ON EXTENDED CHEBYSHEV SYSTEMS WITH POSITIVE ACCURACY

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Abstract. A classical necessary condition for an ordered set of $n+1$ functions $\mathcal{F}$ to be an ECT-system in a closed interval is that all the Wronskians do not vanish. With this condition all the elements of Span($\mathcal{F}$) have at most n zeros taking into account the multiplicity. Here the problem of bounding the number of zeros of Span($\mathcal{F}$) is considered when some of the Wronskians vanish. An application to counting the number of isolated periodic orbits for a family of non-smooth systems is done.

1. Introduction and statement of the main results

Let $\mathcal{F}=[u_0,\dots,u_n]$ be an ordered set of functions of class $\mathcal{C}^n$ on the closed interval $[a,b]$. We denote by $Z(\mathcal{F})$ the maximum number of zeros counting multiplicity that any nontrivial function $v\in\mathrm{Span}(\mathcal{F})$ can have. Here Span($\mathcal{F}$) is the set of functions generated by liner combinations of elements of $\mathcal{F}$, that is $v(s)=a_0u_0(s)+a_1u_1(s)+\cdots+a_nu_n(s)$ where a_i for $i=0,1,\dots,n$ are real numbers.

The theory of Chebyshev systems is a classical tool to study the quantity $Z(\mathcal{F})$. In this theory, when $Z(\mathcal{F})\leq n$, the set $\mathcal{F}$ is called an *extended Chebyshev system* or ET-system on $[a,b]$ (see [6]); and when $Z(\mathcal{F})\leq n+k$, the set $\mathcal{F}$ is called an extended Chebyshev system with *accuracy* k on $[a,b]$ (see [5]). Following the book of Karlin and Studden [6] we can see that the condition $W(u_0,u_1,\dots,u_n)(t)\neq 0$ implies that $\mathcal{F}$ is an ET-system, the converse implication, in general, is not true. Here $W(u_0,u_1,\dots,u_k)(t)$ denotes the Wronskian of the ordered set of functions $[u_0,u_1,\dots,u_k]$ with respect to t. We recall the definition of the Wronskian of a set of functions:

$$W_k(t)=W_k(u_0,\dots,u_k)(t)=\det\big(M(u_0,\dots,u_k)(t)\big), \tag{1}$$

where

$$M(u_0,\dots,u_k)(t)=\begin{pmatrix} u_0(t) & \cdots & u_k(t)\\ u_0'(t) & \cdots & u_k'(t)\\ \vdots & \ddots & \vdots\\ u_0^{(k)}(t) & \cdots & u_k^{(k)}(t)\end{pmatrix}.$$

We say that $\mathcal{F}$ is an *Extended Complete Chebyshev* system or an ECT-system on a closed interval $[a,b]$ if and only if for any k, $0\leq k\leq n$, $[u_0,u_1,\dots,u_k]$ is an ET-system. In order to prove that $\mathcal{F}$ is a ECT-system on $[a,b]$ is sufficient

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2010 *Mathematics Subject Classification.* Primary: 37G15, 37C27.

Key words and phrases. Number of zeros of real functions, ECT-System, Zeros of Melnikov functions for non-smooth systems.

and necessary to show that $W(u_0, u_1, \dots, u_k)(t) \neq 0$ on $[a, b]$ for $0 \leq k \leq n$, see [6].

Initially Chebyshev systems were used in approximation theory in the study of spline functions and in the theory of fine moment, see [6] and [1] for more recent results on this field. Lately they where used in the theory of differential equations to study versal unfoldings of singularities of vector fields, see [11, 13]. Recently it has also been used to study the period function of centers of potential systems, see [9]. ECT systems are used, in qualitative theory of differential equations, to study the number of isolated periodic orbits (limit cycles) bifurcating from a period annulus, see also [13]. More concretely, this technique is useful to get upper bounds for the number of zeros of the Poincaré–Pontryaguin–Melnikov function. In fact these studies provide lower bounds for the so called weak Hilbert 16th problem, see [2, 14]. Nevertheless, when this set of functions is not an ECT-system, as far as we know, there are no well developed tools to deal with this problem. Our main goal in this paper is to establish results, similar to that ones in the Chebyshev theory, for systems which are not an ECT-system.

The next theorem, that we will prove in Section 3 extends the results of ECT-system when some Wronskians of the ordered set $\mathcal{F}$ vanish. Other extensions of this theory can be found in [10, 15], where the initial set is embedded into an ECT-system.

Theorem 1.1. *Let $\mathcal{F} = [u_0, u_1, \dots, u_n]$ be an ordered set of smooth functions on $[a, b]$. Assume that, for some $0 < \zeta < n$, all the Wronskians are nonvanishing except $W_{n-\zeta}(x)$ and $W_n(x)$, which have k and ℓ zeros on (a, b), respectively, and all the zeros are simple. Then $n \leq Z(\mathcal{F}) \leq n + \zeta\, k$, when $\ell = 0$, and $n + 1 \leq Z(\mathcal{F}) \leq n + \zeta\, k + \ell$, when $\ell \geq 1$.*

We shall see in Section 2 that the lower bounds of the above result can be realized by simple zeros. In Section 4 we give some examples showing that these lower bounds cannot be improved in general.

We also have that the upper bounds of the above result are still true if we assume only the n-differentiability of the functions in $\mathcal{F}$. Moreover we show in the next two results, for the particular case $\zeta = 1$, that these upper bounds cannot be improved. We do note prove the optimality of them for all values of k, ℓ, and ζ.

