Dynamic rays for transcendental holomorphic self-maps of $\mathbb{C}$ and $\mathbb{C}^*$

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Master's thesis in the Research pathway directed by
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2010 – 2011 Academic Year

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Introduction

This Master's thesis belongs to the field of complex dynamical systems, those generated by the iteration of self holomorphic maps of a Riemann surface. The theory is interesting in 3 cases: the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (rational maps), the complex plane (entire transcendental maps) and the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (maps with two essential singularities). In any other case the Julia set is empty and the dynamical study becomes trivial. By Montel's theorem, if we iterate a map omitting at least three points of $\widehat{\mathbb{C}}$ then every point must be normal.
The main goal of this project is twofold. On the one hand we study the recent article *Dynamic rays of bounded-type entire functions* by Günter Rottenfußer, Johannes Rückert, Lasse Rempe and Dierk Schleicher published in Annals of Mathematics (Second Series) in 2011, [RRRS11]. The results in this paper are a serious advance in the theory of iteration of entire transcendental maps, since they apply to a wide class of maps, setting the basis for further work in the field. The tools used in the paper are many and of varied nature, and we have made an effort to introduce them properly and fill in all details. On the other hand this project also contains original work by the author, namely the initial steps necessary to extend the above mentioned theory to self holomorphic maps of $\mathbb{C}^*$.
We start with a brief historical note about complex dynamics and afterwards we will motivate the project and present our main results. At the end there is a section to clarify the notation we are going to use throughout this work.

A bit of history

Given a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ and a seed $z_0 \in \mathbb{C}$ we are interested in the behaviour of the sequence

$$z_n = f(z_{n-1}) = f^n(z_0), \quad n \geqslant 1$$

called the (forward) orbit of z_0 under f. The origins of complex dynamics go back to the first studies of the Newton's method, one of the oldest root-finding algorithm but at the same time very efficient. Given a holomorphic function f and a seed z_0 close enough to a zero α of f then when we iterate the function

$$N_f(z) = z - \frac{f(z)}{f'(z)}$$

the orbit of z_0 converges to α. The first time that the iteration of holomorphic functions is mentioned is in 1870 in the studies of Ernst Schröder (1841-1902). A few years later, Arthur Cayley (1821-1895) also became interested in this topic. Both Cayley and

Schröder developed greatly the local study of the method. They were worried about questions like finding sufficient conditions for the local convergence or improving the speed of convergence. However, they also considered global questions (separating the plane into different attracting basins) but they only solved the polynomial case of degree 2. If you apply the Newton's method to $P(z) = (z - \alpha)(z - \beta)$, $\alpha \neq \beta$, you obtain two half planes of initial conditions converging respectively to α and β divided by the line bisecting the segment $\overline{\alpha\beta}$. These intermediate points, when used as initial conditions, produce orbits which do not converge to any of the roots. Cayley already noticed the difficulty of the degree 3 case. When we consider a polynomial with three roots there appear fractal structures as you can see in Figure 1, hence it was really difficult to solve it analytically with the tools that they had by that time.

