# Invariant circles for homogeneous polynomial vector fields on $\mathbb{S}^2$

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# Introduction and statement of the main results

A *polynomial vector field $X$ in* $\mathbb{R}^3$ is a vector field of the form

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}, \tag{1}$$

where $P$, $Q$, $R$ are polynomials in the variables $x$, $y$ and $z$ with real coefficients. We denote $m = \max\{\deg P, \deg Q, \deg R\}$ the *degree* of the polynomial vector field $X$. In what follows $X$ will denote the above polynomial vector field.

Let $\mathbb{S}^2$ be the 2–dimensional sphere $\{(x,y,z) \in \mathbb{R}^3 \ : x^2+y^2+z^2=1\}$. A *polynomial vector field $X$ on* $\mathbb{S}^2$ is a polynomial vector field in $\mathbb{R}^3$ such that restricted to the sphere $\mathbb{S}^2$ defines a vector field on $\mathbb{S}^2$; i.e. it must satisfy the following equality

$$xP(x,y,z) + yQ(x,y,z) + zR(x,y,z) = 0, \tag{2}$$

for all points $(x,y,z)$ of the sphere $\mathbb{S}^2$.

Let $f \in \mathbb{R}[x,y,z]$, where $\mathbb{R}[x,y,z]$ denotes the ring of all polynomials in the variables $x$, $y$ and $z$ with real coefficients. The algebraic surface $f=0$ is an *invariant algebraic surface* of the polynomial vector field $X$ if for some polynomial $K \in \mathbb{R}[x,y,z]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf.$$

The polynomial $K$ is called the *cofactor* of the invariant algebraic surface $f=0$. We note that since the polynomial system has degree $m$, then any cofactor has at most degree $m-1$.

The algebraic surface $f=0$ defines an *invariant algebraic curve* $\{f=0\}\cap\mathbb{S}^2$ of the polynomial vector field $X$ on the sphere $\mathbb{S}^2$ if

(i) for some polynomial $K \in \mathbb{R}[x,y,z]$ we have $Xf = P\dfrac{\partial f}{\partial x} + Q\dfrac{\partial f}{\partial y} + R\dfrac{\partial f}{\partial z} = Kf$, on all the points $(x,y,z)$ of the sphere $\mathbb{S}^2$, and

(ii) the intersection of the two surfaces $f=0$ and $\mathbb{S}^2$ is transversal; i.e. for all points $(x,y,z) \in \{f=0\}\cap\mathbb{S}^2$ we have that $(x,y,z) \wedge \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right) \neq 0$, where $\wedge$ denotes the vector cross product in $\mathbb{R}^3$.

Again the polynomial $K$ is called the *cofactor* of the invariant algebraic curve $\{f = 0\} \cap \mathbb{S}^2$.

Since on the points of the algebraic curve $\{f = 0\} \cap \mathbb{S}^2$ the gradient $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ of the surface $f = 0$ is orthogonal to the polynomial vector field $X = (P, Q, R)$, and the vector field $X$ is tangent to the sphere $\mathbb{S}^2$, it follows that the vector field $X$ is tangent to the curve $\{f = 0\} \cap \mathbb{S}^2$. Hence, the curve $\{f = 0\} \cap \mathbb{S}^2$ is formed by trajectories of the vector field $X$. This justifies to call $\{f = 0\} \cap \mathbb{S}^2$ an invariant algebraic curve, since in this case it is invariant under flow defined by $X$ on $\mathbb{S}^2$.