Proposition 1.2. *Let n and ℓ be positive integers. There exists an ordered polynomial set, $\mathcal{F} = [u_0, u_1, \dots, u_n]$, such that all the Wronskians are nonvanishing except $W_n(x)$, which has exactly ℓ simple zeros, and with an element in* Span$(\mathcal{F})$ *having exactly $n + \ell$ simple zeros. In particular $Z(\mathcal{F}) = n + \ell$.*

Proposition 1.3. *Let n and k be positive integers. There exists an ordered polynomial set, $\mathcal{F} = [u_0, u_1, \dots, u_n]$, such that all the Wronskians are nonvanishing except $W_{n-1}(x)$, which has exactly k simple zeros, and with an element in* Span$(\mathcal{F})$ *having exactly $n + k$ simple zeros. In particular $Z(\mathcal{F}) = n + k$.*

Proposition 1.4. *Let n, k, and ℓ be positive integers. There exists an ordered polynomial set, $\mathcal{F} = [u_0, u_1, \dots, u_n]$, such that all the Wronskians are* *nonvanishing except $W_{n-1}(x)$ and $W_n(x)$, which have k and ℓ zeros on (a,b), respectively, and with an element in* Span$(\mathcal{F})$ *having exactly $n+k+\ell$ simple zeros. In particular $Z(\mathcal{F}) = n+k+\ell$.*

In [4] is proved that the set $[1, x, \sqrt{x+1}, x\sqrt{x+1}, \sqrt{x}, x\sqrt{x}, x^2\sqrt{x}]$ is an ET-system with accuracy 1 on $(0,\infty)$. This fact can be obtained from Proposition 1.2 with $k=1$ because all the ordered Wronskians are nonvanishing except the last one that has exactly one positive zero taking as the interval of definition any closed interval in $(0,\infty)$. Another set considered in [4] is $\mathcal{F} = [\cup_{i=0}^{2k-1}\{\sqrt{x+a_i}\} \cup_{i=k}^{2k-1} \{x\sqrt{x+a_i}\}]$. In that paper, taking $k=3$ for example, the authors prove that the number of zeros of the span of $\mathcal{F}$ is lower than or equal to $4k-1=11$. For the concrete values $a_0 = 0, a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 2, a_5 = -7$, for example, this upperbound can be obtained from Theorem 1.2 with $\zeta = 2$, $k = \ell = 1$ because all the Wronskians are nonvanishing except W_6 and W_8 that vanish exactly ones. Moreover we can also prove that $9 \leq Z(\mathcal{F})$. Another direct application of this work can be found in [7] where the set of functions is an ET-system with accuracy 1.

Finally in Section 5, as a nontrival application of the above results, we improve the results of [8] where the maximum number of limit cycles for a class of nonsmooth systems is studied. Here we prove that this maximum is three.

2. Lower bounds of the number of zeros

In [3] it is proved that for a family of $n+1$ linearly independent analytical functions, such that at least one of that has constant sign in its domain, there exists a linear combination of these functions having at least n simple zeros. The next theorem extends this result showing that for each configuration of $m \leq n$ zeros, taking into account their multiplicity, there exists a linear combination of those function having this configuration. We also provide sufficient conditions which assure the same result but for each configuration of $m \leq n+1$ zeros.

Theorem 2.1. *Let $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ be an ordered set of real smooth functions on (a,b) such that there exists $\xi \in (a,b)$ with $W_{n-1}(\xi) \neq 0$. Then next properties hold:*

- (a) *If $W_n(\xi) \neq 0$ then for each configuration of $m \leq n$ zeros, taking into account their multiplicity, there exists $F \in$* Span$(\mathcal{F})$ *with this configuration of zeros.*
- (b) *If $W_n(\xi) = 0$ and $W_n^{(p)}(\xi) \neq 0$ for some $p \geq 1$ then for each configuration of $m \leq n+1$ zeros, taking into account their multiplicity, there exists $F \in$* Span$(\mathcal{F})$ *with this configuration of zeros.*

Proof. First we will look for an element in the Span$(\mathcal{F})$ with a zero of the highest multiplicity. Secondly we will perturb it inside Span$(\mathcal{F})$ in order to have the prescribed configuration of zeros. We point out that the first part is common for both statements but the second is not.

As $W_{n-1}(\xi) \neq 0$ there exists a unique function $F_0(x) = \sum_{i=0}^{n} a_i u_i(x) \in$ Span$(\mathcal{F})$ such that $F_0(\xi) = 0$, $F_0^{(i)}(\xi) = 0$ for $i = 1, \ldots, n-1$ and $a_n = 1$.

The coefficients a_i's can be obtained from the linear system of equations

$$\begin{pmatrix} u_0(\xi) & \cdots & u_{n-1}(\xi) \\ u_0'(\xi) & \cdots & u_{n-1}'(\xi) \\ \vdots & \ddots & \vdots \\ u_0^{(n-1)}(\xi) & \cdots & u_{n-1}^{(n-1)}(\xi) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = - \begin{pmatrix} u_n(\xi) \\ u_n'(\xi) \\ \vdots \\ u_n^{(n-1)}(\xi) \end{pmatrix}.$$

Using the Cramer's rule we get

$$a_i = -\frac{W_{n-1}(u_0, \ldots, \overset{(i)}{u_n}, \ldots, u_{n-1})(\xi)}{W_{n-1}(\xi)} = (-1)^{n-i} \frac{W_{n-1}(u_0, \ldots, \widehat{u_i}, \ldots, u_n)(\xi)}{W_{n-1}(\xi)}. \tag{2}$$

The notation $(u_0, \ldots, \widehat{u_i}, \ldots, u_n)$ means that the element u_i is removed.

