{s^\ell} \tag{15}$$

of the asymptotic expansions in Theorem A. Notice that in the case of assertion (b), $h_\ell(s) = s^{-\ell}\mathcal{D}(s)$, where we denote

$$\mathcal{D}(s;\text{\textcent}) := D(s+\vartheta_\varepsilon) = \exp\left(\lambda \int_1^{s+\vartheta_\varepsilon} \frac{V_a(t)}{P_\varepsilon(t)} dt\right). \tag{16}$$

The following two lemmas give the basis of induction $k = 0$ in assertion (b) and (a) of Theorem A, respectively.

Lemma 2.8. *For each $\ell \in \mathbb{Z}^+$ and $y > 0$ small enough, there exists $\varepsilon_0 > 0$ such that $\frac{y^\ell \mathcal{D}(s)}{s^\ell \mathcal{D}(y)}$ and $s^{-\ell}\mathcal{D}(s)$ tend to zero, as $s \to 0^+$, uniformly on $[0, \varepsilon_0] \times K_a \times [\lambda_0, \infty)$.*

Proof. Note that it suffices to prove the first limit as $y < 1$ is fixed and $\mathcal{D}(y) < 1$. By definition we have that $\frac{y^\ell \mathcal{D}(s)}{s^\ell \mathcal{D}(y)} = \exp(-B(s, y; \varepsilon, a, \lambda))$, with

$$B(s, y; \varepsilon, a, \lambda) := \ell \log(s/y) + \lambda \int_s^y \frac{V_a(x+\vartheta_\varepsilon)}{P_\varepsilon(x+\vartheta_\varepsilon)}\, dx.$$

We must prove that there exists $\varepsilon_0 > 0$ such that $\lim\limits_{s\to 0^+} B(s, y; \varepsilon, a, \lambda) = +\infty$, uniformly on $[0, \varepsilon_0] \times K_a \times [\lambda_0, \infty)$. By hypothesis, due to the compactness of K_a, there exists a positive constant m_1 such that $V_a(x) \geqslant m_1$, for any $x \in [s - \vartheta_\varepsilon, y - \vartheta_\varepsilon]$ and $a \in K_a$. Recall that ϑ_ε is the biggest root of P_ε, which tends to zero as $\varepsilon \to 0$, and that $\vartheta_\varepsilon = \sigma(\text{\textcent})$, with σ analytic at zero. We have that $P_\varepsilon(x+\vartheta_\varepsilon) = x\mathcal{Q}(x, \text{\textcent})$. Due to $\mathcal{Q}(0,0) = 0$, for every $m_0 > 0$ there exists $\varepsilon_0 > 0$ such that $|\mathcal{Q}(x, \text{\textcent})| \leqslant m_0$, for all $\varepsilon, x \in [0, \varepsilon_0]$. Hence,

$$B(s, y; \varepsilon, a, \lambda) = \ell \log(s/y) + \lambda \int_s^y \frac{V_a(x+\vartheta_\varepsilon)}{x\mathcal{Q}(x+\vartheta_\varepsilon)}\, dx \geqslant \left(\lambda_0 \frac{m_1}{m_0} - \ell\right) \log(y/s).$$

Taking m_0 small enough, we see that the right hand side tends to $+\infty$, as $s \to 0^+$. ■

We show now the case $k = 0$ in assertion (a) of Theorem A. To this end we write, see (15), $h_\ell = \frac{f_\ell}{g_\ell}$ with

$$f_\ell(s) := \frac{\mathcal{T}(s) - \Sigma_\ell(s)}{\mathcal{D}(s)} \quad \text{and} \quad g_\ell(s) := \frac{s^\ell}{\mathcal{D}(s)}, \tag{17}$$

where $\mathcal{D}(s)$ is defined in (16).

Lemma 2.9. *For each $\ell \in \mathbb{Z}^+$, there exists $\varepsilon_0 > 0$ such that $h_\ell(s)$ tends to zero, as $s \to 0^+$, uniformly on $K_x \times [0, \varepsilon_0] \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$.*

Proof. This will follow by applying the uniform L'Hôpital's rule stated in Appendix A taking f_ℓ and g_ℓ as in (17). To this end, we must check that these functions verify the five conditions in Proposition 4.1. Condition (a) is obvious because f_ℓ and g_ℓ are differentiable for $s > 0$. Using that $\mathcal{Q}\Theta_\lambda\mathcal{D} = \mathcal{V}\mathcal{D}$ and applying Lemma 2.3, we deduce that

$$\Theta_\lambda f_\ell = \frac{-s^{\ell+1}F_{\ell+1}}{\mathcal{Q}\mathcal{D}} \quad \text{and} \quad \Theta_\lambda g_\ell = \frac{-s^\ell \mathcal{V}_\ell}{\mathcal{Q}\mathcal{D}}.$$

In particular, $\partial_s g_\ell = -\frac{\lambda s^{\ell-1}\mathcal{V}_\ell}{\mathcal{Q}\mathcal{D}} < 0$, for $s > 0$, which shows condition (b). Moreover,

$$\frac{\partial_s f_\ell}{\partial_s g_\ell} = \frac{\Theta_\lambda f_\ell}{\Theta_\lambda g_\ell} = s\frac{F_{\ell+1}}{\mathcal{V}_\ell}$$

tends to zero, as $s \to 0^+$ uniformly on $[0, \varepsilon_0] \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$, for some $\varepsilon_0 > 0$ small enough. This follows from Lemma 2.5, taking into account that $F_{\ell+1} \in \mathcal{F}_0^1$, thanks to Proposition 2.7. This shows that (c) and (d) are verified. It only remains to check (e). The first part follows from Lemma 2.8. To see the second part, we must verify that, for each fixed $s > 0$ small enough, $\frac{f_\ell(s)}{g_\ell(s)} = \frac{\mathcal{T} - \Sigma_\ell}{s^\ell}$ is bounded on $K_x \times [0, \varepsilon_0] \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$. This follows from Proposition 2.7 and the expression

$$\mathcal{T}(s) = \mathcal{D}(s) \int_{s+\vartheta_\varepsilon}^{x_0} \frac{U(x)}{P_\varepsilon(x)} \frac{dx}{\mathcal{D}(x - \vartheta_\varepsilon)},$$

on account of $\sup\{|U(x)|; x \in [s + \vartheta_\varepsilon, x_0]\} \leqslant \|U\|$, the monotonicity of the Dulac map $\mathcal{D}(x)$ and the inequalities $s + \vartheta_\varepsilon \leqslant x \leqslant x_0 \leqslant 1$. We can thus apply Proposition 4.1, which shows that $h_\ell(s) = \frac{f_\ell(s)}{g_\ell(s)}$ tends to zero, as $s \to 0^+$, uniformly on $K_x \times [0, \varepsilon_0] \times K_a \times [\lambda_0, \infty) \times \mathscr{U}_1$, as desired. ■

