On the Chebyshev property of certain Abelian integrals near a polycycle

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Abstract. F. Dumortier and R. Roussarie formulated in [*Birth of canard cycles*, Discrete Contin. Dyn. Syst. **2** (2009) 723–781] a conjecture concerning the Chebyshev property of a collection $I_0, I_1, \ldots, I_n$ of Abelian integrals arising from singular perturbation problems occurring in planar slow-fast systems. The aim of this note is to show the validity of this conjecture near the polycycle at the boundary of the family of ovals defining the Abelian integrals. As a corollary of this local result we get that the linear span $\langle I_0, I_1, \ldots, I_n\rangle$ is Chebyshev with accuracy $k = k(n)$.

1 Introduction and statement of the main result

This paper is concerned with the problem of studying when a collection of Abelian integrals form an extended complete Chebyshev system (see Definition 2.1). This type of problem arises in the context of the so-called *infinitesimal Hilbert's 16th problem* proposed by Arnold [1]. In the present paper we are interested in a conjecture formulated by Dumortier and Roussarie in [4], where the authors consider singular perturbation problems occurring in planar slow-fast systems depending on parameters. They investigate the number of limit cycles that appear near a slow-fast Hopf point, i.e., its *cyclicity*. Their main results show that under very general conditions this cyclicity is finite and, modulo the aforementioned conjecture, provide its sharp upper bound. In order to give a precise statement of their conjecture let us consider the function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $H(x, y) = e^{-x}(1 + x - \frac{1}{2}y^2)$. It can be checked that the level sets $\{H(x, y) = h\}$ for $h \in (0, 1)$ are ovals γ_h surrounding the origin. The family $\{\gamma_h\}_{h\in(0,1)}$ form a *period annulus* and its boundary has two connected components, the parabola $y^2 = 2(x + 1)$ and the origin $(0, 0)$, which are the level sets $h = 0$ and $h = 1$, respectively (see Figure 1). Let us define the family of Abelian integrals

$$I_k(h) := \int_{\gamma_h} y^{2k-1} dx, \quad k \in \mathbb{Z}^+.$$

With this notation, the conjecture posed by Dumortier and Roussarie in [4] is the following:

Conjecture. *For each* $n \geqslant 0$, $(I_0, I_1, \ldots, I_n)$ *is an ECT-system on* $[h_0, 1]$ *for any* $h_0 \in (0, 1)$.

The validity of this conjecture for $n \leqslant 2$ is proved in [5]. Moreover, taking advantage of the analyticity of I_k at $h = 1$, in that paper is also proved that for each $n \geqslant 0$ there exists $\varepsilon > 0$ such that $(I_0, I_1, \ldots, I_n)$ is an ECT-system on $(1 - \varepsilon, 1]$, see [5, Corollary 3.5]. In other words, that the conjecture is true near the

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