Inverse problems of the Darboux theory of integrability for planar polynomial differential systems.

In this work we are interested in the integrability of the planar polynomial differential systems. The algebraic theory of integrability is a classical one, which is related with the first part of the Hilbert's 16th problem. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of polynomial systems and the number of invariant algebraic curves they have.

A polynomial system is Darboux integrable if it has a first integral or a an integrating factor given by a Darboux function. In 1878 Darboux showed that planar polynomial differential systems having an adequate number of invariant algebraic curves have a first integral which can be constructed using such curves. The Darboux theory of integrability has been improved since its beginning. We present a survey on the Darboux theory of integrability for the planar polynomial differential systems. We also present the recent extensions which additionally to the concept of the invariant algebraic curve, incorporate the notions of the exponential factors, independent singular points, the multiplicity of the invariant algebraic curves and the invariants. In this work we also introduce the concept of generalized invariant.

Since the existence of invariant algebraic curves is the key point for the application of the Darboux theory of integrability we also consider the following reciprocal questions to the Darboux theory of integrability:

Question 1 Given a set of algebraic curves which are the planar polynomial differential systems having these curves invariant by the flow?

We present a complete answer to this question in the generic case due to a Theorem that we obtain in collaboration with Christopher, Llibre and Zhang.

We note that in this Theorem it appears a strong relation between the degrees of the curve and the degrees of the system. This relation is due to the generic nature of the curves. In particular, when the total degree of the generic curves is the degree of the system increased by one, then the vector field has a very simple form and it has always a Darboux first integral. Hence, this relation between the degrees and the nature of the curves guarantee the Darboux integrability. Finally, a more general answer to this question is given by a Thorem due to Walcher.

Question 2 Given a Darboux first integral which are the planar polynomial differential systems having such a first integral?

We present a complete answer to this question as we connect the degree of the curves with the degree of the system. As far as we know, this is the first time that information about the degree of the invariant algebraic curves, instead of the number of these curves, is used for studying the integrability of a polynomial vector field. We also improve a previous result due to Prelle and Singer about the existence of integrating factors.

Question 3 Given a Darboux integrating factor which are the planar polynomial differential systems having such a Darboux integrating factor?

First, we present a general family of polynomial differential systems having a Darboux function as an integrating factor. Although, it is a large amily of systems, it is not the most general one. Second, we characterize polynomial differential systems having a generic Darboux integrating factor, i.e. an integrating factor formed by generic curves. Moreover, for such systems we can always obtain a Darboux first integral.

Rudolf Winkel conjectured: *For a given algebraic curve* $f = 0$ *of degree* $m \geq 4$ *there is in general no polynomial vector field of degree less than* $2m-1$ *leaving invariant* $f = 0$ *and having exactly the ovals of* $f = 0$ *as limit cycles.*

We show that this conjecture is not true.