The induction step, for assertions (a) and (b) in Theorem A, will be treated jointly:

Proposition 2.10. *For each $\ell, k \in \mathbb{Z}^+$, there exist $v_{\ell k} \in \mathcal{F}_k^0$ and $w_{\ell k} \in \mathcal{F}_k^1$ such that*

$$\Theta_\lambda^k h_\ell = v_{\ell k} h_{\ell+k\mu} + s w_{\ell k}.$$

Proof. We proceed by induction on k. For $k = 1$, we have that

$$\Theta_\lambda h_\ell = \frac{\Theta_\lambda f_\ell}{g_\ell} - h_\ell \frac{\Theta_\lambda g_\ell}{g_\ell} = \frac{-sF_{\ell+1} + h_\ell \mathcal{V}_\ell}{\mathcal{Q}}.$$

Since $\mathcal{T} = \Sigma_\ell + s^\ell h_\ell = \Sigma_{\ell+\mu} + s^{\ell+\mu} h_{\ell+\mu}$, we get $h_\ell = s^\mu h_{\ell+\mu} + s\Sigma_\ell^\mu$, with $\Sigma_\ell^\mu := \frac{\Sigma_{\ell+\mu} - \Sigma_\ell}{s^{\ell+1}} = \sum_{j=0}^{\mu-1} c_{\ell+1+j} s^j$. Therefore,

$$\Theta_\lambda h_\ell = \underbrace{\frac{s^\mu \mathcal{V}_\ell}{\mathcal{Q}}}_{v_{\ell,1}} h_{\ell+\mu} + s \underbrace{\frac{\mathcal{V}_\ell \Sigma_\ell^\mu - F_{\ell+1}}{\mathcal{Q}}}_{w_{\ell,1}}.$$

It is clear that $v_{\ell,1} \in \mathcal{F}_1^0$, because $\frac{s^\mu}{\mathcal{Q}} \in \mathcal{F}_1^0$ and $\mathcal{V}_\ell \in \mathcal{F}_0^0$, by Lemma 2.5. On the other hand, by Lemma 2.3,

$$\begin{aligned}
\mathcal{V}_\ell \Sigma_\ell^\mu &= s^{-\ell-1}\big(\mathcal{V}\Sigma_{\ell+\mu} - \mathcal{V}\Sigma_\ell\big) - \frac{\ell}{\lambda}\mathcal{Q}\Sigma_\ell^\mu \\
&= s^{-\ell-1}\big(\mathcal{Q}\Theta_\lambda\Sigma_{\ell+\mu} - \mathcal{Q}\Theta_\lambda\Sigma_\ell - s^{\ell+\mu+1}F_{\ell+\mu+1} + s^{\ell+1}F_{\ell+1}\big) - \frac{\ell}{\lambda}\mathcal{Q}\Sigma_\ell^\mu \\
&= \mathcal{Q}s^{-\ell-1}\Theta_\lambda(s^{\ell+1}\Sigma_\ell^\mu) - s^\mu F_{\ell+\mu+1} + F_{\ell+1} - \frac{\ell}{\lambda}\mathcal{Q}\Sigma_\ell^\mu = \mathcal{Q}\big(\frac{1}{\lambda}\Sigma_\ell^\mu + \Theta_\lambda\Sigma_\ell^\mu\big) - s^\mu F_{\ell+\mu+1} + F_{\ell+1}.
\end{aligned}$$

Hence $w_{\ell,1} = \frac{1}{\lambda}\Sigma_\ell^\mu + \Theta_\lambda\Sigma_\ell^\mu - \frac{s^\mu}{\mathcal{Q}}F_{\ell+\mu+1} \in \mathcal{F}_1^1$, thanks to Lemma 2.5 and Proposition 2.7. We now complete the inductive step:

$$\begin{aligned}
\Theta_\lambda^{k+1} h_\ell &= \Theta_\lambda(\Theta_\lambda^k h_\ell) = \Theta_\lambda(v_{\ell k} h_{\ell+k\mu} + s w_{\ell k}) = \Theta_\lambda(v_{\ell k}) h_{\ell+k\mu} + v_{\ell k}\Theta_\lambda h_{\ell+k\mu} + s\big(\frac{w_{\ell k}}{\lambda} + \Theta_\lambda w_{\ell k}\big) \\
&= \Theta_\lambda(v_{\ell k}) s^\mu h_{\ell+(k+1)\mu} + s\Theta_\lambda(v_{\ell k})\Sigma_{\ell+k\mu}^\mu + v_{\ell k}[v_{\ell+k\mu,1} h_{\ell+(k+1)\mu} + s w_{\ell+k\mu,1}] + s\big(\frac{w_{\ell k}}{\lambda} + \Theta_\lambda w_{\ell k}\big) \\
&= \underbrace{\big(\Theta_\lambda(v_{\ell k}) s^\mu + v_{\ell k} v_{\ell+k\mu,1}\big)}_{v_{\ell,k+1}} h_{\ell+(k+1)\mu} + s \underbrace{\big(\Theta_\lambda(v_{\ell k})\Sigma_{\ell+k\mu}^\mu + v_{\ell k} w_{\ell+k\mu,1} + \frac{w_{\ell k}}{\lambda} + \Theta_\lambda w_{\ell k}\big)}_{w_{\ell,k+1}}.
\end{aligned}$$

Here $v_{\ell,k+1} \in \mathcal{F}_{k+1}^0$ and $w_{\ell,k+1} \in \mathcal{F}_{k+1}^1$ on account of the inductive hypothesis, Lemma 2.5 and Proposition 2.7. This completes the proof. ■