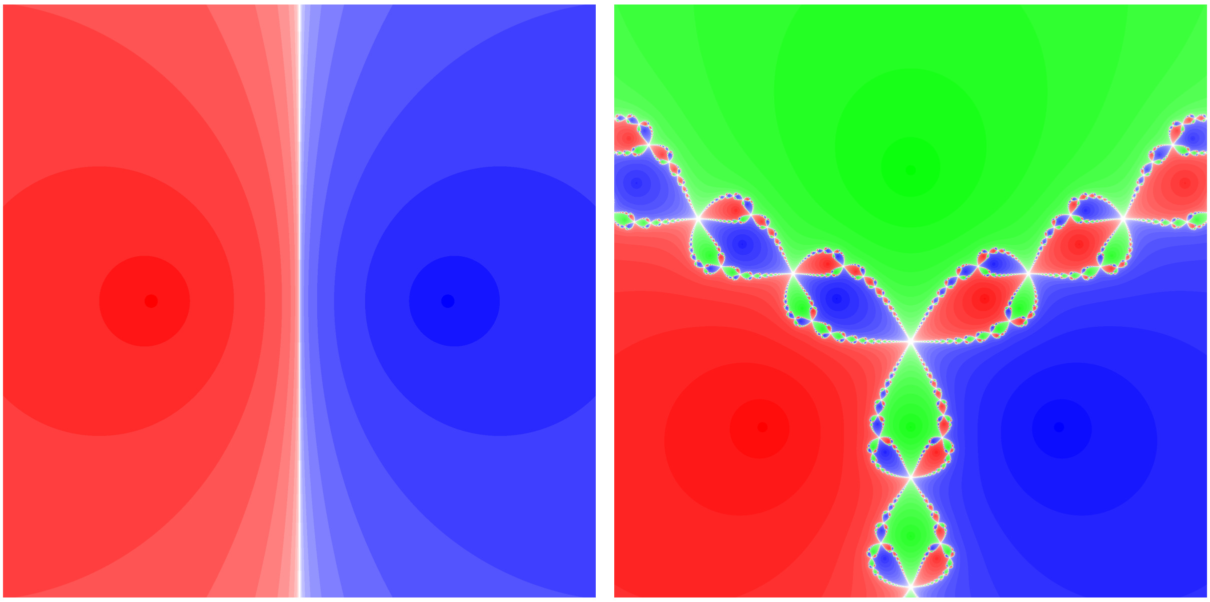

Figure 1: Phase space of the Newton's method applied to a polynomial. **L:** degree 2; **R:** degree 3. The color indicates which is the limit point of every seed and the number of iterates needed to enter a certain neighbourhood of the root, the white points belong to the Julia set.

There were no significant contributions to the global study until the beginning of the 20th century with the works by Pierre Fatou (1878-1929) and Gaston Julia (1893-1978) about the iteration of holomorphic functions on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. On 1915, the *Académie des sciences* from France announced that the *Grand Prix des Sciences Mathématiques* of 1918 was going to be awarded to the best work on iteration, specifying that it needed to be a global study. This choice could have been motivated by the works of Henri Poincaré. This contest led to a strong rivalry between Fatou and Julia. In the last moment Fatou decided not to participate in the contest and the *prix* was awarded to Julia. However, both produced excellent works which are basic to understand complex dynamics as we do nowadays.

They introduced the use of normal families to decompose the phase space. Every normal point has a neighbourhood of points which behave in a similar fashion when iterated: these points are in the stable set. Conversely, every point in the complement of this set has a chaotic behaviour. Today, the stable set is known as the Fatou set and its complement

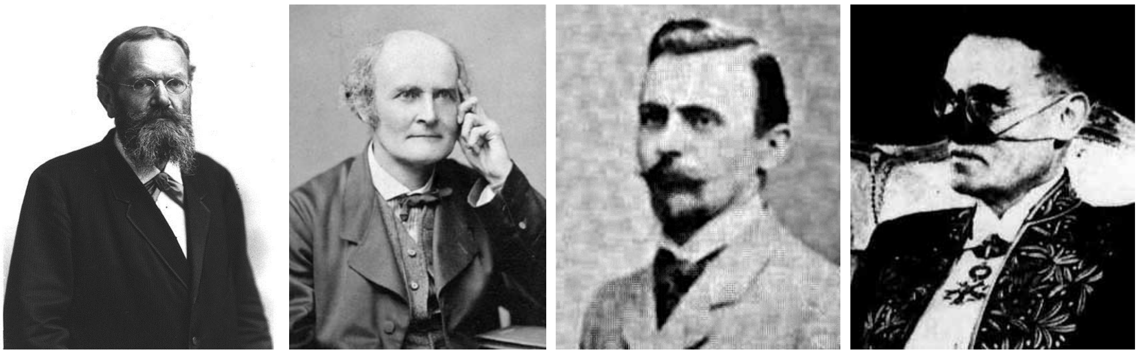

Figure 2: From left to right, Ernst Schröder, Arthur Cayley, Pierre Fatou and Gaston Julia. Pictures from the internet.

is called the Julia set. When they tried to study the Julia set they encountered the same difficulty as Cayley, it is a fractal set. In Figure 3 you can see how Julia tried to draw it.