If the invariant algebraic curve $\{f = 0\} \cap \mathbb{S}^2$ is contained in some plane, then we say that $\{f = 0\} \cap \mathbb{S}^2$ is an *invariant circle* of the polynomial vector field $X$ on the sphere $\mathbb{S}^2$. Moreover, if the plane contains the origin, then $\{f = 0\} \cap \mathbb{S}^2$ is a *invariant great circle.*

Let $U$ be an open subset of $\mathbb{R}^3$. Here a nonconstant analytic function $H : U \to \mathbb{R}$ is called a *first integral* of the system on $U$ if it is constant on all solutions curves $(x(t), y(t), z(t))$ of the vector field $X$ on $U$; i.e. $H(x(t), y(t), z(t)) =$ constant for all values of $t$ for which the solution $(x(t), y(t), z(t))$ is defined in $U$. Clearly $H$ is a first integral of the vector field $X$ on $U$ if and only if $XH \equiv 0$ on $U$. If $X$ is a vector field on $\mathbb{S}^2$, the definition of *first integral* on $\mathbb{S}^2$ is the same substituting $U$ by $U \cap \mathbb{S}^2$.

In what follows we say that two phase portraits of the vector fields $X_1$ and $X_2$ on $\mathbb{S}^2$ are (*topologically* ) *equivalent*, if there exists a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h$ applies orbits of $X_1$ into orbits of $X_2$, preserving or reversing the orientation of all orbits.

Let $C_1$ and $C_2$ be invariant circles of a homogeneous polynomial vector field $X$ on $\mathbb{S}^2$. We say that $C_1$ and $C_2$ are *parallel invariant circles* if the planes that contains them are parallel. In particular, we say that $C_1$ and $C_2$ have the same *director vector* of the planes that contain them.

In 2002 Gutierrez and Llibre [9] extended the Darbouxian theory of integrability from polynomial vector fields on $\mathbb{R}^2$ (see [6]) to polynomial vector fields on $\mathbb{S}^2$. The Darbouxian theory of integrability analyze how to construct a first integral of a polynomial vector field by using a sufficient number of invariant algebraic curves. Therefore, to study the existence and number of invariant algebraic curves of a polynomial vector field $X$ in dimension 2 (and in particular in $\mathbb{S}^2$), is an interesting subject of recent papers [2,4,5,6,10,11]. The first step in this direction is to determine the maximum number of invariant circles (invariant algebraic curves of degree 1 on $\mathbb{S}^2$) for a polynomial vector fields $X$ on $\mathbb{S}^2$, when $X$ has finitely many invariant circles. The main motivation for study this problem is the paper [2]. In [2] the authors study the maximum number of invariant straight lines for a polynomial vector fields $X$ on $\mathbb{R}^2$, when $X$ has finitely many invariant straight lines. They solve this problem for the cases in that $X$ has degree 1–5 and they show that for degree $n$ the upper bound is $3n-1$. Moreover, for degrees 6–20 lower bounds are find for the maximum number of invariant straight lines.

In Chapter 1 we consider homogeneous polynomial vector fields of degree two on $\mathbb{S}^2$, and we determine the number of invariant circles when it has finitely many invariant circles. Moreover, we characterize the global phase portrait of these vector fields modulo limit cycles. Camacho [3] in 1981 proved some properties of this kind of vector fields. More precisely, in her work we can find a classification of the Morse–Smale homogeneous polynomial vector fields of degree two on $\mathbb{S}^2$ with no limit cycles.

In Chapter 2 we study the problem for polynomial vector fields of degree $n$ on $\mathbb{S}^2$ and we obtains upper bounds for the number of invariant circles, invariant great circles, invariant circles intersecting at a same point and parallel circles with the same director vector, when the vector fields have a finite number of these objects.

In Chapter 3 we give examples of homogeneous polynomial vector fields of degree 3 on $\mathbb{S}^2$ having finitely many invariant circles which are not great circles, which are limit cycles, but are not great circles and invariant great circles that also are limit cycles. These examples show that for degree 3 the problem is more complicate than for degree 2 and there exist many others phenomena.

The main results of this work are the following theorems.

**Theorem 1.** *Let $X$ be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. If $X$ has finitely many invariant circles, then every invariant circle is a great circle of $\mathbb{S}^2$.*

**Theorem 2.** *Let $X$ be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that $X$ has invariant circles on $\mathbb{S}^2$, then it has either at most two invariant circles, or it has infinitely many invariant circles on $\mathbb{S}^2$. Moreover, the invariant circles are never limit cycles of $X$.*

Theorems 1 and 2 will be proved in Section 1.9.