Proof of Theorem A. The coefficients c_j, for $j \in \mathbb{Z}^+$, are given in Definition 2.1. Proposition 2.7 shows that there exists $\varepsilon_0 > 0$, such that $c_0, c_1, \ldots, c_\ell$ are analytic on $[0, \varepsilon_0] \times K_a \times [\lambda_0, \infty)$ and are uniformly bounded linear operators on $\mathscr{U}$. By Proposition 2.10, we get

$$\Theta_\lambda^r h_\ell = v_{\ell r} h_{\ell + r\mu} + s\, w_{\ell r},$$

with $v_{\ell r}$ and $w_{\ell r}$ bounded on $[0, s_0] \times [0, \varepsilon_0] \times [\lambda_0, \infty) \times K_a \times \mathscr{U}_1$, for some $s_0 > 0$ and $\varepsilon_0 > 0$, thanks to assertion (e) in Lemma 2.5. Notice that the linearity of $w_{\ell r}$ on U implies that in the case (b), where $U \equiv 0$, we have $w_{\ell r} \equiv 0$. We conclude that the limits, as s tends to zero, in assertions (a) and (b) of Theorem A are zero uniformly on the corresponding parameters, using Lemma 2.9 and Lemma 2.8, respectively. ∎

Proof of Corollary A. It is easy to check that on the half planes $\varepsilon \geqslant 0$ and $\varepsilon \leqslant 0$ the corresponding functions $\mathcal{Q}(s; \text{⨍})$, given by (12), satisfy hypothesis (H1) and (H2). To show assertion (a), we apply twice assertion (a) of Theorem A, with $\rho_- = 1$ and $\rho_+ = \mu$, to deduce that

$$y(s + \vartheta_\varepsilon) = \begin{cases} \sum_{j=0}^{\ell} c_j^-\big(\varepsilon, a, \lambda, U\big) s^j + s^\ell h_\ell^-(s), & \text{for } \varepsilon \leqslant 0, \\ \sum_{j=0}^{\ell} c_j^+\big(\varepsilon^{1/\mu}, a, \lambda, U\big) s^j + s^\ell h_\ell^+(s), & \text{for } \varepsilon \geqslant 0, \end{cases}$$

where the functions $h_\ell^\pm(s)$, depending on the parameters $(x_0, \varepsilon, a, \lambda, U)$, satisfy

$$\Theta_\lambda^r h_\ell^\pm(s) \to 0, \quad \text{as} \quad s \to 0^+,$$

uniformly on $K_x \times [-\varepsilon_0, \varepsilon_0] \times [\lambda_0, \infty) \times K_a \times \mathscr{U}_1$, for $r = 0, 1, \ldots, k$. The flatness property of $h_\ell^\pm$, together with the analyticity of $c_j^\pm\big(\varepsilon, a, \lambda, U\big)$ on $\big([-\varepsilon_0, \varepsilon_0] \cap \{\pm\varepsilon \geqslant 0\}\big) \times K_a \times [\lambda_0, \infty) \times \mathscr{U}$, easily implies that, for all $j \in \mathbb{Z}^+$, the functions

$$c_j(\varepsilon, a, \lambda, U) := \begin{cases} c_j^-(\varepsilon, a, \lambda, U), & \text{for } \varepsilon \leqslant 0, \\ c_j^+(\varepsilon^{1/\mu}, a, \lambda, U), & \text{for } \varepsilon \geqslant 0, \end{cases}$$

are continuous at $\varepsilon = 0$. Moreover, the coefficients $c_0, \ldots, c_{m-1}$ are identically zero, for $\varepsilon \leqslant 0$. This follows from the fact that $U(x) = x^m \bar{U}(x)$ and $\vartheta_\varepsilon = 0$, for $\varepsilon \leqslant 0$, by using the recursive definition of c_j and Remark 2.2. Finally, the derivative properties of the function

$$h_\ell(s; x_0, \varepsilon, a, \lambda, U) := \begin{cases} h_\ell^-(s; x_0, \varepsilon, a, \lambda, U), & \text{for } \varepsilon \leqslant 0, \\ h_\ell^+(s; x_0, \varepsilon, a, \lambda, U), & \text{for } \varepsilon \geqslant 0, \end{cases}$$

follow from the corresponding properties of $h_\ell^\pm$. Assertion (b) in Corollary A is deduced from assertion (b) in Theorem A in a similar way. ∎

3 Temporal results

This section is dedicated to the proof of Theorem B, which follows by applying Theorem A. It will be clear now why we need uniformity on $\lambda \in [\lambda_0, \infty)$ and U varying in the Banach space $\mathscr{U}$.

Proof of Theorem B. Consider $\ell, k \in \mathbb{Z}^+$ and a compact set $K_a \subset A$. We decompose the given function $U_a(x, y) = \sum_{n \geqslant 1} U_{n,a}(x) y^{n-1}$, with $U_{n,a} \in \mathscr{U}$, for all $n \in \mathbb{N}$ and $a \in K_a$. Since $U_a(x, y)$ is absolutely convergent on $|x|, |y| \leqslant 1$, the series $\sum_{n \geqslant 1} \|U_{n,a}\| y^n$ and all its $y\partial_y$ derivatives have convergence radius at least 1. Consequently,

$$\sum_{n \geqslant 1} n^r \|U_{n,a}\| < \infty, \text{ for all } r \in \mathbb{Z}^+ \text{ and } a \in K_a. \tag{18}$$

Let $y(x;s)$ be the trajectory of the vector field $P_\varepsilon(x)\partial_x - V_a(x)y\partial_y$, with initial condition $y(s;s) = 1$. Note that the Dulac time in the statement is given by

$$\mathcal{T}(s;\ddagger,a) = \int_{s+\vartheta_\varepsilon}^1 \frac{U_a\big(x,y(x;s)\big)y(x;s)}{P_\varepsilon(x)}\,dx = \int_{s+\vartheta_\varepsilon}^1 \sum_{n\geqslant 1} \frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}\,dx.$$