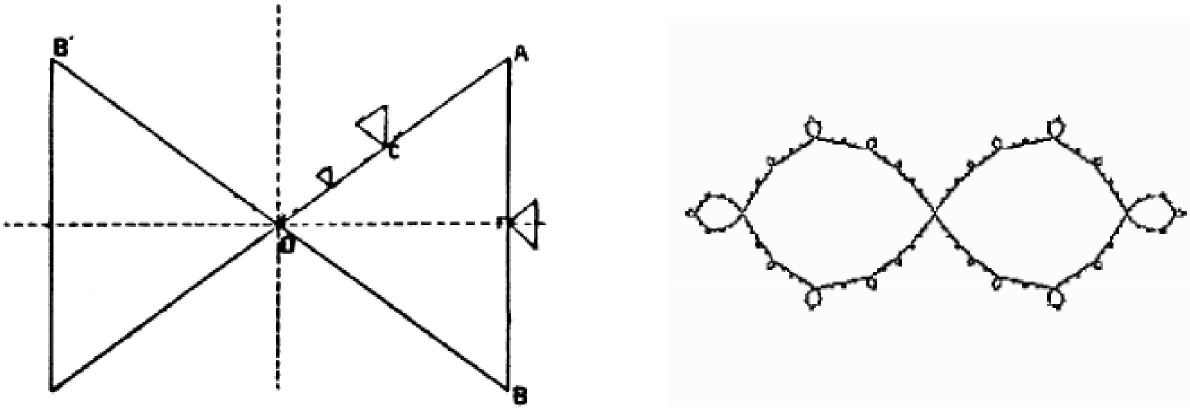

Figure 3: **L:** the sketch that Gaston Julia made of a Julia set; **R:** the actual Julia set. Picture from [Ale94].

Fortunately, the use of computers was a major breakthrough for complex dynamics. Now we can visualize the Fatou and Julia sets, and hence we have a much better intuition of what is happening. The phase space of polynomials and rational functions has been widely studied and is fairly well understood. Nevertheless there are still many relevant open problems, specially those related about parameter spaces.

In parallel, but with a lesser speed, the theory of transcendental maps has been developed. Maps with an essential singularity show a substantial increase of difficulty. For instance, Great Picard's theorem tells us that every neighbourhood of the essential singularity must be mapped to the whole plane with the exception, at most, of one point. This simle fact, adds plenty of chaos to the system producing unbounded Julia sets with very interesting topology. Fatou was the first to consider the dynamics of entire transcendental functions in his article [Fat26]. Since then, transcendental dynamics has become a very active area of research.

If you are interested in the history of complex dynamics, you may like the book [Ale94].

Transcendental maps: Motivation and contents

In his article [Fat26], Fatou already observed that the Julia set of certain entire transcendental functions contains curves of points that escape to infinity under iteration and wondered if this was a general property. Alexandre Eremenko introduced the notion of escaping set in his article [Ere89]. Given an entire transcendental function f,

$$I(f) := \{z \in \mathbb{C} \ : \ |f^n(z)| \to +\infty\}.$$

He proved that every component of $\overline{I(f)}$ is unbounded and conjectured that

- each component of $I(f)$ is unbounded (weak Eremenko's conjecture);
- every point in $I(f)$ can be joined with ∞ by a curve in $I(f)$ (strong Eremenko's conjecture).

Such curves are called dynamical rays (or hairs) in analogy to the dynamic rays of polynomials introduced by A. Douady and J. Hubbard [Mil06], These type of invariant objects, governed by symbolic dynamic, proved themselves to be a crucial tool in the development of the theory since they often can be used to define dynamicaly meaningful partitions of the phase space.
In contrast to the polynomial case where dynamic rays belong to the Fatou (or stable) set, dynamic rays for transcendental maps are part of the Julia (or chaotic) set. Studies about the existence and topological description of (transcendental) dynamic rays were developed in the 80's by Robert L. Devaney and his collaborators. They started with the exponential family as the simplest model for these types of maps, and they later moved to entire maps *of finite type* (i.e., with a finite number of singularities of the inverse map). The main reference is by Devaney and Tangerman [referencia] where they gave some conditions under which they could prove the existence of *Cantor Bouquets*, consisting of Cantor sets of dynamic rays. This seminal work gave rise to many intents of generalizations. The most successful one is the recent work of Rempe et al. in [RRRS11], which is partially the object of this thesis. In this paper, the authors set up a general theory which partially proves Eremenko's conjecture for a wide class of functions in class $\mathcal{B}$, i.e., maps with a bounded set of singularities of the inverse. More precisely, their main theorem reads as follows.