From the Leibniz formula for determinants we can compute the ℓ-derivative of the Wronskian $W_n(\xi)$ as

$$\begin{aligned} W_n^{(\ell)}(\xi) &= \begin{vmatrix} u_0(\xi) & \cdots & u_n(\xi) \\ \vdots & & \vdots \\ u_0^{(n-1)}(\xi) & \cdots & u_n^{(n-1)}(\xi) \\ u_0^{(n+\ell)}(\xi) & \cdots & u_n^{(n+\ell)}(\xi) \end{vmatrix} \\ &= \sum_{i=0}^{n} (-1)^{n-i} u_i^{(n+\ell)} W_{n-1}(u_0, \ldots, \widehat{u_i}, \ldots, u_n)(\xi). \end{aligned}$$

For simplificity we have denoted $G^{(0)} = G$. This concludes the first part of the proof. So from (2) we write

$$\begin{aligned} F_0^{(n+\ell)}(\xi) &= \sum_{i=0}^{n-1} a_i u_i^{(n+\ell)}(\xi) + u_n^{(n+\ell)}(\xi) \\ &= \frac{1}{W_{n-1}(\xi)} \sum_{i=0}^{n-1} (-1)^{n-i} u_i^{(n+\ell)}(\xi) W_{n-1}(u_0, \ldots, \widehat{u_i}, \ldots, u_n)(\xi) \\ &= \frac{W_n^{(\ell)}(\xi)}{W_{n-1}(\xi)}. \end{aligned}$$

Consequently, if $q \geq 0$ is the smallest integer such that $W_n^{(q)}(\xi) \neq 0$, we write

$$\begin{aligned} F_0(x) &= \sum_{i=0}^{n+q} \frac{F_0^{(i)}(\xi)}{i!} (x-\xi)^i + O_{n+q+1}(x-\xi) \\ &= \frac{W_n^{(q)}(\xi)}{(n+q)! W_{n-1}(\xi)} (x-\xi)^{n+q} + O_{n+q+1}(x-\xi). \end{aligned}$$

Now we consider the perturbation $F_\varepsilon(x) = \sum_{i=0}^{n-1}(a_i+\varepsilon_i)u_i(x) + u_n(x) \in \mathrm{Span}(\mathcal{F})$ of F_0. Straightforward computations show that

$$
\begin{aligned}
F_\varepsilon(x) = & \sum_{i=0}^{n+q} \frac{F_\varepsilon^{(i)}(\xi)}{i!}(x-\xi)^i + O_{n+q+1}(x-\xi) \\
= & \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} u^{(j)}(\xi)\varepsilon_j\right)\frac{(x-\xi)^i}{i!} + \sum_{i=n}^{n+q}\frac{A_i}{i!}(x-\xi)^i + O_{n+q+1}(x-\xi)
\end{aligned}
\tag{3}
$$

with $A_i = O(\varepsilon)$ for $i = n, \ldots, n+q-1$, $A_{n+q} = \dfrac{W_n^{(q)}(\xi)}{W_{n-1}(\xi)} + O(\varepsilon)$ and $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1})$.

The proof of statement (a), $q = 0$, follows using that $W_{n-1}(\xi) \neq 0$, $W_n(\xi) \neq 0$, and the Malgrange Preparation Theorem, see [12]. In particular, there exists a smooth function h such that

$$
F_\varepsilon(x) = \left(\sum_{i=0}^{n-1}\delta_i(x-\xi)^i + (x-\xi)^n\right) h(x,\delta)
$$

with $\delta = (\delta_0, \ldots, \delta_{n-1})$ and $h(0,0) \neq 0$.

When $q \geq 1$ we consider a second perturbation, $\overline{F}_{\overline{\varepsilon}}(x) = F_\varepsilon(x+\varepsilon_n)$ where $\overline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_n)$. Then, writing this perturbed function in powers of $x-\xi$, the conditions $W_{n-1}(\xi) \neq 0$, $W_n^{(q)}(\xi) \neq 0$ provides a change of variables in the parameter space such that, using the Malgrange Preparation Theorem as above, there exists a smooth function $\overline{h}$ for which

$$
\overline{F}_{\overline{\varepsilon}}(x) = \left(\sum_{i=0}^{n+q-1} B_i(\overline{\delta})(x-\xi)^i + (x-\xi)^{n+q}\right)\overline{h}(x,\overline{\delta}) \tag{4}
$$

with $\overline{\delta} = (\overline{\delta}_0, \ldots, \overline{\delta}_n)$, $B_i = \overline{\delta}_i$, $i = 0, \ldots, n-1$, $B_i = O(\overline{\delta})$, $i = n, \ldots, n+q-2$, $B_{n+q-1} = \overline{\delta}_n$, and $\overline{h}(0,0) \neq 0$. In (4) we do note have enough information about the functions B_i, for $i = n, \ldots, n+q-2$, to ensure the existence of more than n+1 zeros, taking into account their multiplicity. This completes the proof of statement (b). □

3. Proofs of the main results

We start this section recalling some relations between a set of functions and their Wronskians (see [6, sec. 2 chap. XI]). In particular, we link them with the Division-Derivation algorithm (see [13, p. 119]).

Let $w_0, w_1, \dots, w_n$ be nonidentically zero functions such that w_i is of class $\mathcal{C}^{n-i}$ on $[a,b]$. If we define

$$
\begin{aligned}
u_0(x) &= w_0(x),\\
u_1(x) &= w_0(x)\int_a^x w_1(s_1)ds_1,\\
u_2(x) &= w_0(x)\int_a^x w_1(s_1)\int_a^{s_1} w_2(s_2)ds_2ds_1,\\
&\vdots\\
u_n(x) &= w_0(x)\int_a^x w_1(s_1)\int_a^{s_1} w_2(s_2)\cdots w_{n-1}(s_{n-1})\int_a^{s_{n-1}} w_n(s_n)ds_n\cdots ds_1,
\end{aligned}
\tag{5}
$$

then straightforward calculations establishes a first relation between the ordered set $[u_0, u_1, \dots, u_n]$ with their Wronskians,

$$W_k(x) = W(u_0, u_1, \dots, u_k)(x) = (w_0(x))^{k+1}(w_1(x))^k \dots (w_{k-1}(x))^2(w_k(x)) \tag{6}$$

for $k = 0, 1, \dots, n$. These expressions write, recurrently, as

$$w_0(x) = W_0(x), \quad w_1(x) = \frac{W_1(x)}{(W_0(x))^2}, \quad \text{and} \quad w_k(x) = \frac{W_{k-2}(x)W_k(x)}{(W_{k-1}(x))^2}, \tag{7}$$

for $k = 2, 3, \dots, n$. We remark that the functions $\{u_0, u_1, \dots, u_n\}$ are linearly independent. By introducing the differential operators

$$D_j v = \frac{d}{dx}\frac{v}{w_j}, \quad \text{for} \quad j = 0, 1, \dots, n, \tag{8}$$

the Division-Derivation algorithm for the functions u_i, defined in (5), give us the second relation

$$w_{j+1} = D_j D_{j-1} \cdots D_0\, u_{j+1}, \quad \text{for} \quad j = 0, 1, \dots, n-1. \tag{9}$$