We define

$$T_n(s) := \int_s^1 \frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}dx,$$

whose derivative satisfies

$$\partial_s T_n(s) = \int_s^1 \frac{U_{n,a}(x)\partial_s y^n(x;s)}{P_\varepsilon(x)}\,dx - \frac{U_{n,a}(s)}{P_\varepsilon(s)} = \frac{nV_a(s)}{P_\varepsilon(s)}T_n(s) - \frac{U_{n,a}(s)}{P_\varepsilon(s)},$$

by using $\partial_s y(x;s) = y(x;s)\frac{V_a(s)}{P_\varepsilon(s)}$. This shows that $T_n(x)$ is the trajectory with initial condition $T_n(1) = 0$ of the vector field obtained from (6) by replacing $U(x)$ by $U_{n,a}(x)$, $V_a(x)$ by $\frac{V_a(x)}{V_a(0)}$ and λ by $nV_a(0)$. We can thus apply Theorem A, with the given $\ell, k \in \mathbb{Z}^+$, the compact set $K_a \subset A$, $\lambda_0 := \inf\{V_a(0) : a \in K_a\} > 0$ and $U \in \mathscr{U}$ to obtain the asymptotic expansion of $\mathcal{T}_n(s) := T_n(s+\vartheta_\varepsilon)$ at $s = 0$. So, there exists $\varepsilon_0 > 0$ and

$$\mathcal{T}_n(s;\ddagger,a) = \sum_{j=0}^{\ell} c_j\big(\ddagger,a,nV_a(0),U_{n,a}\big)s^j + s^\ell h_\ell\big(s;\varepsilon,a,nV_a(0),U_{n,a}\big),$$

where the coefficients c_j depend analytically on $(\ddagger,a) \in [0,\varepsilon_0]\times K_a$. Moreover,

$$\gamma_j := \sup\big\{\big|c_j\big(\varepsilon,a,\lambda,U\big)\big| : (\varepsilon,a,\lambda,U) \in [0,\varepsilon_0]\times K_a \times [\lambda_0,+\infty)\times \mathscr{U}_1\big\} < +\infty$$

and, for all positive s, small enough,

$$M_\ell^r(s) := \sup\big\{\big|\Theta_\lambda^r h_\ell\big(s;\varepsilon,a,\lambda,U\big)\big| : (\varepsilon,a,\lambda,U) \in [0,\varepsilon_0]\times K_a \times [\lambda_0,+\infty)\times \mathscr{U}_1\big\} < +\infty,$$

with $M_\ell^r(s) \to 0$, as $s \to 0^+$, for $r = 0,1,\dots,k$. In particular, for $(\varepsilon,a,n) \in [0,\varepsilon_0]\times K_a \times \mathbb{N}$ and $r = 0,1,\dots,k$ we have

$$|c_j(\ddagger,a,nV_a(0),U_{n,a})| \leqslant \gamma_j\|U_{n,a}\| \quad\text{and}\quad |\Theta_\lambda^r h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})| \leqslant M_\ell^r(s)\|U_{n,a}\|. \tag{19}$$

Here, it is crucial that Theorem A holds for λ unbounded and U varying in the Banach space $\mathscr{U}$.

We define at this point the coefficients

$$c_j(\ddagger,a) := \sum_{n\geqslant 1} c_j(\ddagger,nV_a(0),a,U_{n,a}),\ \text{for all } j \in \mathbb{Z}^+,$$

which are well-defined because the series are uniformly convergent on $(\ddagger,a) \in [0,\varepsilon_0]\times K_a$ thanks to (18), with $r = 0$ and the first inequality in (19). In particular, these coefficients are analytic on $(\ddagger,a) \in [0,\varepsilon_0]\times K_a$. On the other hand, by using the second inequality in (19), the series

$$h_\ell(s;\varepsilon,a) := \sum_{n\geqslant 1} h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})$$

is uniformly convergent on $(s,\ddagger,a) \in [0,s_0]\times[0,\varepsilon_0]\times K_a$, for s_0 small enough, and it tends to zero, as $s \to 0^+$, uniformly on (ε,a). Hence, the series

$$\begin{aligned}\sum_{n\geqslant 1}\mathcal{T}_n(s;\ddagger,a) &= \sum_{n\geqslant 1}\sum_{j=0}^{\ell} c_j(\ddagger,a,nV_a(0),U_{n,a})s^j + s^\ell\sum_{n\geqslant 1} h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})\\ &= \sum_{j=0}^{\ell} c_j(\ddagger,a)s^j + s^\ell h_\ell(s;\ddagger,a)\end{aligned}$$

is uniformly convergent on $(s,\nu,a)\in[0,s_0]\times[0,\varepsilon_0]\times K_a$, because it is the sum of $\ell+2$ uniformly convergent series. For this reason, we can commute summation and integration in the following expression of the Dulac time

$$\mathcal{T}(s;\nu,a)=\int_{s+\vartheta_\varepsilon}^1\sum_{n\geqslant1}\frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}\,dx=\sum_{n\geqslant1}\mathcal{T}_n(s;\nu,a).$$

Accordingly, $\mathcal{T}(s;\nu,a)=\sum_{j=0}^{\ell}c_j(\nu,a)s^j+s^\ell h_\ell(s;\nu,a)$. Finally, taking $\lambda=nV_a(0)$ and (10) into account, for $r=1,2,\ldots,k$, the series

$$\sum_{n\geqslant1}|\Theta_1^r h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})|=V_a^r(0)\sum_{n\geqslant1}n^r\,|\Theta_\lambda^r h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})|\leqslant V_a(0)^rM_\ell^r(s)\sum_{n\geqslant1}n^r\|U_{n,a}\|$$

is uniformly convergent in (s,ν,a) and tends to zero, as $s\to0^+$, uniformly on (ν,a), thanks to (18) and (19). Recall that uniform convergence of a series of functions does not imply the uniform convergence of its derivatives. However, if $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point $x_0\in[a,b]$ and $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and $f'(x)=\lim_{n\to\infty}f_n'(x)$, for all $x\in[a,b]$ (see [12, Theorem 7.17]). Taking this into account, we can assert that $\Theta_1^r h_\ell(s;\varepsilon,a)=\sum_{n\geqslant1}\Theta_1^r h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})$ tends to zero, as $s\to0^+$ uniformly on $[0,\varepsilon_0]\times K_a$, for all $r=0,1,\ldots,k$. This concludes the proof of the result. ■

The proof of Corollary B is completely analogous to that of Corollary A.

