On the number of limit cycles for some families of planar differential equations

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Introduction

During 17th century, ordinary differential equations were born to explain the movement of the particles in physical systems, such as the translation of the planets around the Sun. Since then they have become a very important tool in many other fields in science and engineering, like biology or electronics. For instance, they are used to model the population growth of species or the evolution of electrical circuits. When the derivation variable just plays an implicit role, the differential equation is said autonomous. The autonomous cases can be considered as dynamical systems. Intuitively, they are rules for the evolution in time of any particle in space. Therefore time is taken as the derivation variable.

Ordinary differential equations of order n take the form

$$F(t, x, x', x'', \dots, x^{(n)}) = 0, \tag{1}$$

where $x^{(n)}$ is the nth derivative of x with respect to t. The autonomous cases take place when F does not depend on t. If x is a vector instead of a real function, equation (1) is called a differential system. In particular, if the space where x is considered is $\mathbb{R}^2$ or any contained open subset, we refer to it as a planar differential system.

In this thesis the equations considered are, or can be seen as, first-order autonomous planar differential systems,

$$\left\{ \begin{array}{rcl} x' & = & f(x, y), \\ y' & = & g(x, y), \end{array} \right. \tag{2}$$

where $x(t)$, $y(t)$, $f(x, y)$ and $g(x, y)$ are real functions.

The aim of the qualitative theory is to understand the behavior of the solutions of any differential system without obtaining their specific expression. The qualitative theory was first introduced by Henri Poincaré in his "Mémoire sur les courbes définies par une équation différentielle", [Poi81], which was a great breakthrough in the study of differential systems. Poincaré mainly studies the case of planar differential systems and proposes a geometric framework for studying their solutions.

In order to understand this geometric point of view, let us consider the velocity field X, which is the vector field whose components are f and g, the functions in system (2). The solutions of the differential system are the trajectories of the vector field. It means that at any point the tangent vectors to the solution curves and the vector field are parallel. The trajectories are also known as the orbits of the vector field. The advantage of using orbits lies in the fact that if we change the time parametrization, they remain unchanged.

There are some concrete orbits with particular behaviors. Some of them were characterized by Poincaré, among others singular points and cycles. Singular points are the points where the field vanishes. They are also called critical or fixed points. And, cycles are the trajectories of the vector field that repeat themselves along time. Usually, they are also called closed or periodic orbits. Notice that singular points are a particular type of cycles. For a point in a cycle after a time T, its orbit will be again on him. For a fixed point, its orbit is on him for every time t in $\mathbb{R}$.

The notion of limit cycle was also introduced in the first papers which dealt with qualitative theory. Essentially, a limit cycle, γ, is a periodic orbit such that at least one trajectory of the vector field, different from γ, approaches γ in positive or negative time. See, for example [HS74, Chi06]. Usually, when the vector field is of class $\mathcal{C}^1$ an alternative definition is given. A closed orbit is named limit cycle if it is isolated from the other periodic orbits. See [Sot79]. This definition is, in general, more restrictive than the previous one, but both are equivalent in the analytic case.

In many senses a cornerstone of the qualitative theory of autonomous systems in dimension two is the Poincaré-Bendixson Theorem. It first appeared in the third volume of [Poi81] for analytic vector fields. In [Ben01], it is extended to the case of differential systems defined by functions of class $\mathcal{C}^1$. This result, for a continuous vector field X, establishes that for any orbit contained in positive time in a bounded region, if it approaches a set without singular points, this set is a periodic orbit. Different proofs of this continuous version can be found in [CL55, Har64]. It can be mentioned [Cie02] as a neat summary of the theorem and some related results.

Just for the completeness of the terminology, we should introduce the notions of α and ω limit sets. They were first proposed by George D. Birkhoff in [Bir66]. So the ω-limit of an orbit is the set of points that the orbit approaches in positive time. And the α-limit is the set that the orbit departs from, in negative time.