Furthermore for any set of $\mathcal{C}^n$ linearly independent functions $u_0, u_1, \dots, u_n$, if we take (9) as the definition of the functions w_i for $i = 0, 1, \dots, n$, then the equality (6) holds.

The following result is the first step of the proof of our main result. Moreover, it also provides the equivalence between the definition of an ECT-system of functions and the nonvanishing property of their Wronskians.

Lemma 3.1. *Let $\mathcal{F} = [u_0, u_1, \dots, u_n]$ be an ordered set of $\mathcal{C}^n$ functions on $[a,b]$. Assume that $W_i(x)$ is nonvanishing for $i = 0, 1, \dots, m$ with $m \le n$. For each element $v_0 \in \mathrm{Span}(\mathcal{F})$, we write*

$$v_0(s) = a_0u_0(s) + a_1u_1(s) + \cdots + a_nu_n(s) \tag{10}$$

and, for $m \ge 1$, the Division-Derivation algorithm applied m times to (10) gives

$$v_m(x) = \ a_m w_m + \sum_{j=m+1}^{n} a_j D_{m-1}D_{m-2}\cdots D_1 D_0 u_j. \tag{11}$$

If v_m has at most N zeros then, for $0 \le \mu \le m$, $v_{m-\mu}$ has at most $N + \mu$ zeros.

Proof. We start proving the case $m = 1$. Since $u_0(x) = w_0(x) = W_0(x) \neq 0$ for every $x \in [a, b]$, we can divide (10) by $w_0(x)$ and to study the function

$$\tilde{v}_0(x) = a_0 + a_1 \frac{u_1(x)}{w_0(x)} + \cdots + a_n \frac{u_n(x)}{w_0(x)}.$$

Taking $v_1(x) = (d/dx)\tilde{v}_0(x)$ we obtain from (9) that

$$v_1(x) = a_1 w_1(x) + a_2 D_0 u_2 + \cdots + a_n D_0 u_n.$$

We note that if v_1 has at most N zeros, then v_0 has at most $N + 1$ zeros.

When $m > 1$, we have $W_i \neq 0$, or equivalently $w_i \neq 0$, for $i \leq m$. From (8) and (9), straightforward computations show that the Division-Derivation algorithm applied m times to (10) gives (11). The proof finishes using, recursively, the Rolle's Theorem μ times. □

Now we continue with the proof of the main result.

Proof of Theorem 1.1. Applying the first part of Lemma 3.1 for $m = n - \zeta$ we get

$$v_{n-\zeta}(x) = a_{n-\zeta} w_{n-\zeta}(x) + \sum_{j=n-\zeta+1}^{n} a_j D_{n-\zeta-1} D_{n-\zeta-2} \cdots D_1 D_0 u_j.$$

We note that the maximum number of zeros Z_ζ that $v_{n-\zeta}$ can have among any choose of elements $v_0 \in \mathrm{Span}(\mathcal{F})$, corresponds with the case when it does not vanish at the k zeros of $w_{n-\zeta}$. These zeros define a partition of $[a, b]$, namely $[a, b] = \overline{\cup_{i=0}^{k} I_i}$.

In order to estimate Z_ζ we apply, once more, the first part of the Lemma 3.1, now for $m = n$, to obtain $v_n(x) = a_n w_n(x)$. We note that Z_ζ also corresponds with the case when $w_{n-\zeta}$ does not vanish at the ℓ zeros of w_n. So we assume that each interval I_i contains $0 \leq \ell_i \leq \ell$ zeros of w_n such that $\sum_{i=0}^{k} \ell_i = \ell$.

Applying the second part of Lemma 3.1, for $m = n$ and $\mu = \zeta$ we obtain that each interval I_i contains at most $\zeta + \ell_i$ zeros. Therefore $v_{n-\zeta}$ contains at most $(k + 1)\zeta + \ell$ zeros. Finally, applying the second part of Lemma 3.1 for $\mu = m = n - \zeta$ the maximum number of zeros that v_0 can have is $m + N = (n - \zeta) + (k + 1)\zeta + \ell = n + \zeta\, k + \ell$, so $Z(\mathcal{F}) \leq n + \zeta\, k + \ell$. The lower bounds $n \leq Z(\mathcal{F})$ when $\ell = 0$, and $n + 1 \leq Z(\mathcal{F})$ when $\ell \geq 1$ follow, immediately, from Theorem 2.1. □

Proof of Proposition 1.2. Let α_i, $i = 0, \ldots, k - 1$, be real numbers. Taking $w_i(x) = 1$ for $i = 0, \ldots, n - 1$, and

$$w_n(x) = x^k + \sum_{i=0}^{k-1} \alpha_i x^i,$$

consequently $W_i(x) = w_i(x)$ $i = 0, \ldots, n$. From (5) we get $u_i(x) = x^i$ for $i = 1, \ldots, n - 1$, and

$$u_n(x) = \sum_{i=0}^{k} \frac{i!}{(n + i)!} \alpha_i x^{i+n}.$$

These functions define the ordered set $\mathcal{F} = [u_0, u_1, \ldots, u_n]$. Therefore, the function

$$v = \sum_{j=0}^{n-1} a_j u_j(x) + u_n(x) = \sum_{j=0}^{n+k} b_j \, x^j, \tag{12}$$

is in $\mathrm{Span}(\mathcal{F})$, where

$$b_j = \begin{cases} \dfrac{a_j}{j!}, & \text{for} \quad j = 0, 1, \ldots, n-1, \\ \dfrac{(j-n)!}{j!}\alpha_{j-n}, & \text{for} \quad j = n, \ldots, n+k. \end{cases}$$