4 Application to Loud's system

Proof of Theorem C. To study the passage through the unfolding of saddle-node at infinity we use the chart of $\mathbb{RP}^2$ given by $(z,w)=\left(\frac{1-u}{v},\frac{1}{v}\right)$. In these coordinates the Loud differential system (1) writes as

$$\frac{1}{w}\Big(z\big(1-F-Dz^2+(2D+1)zw-(D+1)w^2\big)\partial z+w\big(-F-Dz^2+(2D+1)zw-(D+1)w^2\big)\partial w\Big),$$

which is a meromorphic vector field with Darboux first integral

$$I(z,w)=\frac{w}{z}\left(1-2(F-1)\frac{g(z,w)}{z^2}\right)^{\frac{1}{2(F-1)}},$$

where $g(z,w):=\frac{(2\,D+1)}{(2\,F-1)D}zw-\frac{(D+1)}{2FD}w^2-\frac{1}{2D}$. One can verify that the local change of coordinates given by

$$\left\{x=\frac{z}{\sqrt{g(z,w)}},y=\frac{w}{\sqrt{g(z,w)}}\right\}\tag{20}$$

brings the above vector field to (13)

$$\frac{1}{yU_a(x,y)}\big((x^2-\varepsilon)x\partial x-(2F-x^2)y\partial y\big),$$

with $a:=(D,F)$, $U_a(x,y):=\left(\frac{(2D+1)}{2(2F-1)}xy-\frac{(D+1)}{4F}y^2-\frac{D}{2}\right)^{\frac{-1}{2}}$ and particularizing $\varepsilon:=2(F-1)$. In these local coordinates, the period annulus is in the quadrant $y\geqslant0$ and $x\geqslant\vartheta_\varepsilon$, where ϑ_ε is given by (12) with $\mu=2$. Working on a compact subset K_a of $\{D\in(-1,0),F>\frac{1}{2}\}$, we see that $U_a(x,y)$ has an absolutely convergent Taylor series at $(x,y)=(0,0)$ on $|x|,|y|\leqslant r$ for some $r>0$ depending only on K_a. By rescaling the local coordinates, we can assume that $r=1$. Let Φ be the local diffeomorphism such that $(z,w)=\Phi(x,y)$, i.e. the one obtained by inverting (20).

Since the Loud system (1) is invariant by the symmetry $(u,v) \longmapsto (u,-v)$, half of the period function is the Dulac time T of the singular point at infinity between transverse sections $\Sigma_1 := \{v=0, u \approx 1\}$ and $\Sigma_2 := \{v=0, u \approx -\infty\}$. We decompose it in three parts. Let $T_2(s)$ be the local Dulac time between the normalized transverse sections $\Sigma_1^n := \Phi(\{y=1\})$ and $\Sigma_2^n := \Phi(\{x=1\})$ starting at the point $\Phi(s+\vartheta_\varepsilon, 0)$, see Figure 4. Let $T_1(s)$ be the time that the trajectory starting at Σ_1 spends to arrive to the point $\Phi(s+\vartheta_\varepsilon, 1)$ in Σ_1^n and let $T_3(s)$ be the time that the trajectory starting at the point $\Phi(1,s)$ spends to arrive to Σ_2. Finally, let $\mathcal{D}(s)$ be the Dulac map between Σ_1^n and Σ_2^n, i.e. $\mathcal{D}(s)$ is defined so that the trajectory starting at $\Phi(s+\vartheta_\varepsilon,1)$ intersects Σ_2^n at $\Phi(1,\mathcal{D}(s))$. By construction, see Figure 4, we have $T(s) = T_1(s) + T_2(s) + T_3(\mathcal{D}(s))$. We now examine the asymptotic expansion of each piece. To this end, we denote by $\mathcal{I}(A)$ the space of functions $h(s;a)$, analytic on $s \in (0, s_0)$, such that $h(s;a)$ and $s\partial_s h(s;a)$ tend to zero, as $s \to 0^+$ uniformly, for a varying in any compact subset of A. Observe that this space is stable with respect to addition and multiplication. We apply Corollary B (with $\ell = k = 1$) to obtain $\varepsilon_0 > 0$ and a uniform asymptotic expansion $T_2(s) = c_{2,0} + c_{2,1}s + sh_2(s)$, with $h_2 \in \mathcal{I}(A)$ and $c_{2,j}$ continuous on $A := K_a \cap \{F \in (1-\varepsilon_0, 1+\varepsilon_0)\}$. As we already remarked just before Theorem A, the graph of the Dulac map $\mathcal{D}(s)$ is a trajectory of the vector field $x(x^2-\varepsilon)\partial x + 2F(1-\frac{x^2}{2F})y\partial y$. Hence, by applying assertion (b) of Corollary A with $\lambda = 2F \geqslant 1$, we deduce that $\mathcal{D}(s) = sh_0(s)$, with $h_0 \in \mathcal{I}(A)$. On the other hand, the time function $T_3(s)$ is analytic in s, whereas $T_1(s)$ is an analytic function on s composed with the continuous function $(s,\varepsilon) \mapsto s + \vartheta_\varepsilon$. Accordingly, they can be written as $T_i(s) = c_{i,0} + c_{i,1}s + sh_i(s)$, with $h_i \in \mathcal{I}(A)$ and $c_{i,0}$ and $c_{i,1}$ continuous on A, for $i = 1,3$. Note that $T_3\big(\mathcal{D}(s)\big) = c_{3,0} + s\hat{h}_3(s)$, with $\hat{h}_3(s) := c_{3,1}h_0(s) + h_0(s)h_3\big(sh_0(s)\big)$ and it can be easily checked that $\hat{h}_3 \in \mathcal{I}(A)$. Summing up the three terms we obtain that the period function of the Loud system is of the form

$$P(s;D,F) = 2T(s;D,F) = c_0(D,F) + c_1(D,F)s + sh(s;D,F),$$

with $h := 2(h_1 + h_2 + \hat{h}_3) \in \mathcal{I}(A)$ and c_i continuous on A. On the other hand, restricting to $A \cap \{F \in (\frac{1}{2}, 1)\}$ the singularity at $(x,y) = (\vartheta_\varepsilon, 0)$ is a linearizable saddle and we can apply [5, Proposition 5.2] to obtain the asymptotic expansion of the period function working with a different parametrization, say $\hat{s}$. The two parametrizations differ by composition with a diffeomorphism $\hat{s} = r(s)$ such that $r(0) = 0$ and $r'(0) = \alpha(D,F) \neq 0$, for $F = 1$. In this other parametrization the coefficient $\hat{c}_1(D,F)$ of $\hat{s}$ is explicitly calculated

$$\hat{c}_1(D,F) = \frac{\sqrt{\pi}(2D+1)}{\sqrt{F(D+1)^3}} \frac{\Gamma((3F-1)/(2F))}{\Gamma((4F-1)/(2F))}.$$

Since $c_1(D,F) = \alpha(D,F)\hat{c}_1(D,F)$ and one can verify that

$$\lim_{F \to 1^-} \hat{c}_1(D,F) = \frac{2(2D+1)}{(D+1)^{\frac{3}{2}}},$$

it follows that $c_1(D,1) \neq 0$, for $D \in (-1,0) \setminus \{-\frac{1}{2}\}$. On account of the continuity of c_1 and $h \in \mathcal{I}(A)$, we conclude that

$$P'(s;D,F) = c_1(D,F) + h(s;D,F) + sh'(s;D,F) \neq 0,$$

in a neighbourhood of any point $(s,D,F) = (0,D_0,1)$ in $(0,1) \times (-1,0) \times (\frac{1}{2}, 2)$, with $D_0 \in (-1,0) \setminus \{-\frac{1}{2}\}$. This concludes the proof of the result. ■

Appendix A

In our approach to the proof of Theorem A, the use of L'Hôpital's rule with uniformity in the parameters is fundamental. We have not found such a version in the literature. For this reason we present here the precise statement that we need together with a proof of it.