As a corollary of the Poincaré-Bendixson Theorem we have that the ω-limit set of a positive orbit contained in a bounded region can only be a fixed point, a cycle or a polycycle, a set of fixed points and regular orbits connecting them. The regular orbits connecting fixed points, in particular the ones in polycycles, are said homoclinic or heteroclinic orbits depending in if they connect just one singular point or several ones, respectively. An analogous result is obtained for the α-limit sets. Therefore the behavior of the solutions of a differential equation can be almost reduced to a local study of the singular points, the closed orbits and the connections of regular orbits and fixed points. In this fact lies the importance of the limit sets. We are particularly interested in the limit cycles. And our results are related with the most famous open problem about limit cycles, that is the second part of the Hilbert 16th problem.

In the International Congress of Mathematics in 1900, David Hilbert proposed 23 problems that in his opinion would motivate advances in mathematics during the 20th century. With the 16th problem, Hilbert asked about the topology of algebraic curves and surfaces. He included a second part asking about the maximum number and the position of the limit cycles of a polynomial planar system,

$$\begin{cases} x' &= P(x,y), \\ y' &= Q(x,y), \end{cases}$$

with P and Q polynomials of degree n. This maximum number depending just on n is usually known as the Hilbert number, $\mathbf{H}(n)$.

There are a lot of published works in relation with this problem. But even the minimum case, $n = 2$, it is yet to be proved. Vladimir I. Arnold in [Arn77, Arn83] establishes a weak version of this problem, that is also still open. Steve Smale, in his list of mathematical problems for the 21st century, includes a modern version of it, see [Sma98]. Moreover, he proposes to found upper bounds of the number of limit cycles of order n^q, where q is a universal constant. In the Smale's list, it occupies the 13th position, so this problem is known as the Smale 13th. For more details in the 16th Hilbert we refer the reader to [Ily02, Li03, CL07].

Smale also says that the computation of the Hilbert number can be notably difficult. So the mathematicians must consider a special class of differential equations 'where the finiteness is simple, but the bounds remain unproved'. In fact, he proposes to seek bounds for a special class of polynomial Liénard systems,

$$\begin{cases} x' &= y - F(x), \\ y' &= -x, \end{cases}$$

where F is a real polynomial of odd degree and satisfying $F(0) = 0$.

In order to achieve the Hilbert number, the lower bounds are as important as the upper bounds. The study of lower bounds for the Hilbert number focus mainly on two standard techniques. One is the computation of the limit cycles that persist after the polynomial perturbation of centers. And the other is the study of necessary conditions for a family of differential systems depending on some parameters such that it succeed the birth of a limit cycle from different kind of singular orbits, such as fixed points or heteroclinic connections. It is known as the bifurcation of a limit cycle. For example, the Hopf bifurcation phenomena implies the birth of a cycle from a singular point satisfying some particular conditions.

This thesis deals with these aspects of limit cycles for some particular families of differential equations. In the first chapter we examine the weak Hilbert 16th problem restricted to the polynomial perturbation of a particular center. The second chapter establishes sufficient conditions for the existence and uniqueness of limit cycles in a generalization of the Liénard equation. Chapter 3 provides a study of the location of a bifurcation curve in the parameter space of the Bogdanov-Takens system. Immediately, we show more details of the problems treated. However, each chapter contains a section devoted to summarize the problems and the main results obtained.

Because of the independence between the different problems proposed, this thesis has been written in a modular form, been each chapter independent to the others. And, therefore, it can be read in any order. To facilitate access to the individual topics, the notation on each chapter is rendered as self-contained as possible.

In Chapter 1 we examine the limit cycles that appear after the perturbation of a linear center with extra singular points. The perturbed system that we consider is

$$\begin{cases} x' &= \quad y\,K(x,y) + \varepsilon P(x,y), \\ y' &= -x\,K(x,y) + \varepsilon Q(x,y), \end{cases} \tag{3}$$

where K is a specific family of polynomials, P and Q are any polynomial and ε a small enough real number. This differential system has a center at the origin, equivalent to the generated by the Hamiltonian $H(x,y) = x^2 + y^2$. And the extra singular points that we referred before are the ones that satisfy $K(x,y) = 0$.

In fact we analyze the Abelian integral associated to (3), as it was proposed by Arnold in the statement of the weak Hilbert 16th problem. There are several previous works dealing with the Abelian integral associated to this system where the set of singular points take different forms, like straight lines [LLLZ02] or quadratic curves [BL07]. We consider the case of finite sets of isolated points.