We note that (12) is a full polynomial of degree $n+k$ in x for which the parameters b_j can be chosen in an arbitrary way. Clearly there exists a function v with exactly $n+k$ simple zeros. It remains to prove that the function W_n has exactly k zeros. If W_n has less zeros, $\kappa < k$, applying Theorem 1.1, we have that the function v has at most $n+\kappa < n+k$ zeros, which contradicts the conclusion about the number of zeros of v. □

Proof of Proposition 1.3. The proofs follows by adding a new polynomial to the family given in Proposition 1.2 in such a way the last Wronskian is nonvanishing. Then we shall show that there is an element in the span of this family with the prescribed zeros of the statement.

Consider the ordered set of polynomials $[u_0, \ldots, u_{n-1}]$ given in the proof of Proposition 1.2. So $W_i(x) = 1$ for $i = 0, \ldots, n-2$ and

$$W_{n-1}(x) = x^k + \sum_{i=0}^{k-1} \alpha_i x^i = \prod_{j=1}^{k} (x - \xi_j),$$

where ξ_j, $j = 1, \ldots k$ are the simple zeros of W_{n-1}. Now, we obtain the function u_n by taking the last nonvanishing Wronskian

$$W_n(x) = \prod_{j=1}^{k} (x-\xi_i)^2 \left(1 + \sum_{i=1}^{k} \frac{1}{(x-\xi_i)^2}\right) = \prod_{j=1}^{k} (x-\xi_i)^2 + \sum_{i=1}^{k} \prod_{j\neq i}^{k} (x-\xi_j)^2.$$

Then using (5), we compute

$$u_n(x) = \frac{(k+1)!}{(n+k)!} x^{n+k} + \alpha_{k-1} \frac{k!}{(n+k-1)!} x^{n+k-1} + \sum_{j=1}^{k-1} (\alpha_{j-1} - B_j) x^{j+n-1} - B_0 x^{n-1},$$

where $B_j = \sum_{i=1}^{k} \beta_j^i$ and $\prod_{j\neq i}^{k} (x - \xi_j) = \sum_{j=0}^{k-1} \beta_j^i x^j$.

Let $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ and let $v_\varepsilon \in \mathrm{Span}(\mathcal{F})$ given by

$$v_\varepsilon(x) = v_0(x) - \varepsilon \, u_n(x), \tag{13}$$

where v_0 is a monic polynomial of degree $n+k-1$ with $n+k-1$ simple zeros provided by Proposition 1.2. Hence, for $\varepsilon > 0$ small enough, we conclude that v_ε has $n+k-1$ simple zeros close to the zeros of v_0 and another zero which bifurcates from infinity. Moreover this zero is simple because the degree of v_ε coincides with the number of zeros. □

Proof of Proposition 1.4. The proof follows by perturbing the family given in Proposition 1.3 in such a way the last Wronskian has ℓ simple zero. Then we shall show that there is an element in the span of this family with the prescribed zeros of the statement.

Consider the ordered set of polynomials $\mathcal{F}_0 = [u_0, \dots, u_n]$ given in the proof of Proposition 1.3. Let $v_\varepsilon \in \mathrm{Span}(\mathcal{F}_0)$ be the function (13) satisfying that for some fixed ε it has $n+k$ simple zeros. We know that the leading coefficient term (x^{n+k}) of v_ε is $-\varepsilon(k+1)!/(n+k)! < 0$. Now we define $\mathcal{F}_1 = [u_0, \dots, u_n - \varepsilon_1 x^{n+k+1}]$. Clearly, $v^1_{\varepsilon_1} = v_\varepsilon + \varepsilon_1 \varepsilon x^{n+k+1} \in \mathrm{Span}(\mathcal{F}_1)$. Hence, analogously to the proof of Proposition 1.3, we conclude that, for $\varepsilon_1 > 0$ small enough, $v^1_{\varepsilon_1}$ has $n+k$ simple zeros close to the simple zeros of v_ε and another zero which bifurcates from infinity. Moreover W_n has exactly one zero. Indeed, if W_n has no zeros, applying Theorem 1.1, we have that the function $v^1_{\varepsilon_1}$ has at most $n+k$ zeros, which contradicts the conclusion about the number of zeros of $v^1_{\varepsilon_1}$. The above procedure can be repeated in order to construct the ordered set of polynomials

$$\mathcal{F} = [u_0, \dots, u_n - \varepsilon_1 x^{n+k+1} + \varepsilon_2 x^{n+k+2} - \cdots + (-1)^\ell \varepsilon_\ell x^{n+k+\ell}]$$

and an element $v^\ell_{\varepsilon_\ell} = v^{\ell-1}_{\varepsilon_{\ell-1}} - (-1)^\ell \varepsilon_\ell \varepsilon_{\ell-1} \cdots \varepsilon_1 \varepsilon x^{n+k+\ell}$ in $\in \mathrm{Span}(\mathcal{F})$ having exactly $n+k+\ell$ simple zeros in such way that the Wronskian W_n has exactly ℓ zeros. □

4. Optimality of the results

The optimality of the upper bounds of Theorem 1.1, in the case $\zeta = 1$, are done by Propositions 1.2 and 1.3. The results of this section provide examples of families assuring the optimality of their lower bounds. Indeed for these families $Z(\mathcal{F})$ coincides with the lower bound given in Theorem 1.1.