Proposition 4.1. *Consider two functions $f_\nu, g_\nu \colon (a,b) \longrightarrow \mathbb{R}$ depending on a parameter ν belonging to an arbitrary topological space Λ and verifying the following:*

(a) *f_ν and g_ν are differentiable on (a,b),*

(b) *$g'_\nu(x) \neq 0$, for all $x \in (a,b)$ and $\nu \in \Lambda$,*

(c) *for all $\nu \in \Lambda$, there exists $L_\nu \in \mathbb{R}$ such that $\lim\limits_{x \to a^+} \frac{f'_\nu(x)}{g'_\nu(x)} = L_\nu$ uniformly on $\nu \in \Lambda$,*

(d) $\sup\{|L_\nu|; \nu \in \Lambda\} < +\infty$,

(e) *there exists $c \in (a,b)$ such that, for each $y \in (a,c)$, we have that $\lim\limits_{x \to a^+} \left|\frac{g_\nu(x)}{g_\nu(y)}\right| = +\infty$, uniformly on $\nu \in \Lambda$ and $\sup\left\{\left|\frac{f_\nu(y)}{g_\nu(y)}\right|; \nu \in \Lambda\right\} < +\infty$.*

Then $\lim\limits_{x \to a^+} \frac{f_\nu(x)}{g_\nu(x)} = L_\nu$, uniformly on $\nu \in \Lambda$.

Proof. For a given $\varepsilon > 0$, we must find $\delta > 0$ such that, if $x \in (a, a+\delta)$, then $\left|\frac{f_\nu(x)}{g_\nu(x)} - L_\nu\right| < \varepsilon$, for all $\nu \in \Lambda$. Let us take $\varepsilon_1 := \min\left(\frac{\varepsilon}{M+3}, 1\right)$, where $M := \sup\limits_{\nu \in \Lambda} |L_\nu|$, which is well defined thanks to assumption (d). From (c) there exists $\delta_1 > 0$ such that, if $c \in (a, a+\delta_1)$, then $\left|\frac{f'_\nu(c)}{g'_\nu(c)} - L_\nu\right| < \varepsilon_1$, for all $\nu \in \Lambda$. Let us fix any $y \in (a, a+\delta_1)$. By the Mean Value Theorem, for each $x \in (a,y)$, there exists $c = c_{x,y,\nu} \in (x,y) \subset (a, a+\delta_1)$ such that $\frac{f_\nu(x) - f_\nu(y)}{g_\nu(x) - g_\nu(y)} = \frac{f'_\nu(c)}{g'_\nu(c)}$. Accordingly,

$$\left| \frac{\frac{f_\nu(x)}{g_\nu(x)} - \frac{f_\nu(y)}{g_\nu(x)}}{1 - \frac{g_\nu(y)}{g_\nu(x)}} - L_\nu \right| = \left| \frac{f'_\nu(c)}{g'_\nu(c)} - L_\nu \right| < \varepsilon_1. \tag{21}$$

On the other hand, the assumption (e) guarantees that there exists $z_y \in (a,y)$ such that, if $x \in (a, z_y)$, then

$$\left|\frac{f_\nu(y)}{g_\nu(x)}\right| < \varepsilon_1 \text{ and } \left|\frac{g_\nu(y)}{g_\nu(x)}\right| < \varepsilon_1\,, \text{ for all } \nu \in \Lambda. \tag{22}$$

Here, we also used that $\frac{f_\nu(y)}{g_\nu(x)} = \frac{f_\nu(y)}{g_\nu(y)} \frac{g_\nu(y)}{g_\nu(x)}$ tends to zero uniformly on $\nu \in \Lambda$, as $x \to a^+$. Note then that $\left|(L_\nu \pm \varepsilon_1)\frac{g_\nu(y)}{g_\nu(x)}\right| < \big(|L_\nu| + \varepsilon_1\big)\varepsilon_1$, and thus

$$-\big(|L_\nu| + \varepsilon_1\big)\varepsilon_1 < (L_\nu \pm \varepsilon_1)\frac{g_\nu(y)}{g_\nu(x)} < \big(|L_\nu| + \varepsilon_1\big)\varepsilon_1. \tag{23}$$

The second inequality in (22) shows in particular that $1 - \frac{g_\nu(y)}{g_\nu(x)} > 0$, because $\varepsilon_1 < 1$, so that, from (21), we get

$$(-\varepsilon_1 + L_\nu)\left(1 - \frac{g_\nu(y)}{g_\nu(x)}\right) + \frac{f_\nu(y)}{g_\nu(x)} < \frac{f_\nu(x)}{g_\nu(x)} < (\varepsilon_1 + L_\nu)\left(1 - \frac{g_\nu(y)}{g_\nu(x)}\right) + \frac{f_\nu(y)}{g_\nu(x)}.$$

Therefore,

$$-\varepsilon_1 - (L_\nu - \varepsilon_1)\frac{g_\nu(y)}{g_\nu(x)} + \frac{f_\nu(y)}{g_\nu(x)} < \frac{f_\nu(x)}{g_\nu(x)} - L_\nu < \varepsilon_1 - (L_\nu + \varepsilon_1)\frac{g_\nu(y)}{g_\nu(x)} + \frac{f_\nu(y)}{g_\nu(x)}.$$

From this, on account of (23) and the first inequality in (22), we get that

$$-2\varepsilon_1 - (|L_\nu| + \varepsilon_1)\varepsilon_1 < \frac{f_\nu(x)}{g_\nu(x)} - L_\nu < 2\varepsilon_1 + (|L_\nu| + \varepsilon_1)\varepsilon_1.$$