One of the key points of the work included in Chapter 1 is that we are able to consider several period annulus where we can seek limit cycles. Hence, we develop a study of the simultaneity of the limit cycles in the different annuli at the same time. Due to their difficulty, the papers on the simultaneity of limit cycles in several regions are not common at all. But some examples are [CL95, CLP09]. Other key point of the first chapter is the study on the dependence of the number of limit cycles on the location of the parameters included in Section 1.4.3. We have constructed a bifurcation diagram that distinguishes different regions on the parameter space where we have different number of limit cycles.

In this chapter the lack of a partial fraction decomposition for two variable rational functions does not allow us to obtain better upper bounds in the more general cases. But the main difficulty in carrying out better results for this chapter is the amount and the size of the computations that should be done. Some of the results of this chapter appear in [PGT12].

In the second chapter of this thesis we extend some of the classical results about existence and uniqueness of limit cycles for the Liénard equation [YCC+86, ZDHD92] to the φ-laplacian case,

$$(\varphi(x'))' + f(x)\psi(x') + g(x) = 0. \tag{4}$$

This equation appears in models that consider definitions of the derivative different from the classic one, such as the relativistic one. In fact, the harmonic relativistic oscillator can be modeled as

$$\left(\frac{x'}{\sqrt{1 - \frac{x'^2}{c^2}}} \right)' + x = 0,$$

see [Gol57, Mic98]. So, our results apply to the relativistic van der Pol equation,

$$\left(\frac{x'}{\sqrt{1 - \frac{x'^2}{c^2}}} \right)' + \mu(x^2 - 1)x' + x = 0,$$

as well as to the ordinary one.

The results included in this chapter involve an ad hoc compactification designed with two objectives. First, to unify the different behaviors of the functions in (4) satisfying our hypothesis. And second, to make possible the comprehension of the global phase portrait. The results of this chapter have also been done in collaboration with Pedro J. Torres. And they appear in [PGTT12].

The aim of the third chapter is to obtain a global knowledge of the homoclinic connection curve in the first quadrant of parameter space, where the limit cycles can appear, in the system associated to this Bogdanov-Takens normal form,

$$\left\{ \begin{array}{rcl} x' & = & y, \\ y' & = & -n + by + x^2 + xy, \end{array} \right. \tag{5}$$

where the parameters, n and b, are real numbers. When the parameters vanish, the origin shows a local structure of cusp point, a kind of degenerate singular point. But if we unfold this vector field, they appear several bifurcation curves, a Hopf bifurcation, a saddle-node bifurcation and a homoclinic connection curves. See [GH02] for more details.

It is worth pointing out that the system (5) is a semi-complete family of rotated vector fields with parameter b. So, for any fixed n, the limit cycle grows until it disappears in the homoclinic connection when we range b from its birth in the Hopf bifurcation.

The study of the bifurcation curve of the homoclinic connection is usually restricted to a local region near the origin. It is known, [GH02], that for $n > 0$ there exists a value $b^*(n)$ such that the system has a unique limit cycle if and only if $b^*(n) < b < \sqrt{n}$. Moreover $b^*(n) = \frac{5}{7}\sqrt{n} + \ldots$ for n small enough. In [Per92] the quadratic differential system is considered in the whole space and it is proved that $b^*(n)$ is analytic and that the previous inequalities are satisfied for all $n > 0$. A detailed study of the curve $b^*(n)$ for n small enough is presented in [GGT10].

The results obtained in [GGT10] are based on an algebraic method for the location of bifurcation curves. In our work we adapt this procedure to our needs. Finally, we obtain explicit curves such that $b_d(n) < b^*(n) < b_u(n)$ for all $n > 0$. With this result we prove a conjecture proposed by Perko in [Per92], where he predicts that $b^*(n)$ goes to infinity as $\sqrt{n}-1$. In particular, we prove that $b^*(n)$ goes to infinity as $\sqrt{n}-1+O(1/n)$.

This chapter has also been developed in collaboration with Armengol Gasull and Héctor Giacomini.