**Theorem 3.** *Let $X$ be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that $X$ has exactly two invariant circles on $\mathbb{S}^2$, then the phase portrait of $X$ is equivalent to one of Figures 1.6, 1.8, 1.9 or 1.10.*

**Theorem 4.** *Let $X$ be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that $X$ has exactly one invariant circle on $\mathbb{S}^2$, then the phase portrait of $X$ is equivalent to one of the phase portraits of Figures 1.11, 1.12, 1.13, 1.14, 1.16, 1.17 or 1.18.*

Theorems 3 and 4 will be proved in Sections 1.13 and 1.15, respectively.

**Proposition 5.** *Let $X$ be a homogeneous polynomial vector field on $\mathbb{S}^2$ of degree $n$. If $X$ has finitely many invariant circles, then $X$ has at most $15n-3$ invariant circles on $\mathbb{S}^2$ if $n$ is odd, or $15n-4$ if $n$ is even.*

**Proposition 6.** *Let $X$ be a homogeneous polynomial vector field on $\mathbb{S}^2$ of degree $n$. If $X$ has finitely many invariant great circles, then $X$ has at most $3n$ invariant great circles on $\mathbb{S}^2$.*

**Proposition 7.** *Let $X$ be a homogeneous polynomial vector field on $\mathbb{S}^2$ of degree $n$. If $X$ has finitely many parallel invariant circles having the same director vector, then $X$ has at most $n+3$ parallel invariant circles on $\mathbb{S}^2$ having the same director vector if $n$ is odd, or $n+2$ if $n$ is even.*

**Proposition 8.** *Let $X$ be a homogeneous polynomial vector field of degree $n$ on $\mathbb{S}^2$. Suppose that $X$ has finitely many invariant circles on $\mathbb{S}^2$, then a single point can belongs at most to $6n+2$ different invariant circles.*

**Proposition 9.** *Let $X$ be a homogeneous polynomial vector field of degree $n$ on $\mathbb{S}^2$. Suppose that $X$ has finitely many invariant great circles on $\mathbb{S}^2$, then a single point can belongs at most to $n+1$ different invariant great circles.*

Propositions 5–9 will be proved in Section 2.3.

**Proposition 10.** *Consider the homogeneous polynomial vector field $X$ on $\mathbb{S}^2$ given by the system*

$$\begin{aligned} \dot{x} &= P(x,y,z) = -x^2z + xy^2 + 3yz^2 + 3z^3, \\ \dot{y} &= Q(x,y,z) = -x^2y - xyz - 3xz^2, \\ \dot{z} &= R(x,y,z) = x^3 + xy^2 - 3xz^2. \end{aligned} \tag{3}$$

*Then $X$ has only two invariant circles which are not great circles. Moreover, these invariant circles also are limit cycles.*

**Proposition 11.** *Consider the homogeneous polynomial vector field $X$ on $\mathbb{S}^2$ given by the system*

$$\begin{aligned} \dot{x} &= P(x,y,z) = x^2y + y^3 + y^2z + z^3, \\ \dot{y} &= Q(x,y,z) = -x^3 - xy^2 - xyz + z^3, \\ \dot{z} &= R(x,y,z) = -xz^2 - yz^2. \end{aligned} \tag{4}$$

*Then the equator $\mathbb{S}^1$ of $\mathbb{S}^2$ is the unique invariant circle of $X$. Moreover, $\mathbb{S}^1$ is also a limit cycle.*

**Proposition 12.** *Consider the homogeneous polynomial vector field $X$ on $\mathbb{S}^2$ given by the system*

$$\begin{aligned} \dot{x} &= z^3 - \frac{1}{3}x^2z + y^2z, \\ \dot{y} &= -\frac{4}{3}xyz, \\ \dot{z} &= \frac{1}{3}x^3 + \frac{1}{3}xy^2 - xz^2. \end{aligned} \tag{5}$$

*Then $X$ has seven invariant circles on $\mathbb{S}^2$.*

Propositions 10, 11 and 12 will be proved in Chapter 3.