Accordingly,

$$\left|\frac{f_\nu(x)}{g_\nu(x)} - L_\nu\right| < \varepsilon_1(2 + |L_\nu| + \varepsilon_1) < \varepsilon_1(3 + |L_\nu|) < \varepsilon_1(3 + M) < \varepsilon,$$

as desired, and so, taking $\delta = z_y - a$, the result follows. ■

Appendix B

In this section we recall the notions of Darboux and Liouville local integrability of an analytic family $Y_\lambda = A_\lambda(x, y)\partial_x + B_\lambda(x, y)\partial_y$ of planar vector fields defined on some open subset $U \subset \mathbb{R}^2$ and parametrized by λ belonging to another open subset $V \subset \mathbb{R}^n$. For each $\lambda \in V$, we consider the dual 1-form $\omega_\lambda = -B_\lambda(x, y)dx + A_\lambda(x, y)dy$. The family $\{Y_\lambda, \lambda \in V\}$ can also be thought of as a single analytic vector field Y defining a one-dimensional foliation on $U \times V$. In the same vein we can also think of the family $\{\omega_\lambda, \lambda \in V\}$ as a single 1-form ω on $U \times V$, but in general it is not integrable (i.e. it does not define a codimension one foliation). From this point of view, ω_λ is the restriction of ω to the slice λ.

Definition 4.2. The unfolding Y_λ is *locally Darboux integrable* if the dual 1-form ω_λ admits a local meromorphic integrating factor $g_\lambda(x, y)$ which is analytic in λ, i.e. such that $d(g_\lambda\omega_\lambda) = 0$. The unfolding Y_λ is *locally Liouville integrable* if there exists an analytic family η_λ of local meromorphic differential 1-forms such that $d\omega_\lambda = \omega_\lambda \wedge \eta_\lambda$ and $d\eta_\lambda = 0$. □

Remark 4.3. Notice that Darboux integrability implies Liouvillian integrability because the logarithmic derivative $\eta_\lambda := d \log g_\lambda = \frac{dg_\lambda}{g_\lambda}$ of a meromorphic integrating factor g_λ of ω_λ is closed and satisfies $d\omega_\lambda = \omega_\lambda \wedge \eta_\lambda$. Moreover, the equality $g_\lambda d\omega_\lambda = \omega_\lambda \wedge dg_\lambda$ implies that the zeros and poles of g_λ are invariant by ω_λ. On the other hand, if ω_λ is Liouville integrable, then $g_\lambda = \exp(\int \eta_\lambda)$ is a (not necessarily meromorphic) integrating factor of ω_λ. □

According to [11], any analytic unfolding of a saddle-node (i.e. an analytic family of vector fields with a saddle-node singularity at some parameter value λ_0) of codimension $\mu \geqslant 1$ is analytically orbitally equivalent to a (non unique) unfolding in the following prenormal form:

$$Y_\lambda(x, y) = P_\lambda(x)\partial_x + \big(P_\lambda(x)R_{0,\lambda}(x) + R_{1,\lambda}(x)y + y^2R_{2,\lambda}(x, y)\big)\partial_y, \tag{24}$$

where $P_\lambda(x) = x^{\mu+1} + \nu_{\mu-1}(\lambda)x^{\mu-1} + \cdots + \nu_1(\lambda)x + \nu_0(\lambda)$, with $P_{\lambda_0}(x) = x^{\mu+1}$, $R_{1,\lambda}(x) = 1 + a(\lambda)x^\mu$ and $R_{0,\lambda}$ and $R_{2,\lambda}$ are germs of holomorphic functions. The dual form of the vector field Y_λ is

$$\omega_\lambda = P_\lambda(x)dy - \big(P_\lambda(x)R_{0,\lambda}(x) + yR_{1,\lambda}(x) + y^2R_{2,\lambda}(x, y)\big)dx.$$

Remark 4.4. If $R_{2,\lambda}(x, y) \equiv 0$ (respectively, $R_{0,\lambda}(x) \equiv 0$ and $R_{2,\lambda}(x, y) = y^{k-1}\bar{R}_{2,\lambda}(x)$, for some $k \geqslant 1$) then ω_λ is Liouville integrable with $\eta_\lambda = -\frac{R_{1,\lambda}(x)+P'_\lambda(x)}{P_\lambda(x)}\,dx$ (respectively, $\eta_\lambda = \frac{kR_{1,\lambda}(x)-P'_\lambda(x)}{P_\lambda(x)}dx - (k+1)\frac{dy}{y}$). On the other hand, if $R_{0,\lambda} \equiv 0$ and $R_{2,\lambda} \equiv 0$, then ω_λ is Darboux integrable with inverse integrating factor $g_\lambda(x, y) = yP_\lambda(x)$. □

The next result shows that the converse of the last assertion in the previous remark is also true.

Proposition 4.5. *Any locally Darboux integrable saddle-node unfolding is analytically orbitally equivalent to (24) with $R_{0,\lambda} \equiv 0$ and $R_{2,\lambda} \equiv 0$.*

Proof. By the preparation theorem in [11] we can assume that the saddle-node unfolding has the form (24), although as a matter of fact we will only use that $R_{1,\lambda_0}(0) \neq 0$. Let $g_\lambda(x, y)$ be a meromorphic integrating factor of ω_λ. We claim that the singular point $(0, 0, \lambda_0) \in \mathbb{R}^{n+2}$ of Y possesses an analytic center manifold. In this regard it is well known (see for instance [1]) that there exists a unique formal series $y = \hat{c}_\lambda(x) \in \mathbb{C}[[x,\lambda]]$ satisfying

$$\frac{dy}{dx} = \frac{P_\lambda(x)R_{0,\lambda}(x) + yR_{1,\lambda}(x) + y^2R_{2,\lambda}(x,y)}{P_\lambda(x)}.$$

Thus, by applying the Center Manifold Theorem (see for instance [3, 4]), if the claim is not true then all the invariant analytic hypersurfaces of Y passing through $(0,0,\lambda_0) \in \mathbb{R}^{n+2}$ are contained in $P_\lambda(x) = 0$. Consider the prime decomposition of $P_\lambda = \prod_j P_{j,\lambda}^{r_j}$ in the ring of convergent complex power series at $(x,\lambda) = (0,\lambda_0)$ with $\gcd(P_{i,\lambda}, P_{j,\lambda}) = 1$ if $i \neq j$. Since the zeros and poles of a meromorphic integrating factor define analytic invariant varieties (recall Remark 4.3), we can assert that

$$g_\lambda(x,y) = u_\lambda(x,y)\tilde{P}_\lambda(x) \text{ with } u_\lambda \text{ a unity and } \tilde{P}_\lambda := \prod_j P_{j,\lambda}^{k_j}, \text{ for some } k_j \in \mathbb{Z}.$$