Proposition 4.1. *The ordered set of functions $\mathcal{F} = [1, t, \cos t]$ is an ET-system with accuracy 1 in $[-\pi, \pi]$.*

Proof. Let $f(t) = a + b\,t + c\cos t$ be an element of $\mathrm{Span}(\mathcal{F})$. Clearly, when $b = 0$, $f(t)$ has at most 2 zeros in $[-\pi, \pi]$. For $b \neq 0$, the derivative $f'(t) = b - c\sin t$ has at most 2 zeros in $[-\pi, \pi]$ which implies that $f(t)$ has at most 3 zeros in $[-\pi, \pi]$. □

For the above family $W_0(t) = W_1(t) = 1$ and $W_2(t) = -\cos(t)$ which has two zeros in the interval $[-\pi, \pi]$. We remark that for an arbitrary set of three functions ($n = 2$) such that all the Wronskians are nonvanishing except the last one (W_2) which has exactly two zeros ($k = 2$ and $\ell = 0$), applying Theorem 1.1, we obtain that $3 \leq Z(\mathcal{F}) \leq 4$. But for the set given in the above proposition we have $Z(\mathcal{F}) = 3$.

Proposition 4.2 (See [16])**.** *The ordered set of functions $\mathcal{F} = [1, t\cos t, t\sin t]$ is an ET-system in $[0, \pi]$ which is not an ECT-system.*

For the above family $W_0(t) = 1$, $W_2(t) = t^2 + 2$ and $W_1(t) = \cos(t) - t\sin(t)$ which has one zero in the interval $[0, \pi]$. We note that for an arbitrary set of three functions ($n = 2$) such that all the Wronskians are nonvanishing except $W_1(t)$ which has exactly one zero ($k = 0$, $\zeta = 1$, and $\ell = 1$), applying

Theorem 1.1, we obtain that $2 \le Z(\mathcal{F}) \le 3$. But Zielke in [16] (example (3) of page 363) proves the last proposition, which implies $Z(\mathcal{F}) = 2$.

5. Application

In [8], the maximum number of limit cycles for a class of nonsmooth systems is studied. For $a, b, c > 0$ and $d \in \mathbb{R}$ it is considered the functions

$$\begin{aligned} g_1(y) &= 1, \\ g_2(y) &= \frac{(ay^2 - b\,c^2)}{y} \log\left(\frac{\sqrt{b}\,c + \sqrt{a}\,y}{\sqrt{b}\,c - \sqrt{a}\,y}\right), \\ g_3^1(y) &= \frac{(d^2+y^2)}{y}\left(3\pi + 2\arctan\left(\frac{d^2-y^2}{2dy}\right)\right), \\ g_3^3(y) &= \frac{(d^2+y^2)}{y}\left(\pi + 2\arctan\left(\frac{d^2-y^2}{2dy}\right)\right), \end{aligned} \tag{14}$$

and the sets $\mathcal{G}^i = \{g_1, g_2, g_3^i\}$ for $i = 1, 3$. The initial problem is reduced to study the maximum number of zeros that the elements of $\mathrm{Span}(\mathcal{G}^1)$ (resp. $\mathrm{Span}(\mathcal{G}^3)$) for $d > 0$ (resp. $d < 0$) can have in $(0, \sqrt{b}c/\sqrt{a})$. Here, we denote these maxima by Z_1 and Z_2, respectively. The authors show that $Z_1, Z_2 \ge 2$.

As an application of our theorems we shall prove that $Z_2 = 2$, and $Z_1 = 2$ or $Z_1 = 3$ depending on the parameters.

We define the functions

$$\begin{aligned} u_0(t) &= 1, \\ u_1^\alpha(t) &= \frac{(t^2-\alpha^2)}{t}\log\left(\frac{\alpha+t}{\alpha-t}\right), \\ u_2^\beta(t) &= \frac{t^2+1}{t}\left(\beta\pi + 2(\beta-2)\arctan\left(\frac{1-t^2}{2t}\right)\right). \end{aligned} \tag{15}$$

It is easy to see that $u_1^\alpha(t) = g_2(|d|\,t)/(a|d|)$ when $\alpha = |\sqrt{b}c/(\sqrt{a}d)|$, $u_2^3(t) = g_3^1(|d|\,t)/|d|$, and $u_2^1(t) = g_3^3(|d|\,t)/|d|$. We define the ordered sets of functions $\mathcal{F}(\alpha,\beta) = [u_0, u_1^\alpha, u_2^\beta]$ for $\alpha > 0$ and $\beta > 0$. Using these sets, the numbers Z_1 and Z_2 are now equivalent to $Z(\mathcal{F}(\alpha,3))$ and $Z(\mathcal{F}(\alpha,1))$ in $(0,\alpha)$, respectively. The next result provides them.

Theorem 5.1. *Let* $\mathcal{F}(\alpha,\beta) = [u_0, u_1^\alpha, u_2^\beta]$ *be the ordered set of the functions* (15). *Then*

$$Z(\mathcal{F}(\alpha,1)) = 2 \quad \text{and} \quad Z(\mathcal{F}(\alpha,3)) = \begin{cases} 2 & \text{if } 0 < \alpha \le \alpha^*, \\ 3 & \text{if } \alpha > \alpha^*. \end{cases}$$

Where α^* *is the unique positive solution of* $2(1-\alpha^2)\arctan\left((1-\alpha^2)/(2\alpha)\right) - 3\pi\alpha^2 + 3\pi + 4\alpha = 0$.

