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Perturbed damped pendulum: finding periodic solutions via averaging method

(Perturbações do pêndulo amortecido: encontrando soluções periódicas via método "averaging")

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Using the damped pendulum model we introduce the averaging method to study the periodic solutions of dynamical systems with small non–autonomous perturbation. We provide sufficient conditions for the existence of periodic solutions with small amplitude of the non–linear perturbed damped pendulum. The averaging method provides a useful means to study dynamical systems, accessible to Master and PhD students.
Keywords: averaging method, periodic solutions, nonlinear systems, damped pendulum.

Utilizando o modelo do pêndulo amortecido, introduzimos o método "averaging" no estudo de soluções periódicas de sistemas dinâmicos com pequenas perturbações não autônomas. Considerando perturbações do sistema do pêndulo amortecido, fornecemos condições suficientes para a existência de soluções periódicas de pequena amplitude. O método "averaging" fornece uma ferramenta útil no estudo de sistemas dinâmicos e é acessível a estudantes de pós-graduação.
Palavras-chave: método "averaging", soluções periódicas, sistemas não lineares, pêndulo amortecido.

1. Introduction

Systems derived from the pendulum give to students important and practical examples of dynamical systems. For instance, we can see the *weight-driven pendulum clocks* which had its historical and dynamical aspects studied by Denny in a recent paper [1]. This system has been revisited by Llibre and Teixeira in [2], and using some simple techniques, from averaging theory, they got the same results. Usually, the systems involving pendulums have also been used to introduce mathematical concepts of classical mechanics, as we can see in Ref. [3].

In this paper we attempt to use a simple physical system, as the damped pendulum, to introduce some concepts and techniques of the important and useful averaging theory, which can be used to study the periodic solutions of dynamical systems. For instance, in Ref. [4], Llibre, Novaes and Teixeira have used the averaging theory to provide sufficient conditions for the existence of periodic solutions of the planar double pendulum with small oscillations perturbed by non–linear functions.

2. The damped pendulum

We consider a system composed of a point mass m moving in the plane, under gravity force, such that the distance between the point mass m and a given point P is fixed and equal to l. We also consider that the motion of the particle suffers a resistance proportional to its velocity. This system is called *Damped Pendulum.*

The position of the pendulum is determined by the angle θ shown in Fig. 1. The equation of motion of this system is given by

$$\ddot{\theta} = -a\sin(\theta) - b\dot{\theta}, \tag{1}$$

where $a > 0$ and $b > 0$ are real parameters, with $a = g/l$, g the acceleration of the gravity, l the length of the rod and b the damping coefficient. We shall also assume that the damping coefficient b is a small parameter.

There are many other kinds of resistance that the particle motion can suffer, providing many different dynamical behaviors (see Remark 1). For instance, the *Coulomb Friction* introduces a discontinuous term in the equation of motion (1). For more details about this last issue, see the book of Andronov *et al.* [5].

In the qualitative theory of dynamical systems, a singularity x^* of an autonomous differential system $\dot{x}(t) = F(x)$, *i.e.* $F(x^*) = 0$, is called *Hyperbolic* if the eigenvalues of the linear transformation $DF(x^*)$ (derivative of F in x^*) has non–zero real components. In this case, applying the classical *Hartman-Grobman Theorem* (see Theorem 2.2.3 from Ref. [6]) we can study the local behavior of the system looking to the linearized

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