For convenience, let us write $u_\lambda = \exp(t_\lambda)$ for some analytic function $t_\lambda = t_\lambda(x,y)$. Then some computations show that

$$\begin{aligned} 0 = \frac{1}{g_\lambda} d(g_\lambda\omega_\lambda) &= d\omega_\lambda + \frac{dg_\lambda}{g_\lambda} \wedge \omega_\lambda = d\omega_\lambda + \left(dt_\lambda + \sum_j k_j \frac{P'_{j,\lambda}}{P_{j,\lambda}} dx \right) \wedge \omega_\lambda \\ &= \left(P'_\lambda + R_{1,\lambda} + \partial_y(y^2R_{2,\lambda}) + P_\lambda\partial_x t_\lambda + (R_{0,\lambda}P_\lambda + R_{1,\lambda}y + R_{2,\lambda}y^2)\partial_y t_\lambda + P_\lambda \sum_j k_j \frac{P'_{j,\lambda}}{P_{j,\lambda}} \right) dx \wedge dy. \quad (25) \end{aligned}$$

Taking the limit $(x,y,\lambda) \longrightarrow (0,0,\lambda_0)$ above we get that $R_{1,\lambda_0}(0) = 0$. This is a contradiction and so the claim is true. By a suitable local change of coordinates we can assume that the center manifold is given by $y = 0$. Accordingly, $R_{0,\lambda} \equiv 0$ and the integrating factor must write as $g_\lambda(x,y) = u_\lambda(x,y)\tilde{P}_\lambda(x)y^{k_0}$, with $u_\lambda = \exp(t_\lambda)$ and $\tilde{P}_\lambda$ as before, and $k_0 \in \mathbb{Z}$. Next we will use again that $\frac{1}{g_\lambda}d(g_\lambda\omega_\lambda) = 0$ taking advantage of the previous computation. To this end note that $\frac{dg_\lambda}{g_\lambda} = dt_\lambda + \sum_j k_j \frac{P'_{j,\lambda}}{P_{j,\lambda}}dx + k_0\frac{dy}{y}$. Consequently, in order to obtain $\frac{1}{g_\lambda}d(g_\lambda\omega_\lambda)$, it suffices to add the term $k_0\frac{dy}{y} \wedge \omega_\lambda = k_0(R_{1,\lambda} + yR_{2,\lambda})$ into (25) because $R_{0,\lambda} \equiv 0$. Evaluating at $y = 0$ the equality that we thus obtain we get

$$P'_\lambda + (1+k_0)R_{1,\lambda} + P_\lambda s_\lambda + P_\lambda \sum_j k_j \frac{P'_{j,\lambda}}{P_{j,\lambda}} = 0,$$

where $s_\lambda(x) := \partial_x t_\lambda(x,0)$. Taking the limit $(x,\lambda) \longrightarrow (0,\lambda_0)$ above and using that $R_{1,\lambda_0}(0) \neq 0$, we deduce that $k_0 = -1$. Therefore, on account of $P_\lambda = \prod_j P_{j,\lambda}^{r_j}$, the above expression yields to

$$-s_\lambda = \frac{P'_\lambda}{P_\lambda} + \sum_j k_j \frac{P'_{j,\lambda}}{P_{j,\lambda}} = \sum_j (k_j + r_j)\frac{P'_{j,\lambda}}{P_{j,\lambda}}.$$

Since $s_\lambda(x)$ is analytic and the factors $P_{j,\lambda}$ are pairwise coprime, the above equality implies that $k_j + r_j = 0$ for all j and that $s_\lambda(x) = \partial_x t_\lambda(x,0) = \partial_x \log\big(u_\lambda(x,0)\big)$ is identically zero. The first fact implies that $g_\lambda(x,y) = \frac{u_\lambda(x,y)}{yP_\lambda(x)}$, whereas the second one that $u_\lambda(x,0)$ is a non-zero constant, say v_λ. It is clear then that $\frac{u_\lambda(x,y)}{v_\lambda} = yB_\lambda(x,y) + 1$, with $B_\lambda(x,y)$ an analytic function.

At this point the idea is to use [2, Theorem 2.1] describing what type of first integral admits an arbitrary Darboux integrable 1-form in terms of the integrating factor. However we can not apply directly this result because it does not contemplate the parameter case. Instead, we will adapt its proof to our situation by considering the closed 1-form

$$\Omega_\lambda := \frac{g_\lambda\omega_\lambda}{v_\lambda} - \left(\frac{dy}{y} - \frac{R_{1,\lambda}(x)}{P_\lambda(x)}dx \right).$$

Taking $\frac{g_\lambda(x,y)}{v_\lambda} = \frac{B_\lambda(x,y)}{P_\lambda(x)} + \frac{1}{yP_\lambda(x)}$ into account, we get that $\Omega_\lambda = A_\lambda(x,y)dx + B_\lambda(x,y)dy$ with

$$A_\lambda(x,y) := \frac{y}{P_\lambda(x)}\left(R_{1,\lambda}(x)B_\lambda(x,y) + \frac{1}{v_\lambda}u_\lambda(x,y)R_{2,\lambda}(x,y)\right).$$

In particular, $A_\lambda(x,0) = 0$. This, together with the fact that $\partial_y A_\lambda(x,y) = \partial_x B_\lambda(x,y)$ with $B_\lambda(x,y)$ an analytic function, implies that $A_\lambda(x,y)$ is in turn analytic. Therefore, since a closed analytic 1-form must be exact, there exists an analytic function $\alpha_\lambda(x,y)$ such that $\Omega_\lambda = d\alpha_\lambda$. Hence, we conclude that the function $H_\lambda(x,y) = ye^{\alpha_\lambda(x,y)}\exp\left(-\int \frac{R_{1,\lambda}(x)}{P_\lambda(x)}dx\right)$ is a first integral of ω_λ. Finally, making the local change of coordinates $\bar{y} = ye^{\alpha_\lambda(x,y)}$ we obtain that $H_\lambda(x,\bar{y}) = \bar{y}\exp\left(-\int \frac{R_{1,\lambda}(x)}{P_\lambda(x)}dx\right)$. Consequently ω_λ is proportional to $P_\lambda(x)\,d\bar{y} - R_{1,\lambda}(x)\bar{y}\,dx$. This proves the result. ∎

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