Before proving the above theorem we shall see that, in general, $Z(\mathcal{F}(\alpha,\beta)) \le 3$. To do that we proceed with an easier approach which consist an embedding of this set of three functions (15) into a set of four functions $\{u_0, u_1^\alpha, u_2, u_3\}$, where

$$\begin{aligned} u_0(t) &= 1, \\ u_1^\alpha(t) &= \frac{(t^2-\alpha^2)}{t}\log\left(\frac{\alpha+t}{\alpha-t}\right), \\ u_2(t) &= \frac{t^2+1}{t}\arctan\left(\frac{1-t^2}{2t}\right). \\ u_3(t) &= \frac{t^2+1}{t}. \end{aligned} \tag{16}$$

Proposition 5.2. *For $\alpha > 0$, the ordered set of functions $\mathcal{G}(\alpha) = [u_0, u_2, u_3, u_1^\alpha]$ is an ECT-system in the interval $(0, \alpha)$. Consequently $Z(\mathcal{G}(\alpha)) = 3$.*

Proof. Straightforward computations give us the Wronskians of the ordered set $\mathcal{G}(\alpha)$:

$$\begin{aligned} W_0(t) &= 1, \\ W_1(t) &= \frac{-(1-t^2)}{t^2}\arctan\left(\frac{1-t^2}{2t}\right) - \frac{2}{t}, \\ W_2(t) &= \frac{-8}{t^2(t^2+1)}, \\ W_3(t) &= \frac{16(\alpha^2+1)}{t^4(t^2+1)^2}\log\left(\frac{\alpha+t}{\alpha-t}\right) - \frac{32\alpha(\alpha^2+1)(\alpha^2+t^2)}{t^3(\alpha-t)^2(\alpha+t)^2(t^2+1)^2}. \end{aligned}$$

Clearly, $W_0(t) > 0, W_2(t) < 0$ for $t \neq 0$, and $W_1(t) < 0$ because both summands are negative for $t > 0$. Moreover $W_3(t) = P(t)Q(t)$ where

$$P(t) = \frac{-16(\alpha^2-t^2)^2(1+\alpha^2)}{t^4(1+t^2)^2(t-\alpha)^2(t+\alpha)^2} \text{ and } Q(t) = \frac{2\alpha t(\alpha^2+t^2)}{(\alpha^2-t^2)^2} - \log\left(\frac{\alpha+t}{\alpha-t}\right).$$

We note that $P(t) < 0$ and $Q(t) > 0$ for $t \in (0, \alpha)$, because $Q'(t) = 16\alpha^2 t^2/(\alpha^2 - t^2)^3 > 0$ and $Q(0) = 0$. Consequently, $W_3(t) < 0$ in $(0, \alpha)$. Hence we conclude that $\mathcal{G}(\alpha)$ is an ECT-system in $(0, \alpha)$. □

We stress that Proposition 5.2 by itself represents an improvement to the results obtained in [8] implying that $Z_1, Z_2 \leq 3$. Nevertheless this approach do not use the new theorems developed in this paper. In what follows we show, by proving Theorem 5.1, how to use our main results to give still better estimations for these upper bounds.

Proof of Theorem 5.1. Straightforward computations give us the Wronskians of the ordered set $\mathcal{F}(\alpha, \beta)$:

$$\begin{aligned} W_0(t) =&\ 1, \\ W_1(t) =&\ \frac{\alpha^2+t^2}{t^2}\log\left(\frac{\alpha+t}{\alpha-t}\right) - \frac{2\alpha}{t} := \frac{\alpha^2+t^2}{t^2}P(t), \\ W_2(t) =&\ \frac{Q(t)}{t^3(1+t^2)(\alpha^2-t^2)} := \frac{Q(t)}{R(t)} \end{aligned}$$

where

$$\begin{aligned} Q(t) = \;& 4\alpha t\big(4t(1+\alpha^2)(\beta-2)+\beta\pi(1+t^2)(1-\alpha^2)\big) \\ & +8\alpha(1-\alpha^2)(\beta-2)t(1+t^2)\arctan\left(\frac{1-t^2}{2t}\right) \\ & +2(\alpha^2-t^2)\big(4t(1-\alpha^2)(\beta-2)+\beta\pi(1+t^2)(1+\alpha^2)\big)\log\left(\frac{\alpha+t}{\alpha-t}\right) \\ & +4(1+\alpha^2)(\beta-2)(\alpha^2-t^2)(1+t^2)\arctan\left(\frac{1-t^2}{2t}\right)\log\left(\frac{\alpha+t}{\alpha-t}\right). \end{aligned}$$

Clearly $W_0(t)>0$, and since $P'(t)=8\alpha^3t^2/((\alpha^2-t^2)(\alpha^2+t^2)^2)>0$ and $P(0)=0$, we get that $W_1(t)$ has no zeros in $(0,\alpha)$.

The proof follows studying the zeros of $W_2(t)$ that coincide with the zeros of $Q(t)$ because $R(t)>0$ for $t\in(0,\alpha)$. As we are interested only in the values $\beta=1$ and $\beta=3$, for both we analyze the limits of Q at the boundary of the interval $(0,\alpha)$ and its monotonicity properties in full interval.

The function Q, close to the origin, writes as

$$\begin{aligned} Q(t) = \;& 16\pi\alpha(\beta-1)t+\frac{16\pi(2\alpha^2-1)(\beta-1)}{3\alpha}t^3 \\ & -\frac{16\pi(5\alpha^2+1)(\alpha^2+1)(\beta-1)}{15\alpha^3}t^5+\frac{512(\alpha^2+1)(\beta-2)}{45\alpha}t^6+O(t^7) \end{aligned} \tag{17}$$

and its limit, when t goes to α, is $Q_\alpha=4\alpha^2(\alpha^4-1)L_\beta(\alpha)$ where

$$L_\beta(\alpha)=-2(\beta-2)\arctan\left(\frac{1-\alpha^2}{2\alpha}\right)+\frac{4\alpha(\beta-2)}{\alpha^2-1}-\beta\pi.$$

For $\beta=3$, we will show that Q is an increasing or a unimodal function in $(0,\alpha)$, for $\alpha\le 1$ or $\alpha>1$, respectively. Moreover it is increasing and positive close to the origin, see (17).

Consequently, when $\alpha>1$, the number of zeros only depends on the sign of $L_3(\alpha)$. In fact we shall prove that $L_3(\alpha)$ has a unique zero, called α^*. We have $\lim_{\alpha\to1^+}L_3(\alpha)=\infty$ and $\lim_{\alpha\to\infty}L_3(\alpha)=-2\pi$, so there exists $\alpha^*>1$ such that $L_3(\alpha^*)=0$. The derivative of $L_3(\alpha)$ is given by $-16\alpha^2/((\alpha^2+1)(\alpha^2-1)^2)<0$ for $\alpha\ne1$, which implies that $L_3(\alpha)$ is decreasing. Since $\lim_{\alpha\to0}L_3(\alpha)=-4\pi$ and $\lim_{\alpha\to1^-}L_3(\alpha)=-\infty$ we conclude that $L_3(\alpha)$ has no zeros in $(0,1)$ and that it has at most one zero for $\alpha>1$. Clearly, for $\alpha\ne1$, the same properties about the number of zeros hold for the function Q_α. Finally $Q_1=32$, hence we obtain the uniqueness of the zero α^* and in addition $(\alpha^*-\alpha)Q_\alpha>0$ for $\alpha\ne\alpha^*$.

For simplicity we write $Q(t)=q_0(t)+q_1(t)a(t)+q_2(t)\ell(t)+q_3(t)a(t)\ell(t)$ where $a(t)=\arctan((1-t^2)/(2t))$ and $\ell(t)=\log((\alpha+t)/(\alpha-t))$. Now we will use the Division-Derivation algorithm to study the graph of Q. Straightforward computations shows that the derivative of Q/q_3 is $Q_1=\tilde q_0(t)+\tilde q_1(t)a(t)-\tilde q_2(t)\ell(t)$, the derivative of $Q_1/\tilde q_2$ is $Q_2=\hat q_0(t)+\hat q_1(t)a(t)$ and the derivative of $Q_2/\hat q_1$ is $Q_3=32(s^4+\alpha^2)(\alpha^2+s^2)s^2/((s^2+1)^2(s^4+3\alpha^2s^2-\alpha^2-3s^2)^2)$. Here all the functions q_i, $\tilde q_i$, $\hat q_i$ are rational functions, q_3 and $\tilde q_2$ are positive, and $\hat q_1$ has a unique zero when $\alpha>1$ and it is positive when $\alpha\le1$. Additionally, the limit of the functions Q_i are nonnegative at the origin. When $\alpha \le 1$, Q_3 is positive continuous function in the full interval $(0,\alpha)$, consequently the functions Q_2, Q_1, Q_0 are positive and increasing functions. When $\alpha > 1$, Q_3 is positive but with an asymptote at the unique zero of $s^4+3\alpha^2 s^2-\alpha^2-3s^2$ in $(1,\alpha)$. Consequently the functions Q_2, Q_1, Q are unimodal and they have at most one zero.

Since, for $\alpha < \alpha^*$, $Q_\alpha < 0$, we conclude that there exists at least one zero t^* of $Q(t)$. Applying Theorem 2.1 we conclude that $Z(\mathcal{F}(\alpha, 3\pi)) = 3$.

Analogously, for $\beta = 1$ we have that the function $L_1(\alpha)$ is increasing for $\alpha > 0$ and $\alpha \neq 1$. Since $\lim_{\alpha\to 0} L_1(\alpha) = 0$ we obtain that $L_1(\alpha)$ has no zeros in $(0,1)$. We also have that $\lim_{\alpha\to 1^+} L_1(\alpha) = -\infty$ and $\lim_{\alpha\to\infty} L_1(\alpha) = -2\pi$, therefore $L_1(\alpha)$ has no zeros for $\alpha > 1$. Clearly the same properties about the number of zeros hold for the function Q_α for $\alpha \neq 1$. Finally $Q_1 = -32$, hence we conclude that the function $Q_\alpha < 0$ for every $\alpha > 0$.

Now we will use again the Division-Derivation algorithm to study the graph of $Q(t) = q_0(t) + q_1(t)a(t) + q_2(t)\ell(t) - q_3(t)a(t)\ell(t)$. Straightforward computations show that the derivative of Q/q_3 is $Q_1 = \tilde{q}_0(t) + \tilde{q}_1(t)a(t) - \tilde{q}_2(t)\ell(t)$, the derivative of $Q_1/\tilde{q}_2$ is $Q_2 = \hat{q}_0(t) - \hat{q}_1(t)a(t)$ and the derivative of $Q_2/\hat{q}_1$ is $Q_3 = -32(t^4+\alpha^2)(t^2+\alpha^2)t^2/((t^2+1)^2(t^4+(\alpha^2-1)t^2-\alpha^2)^2)$. Here all the functions q_i, $\tilde{q}_i$, $\hat{q}_i$ are rational functions, q_3 and $\tilde{q}_2$ are positive, and $\hat{q}_1$ has a unique zero when $\alpha > 1$ and it is positive when $\alpha \le 1$. Additionally, the limit of the functions Q_i are zero at the origin. When $\alpha \le 1$, Q_3 is a negative continuous function in the full interval $(0,\alpha)$, consequently the functions Q_2, Q_1, Q_0 are negative and decreasing functions. When $\alpha > 1$, Q_3 is negative but with an asymptote at the unique zero of $t^4+(\alpha^2-1)t^2-\alpha^2$ in $(1,\alpha)$. Consequently the functions Q_2, Q_1, Q are unimodal. Since, for $\alpha > 0$, $Q_\alpha < 0$, we conclude that $Q(t)$ is nonvanishing. Applying Theorem 2.1 (ECT-system case) we conclude that $Z(\mathcal{F}(\alpha, 3\pi)) = 2$. □

Acknowledgements

The first author is supported by a FAPESP-BRAZIL grant 2013/16492-0. The second author is supported by the MINECO/FEDER MTM2008-03437, MTM2013-40998-P and UNAB13-4E-1604 grants, by the AGAUR 2014SGR568 grant, and by the European Community FP7-PEOPLE-2012-IRSES-316338 and FP7-PEOPLE-2012-IRSES-318999 grants.

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