Theorem (Entire functions with dynamic rays)**.** *Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in I(f)$ can be connected to ∞ by a curve γ such that $f^n_{|\gamma} \to \infty$ uniformly.*

Chapter 4 of this project is devoted to prove this theorem. In the same paper they also showed that some assumptions will be necessary, giving in fact a counterexample to the strong Eremenko's conjecture.

Theorem (Entire functions without dynamic rays)**.** *There exists a hyperbolic entire function $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded.*

Furthermore we are interested in studying some structural properties of the Julia set of holomorphic self-maps of the complex punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. A reason which makes this class of self-maps very interesting is that they often arise as complexifications

of analytic maps of the circle, crucial for the study of rotation domains in real and in complex dynamical systems of all types. For instance, in Section 5.4 we will see that the complexification of the Arnol'd standard family is of this type.
If we lift a self-map of $\mathbb{C}^*$, say f, by $h(z) = \exp(z)$ we get an entire transcendental map, say F, satisfying a lift property, i.e. $F(z+2\pi i) = F(z)+2k\pi i$, where $k \in \mathbb{N}$. The relevant dynamical objects (singularities of the inverse like critical points, asymptotic values, etc.) and even the Julia sets for these two maps are in correspondence via the exponential map [Ber95]. A consequence of this fact, because of the periodicity of the exponential, is that such lift necessarily will have infinitely many of such elements, distributed in the whole complex plane. In particular, no such lift belongs to class $\mathcal{B}$.
The theory developed in [RRRS11] therefore cannot be applied directly to self-maps of $\mathbb{C}^*$. It would be desirable however to establish such results, specially having in mind further applications. This is the goal of a long term project as part of the author's future Ph.D. thesis. In the meantime, in this Master's thesis, we construct the right setup under which the results of [RRRS11] can be applied, providing some partial results in the right direction. More precisely we show the following. Let $I_0(f)$ and $I_\infty(f)$ be the sets of points that converge uniformly to 0 and ∞ respectively under iteration by a holomorphic self-map of $\mathbb{C}^*$ f. For this class of functions we must take into account two orders of growth, one at zero and the other one at infinity. We say that f has *finite order* if both

$$\rho_\infty(f) := \lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|}, \quad \rho_0(f) := \rho_\infty(h \circ f \circ h)$$

are finite, where $h(z) := 1/z$ conjugates a neighbourhood of ∞ to a neighbourhood of 0.

Theorem. *Let $f :\in \mathcal{B}^*$ be a function of finite order or a finite composition of such maps. Then every point $z \in I_\infty(f)$ can be connected to ∞ by a curve γ such that $f^n_{|\gamma} \to \infty$ uniformly. Similarly, every every point $z \in I_0(f)$ can be connected to 0 by a curve γ such that $f^n_{|\gamma} \to 0$ uniformly.*

Structure of the project

The first three chapters are preliminary sections which contain the tools used in the main chapters which are 4 and 5.
In the first chapter we introduce hyperbolic geometry. We begin by defining the Poincaré metric in the unit disc $\mathbb{D}$ and study the existence and uniqueness of geodesics, minimal length curves joining two points. Our purpose is to endow arbitrary domains with a hyperbolic metric and for this we introduce the notion of covering space. In the last section we prove Pick's theorem and the standard estimate, which will be used many times during the project.
Chapter 2 is dedicated to Continuum theory and the main goal is to prove the so-called *Non-cut point characterization of the arc* which will be the key point in the proof of the main result in Chapter 4. A *continuum* is a non-empty, compact, connected metric space and we call *cut point* of a topological space S to a point $p \in S$ such that $S \setminus \{p\}$ is disconnected. Otherwise, the point p is called a non-cut point of S. Using some *separation ordering* we show that a continuum is an arc if and only if it has exactly two non-cut

points. On the other hand, this is equivalent to the fact that the continuum has some topological properties and its topology agrees with the separation order topology.
In Chapter 3 we introduce the basic concepts in complex dynamics, stating with the notions of normality and the Fatou and Julia sets. We state their basic topological and dynamical properties and classify the Fatou components for rational and entire transcendental functions. A very characteristic property of transcendental functions is that their Julia set contains Cantor bouquets. At the end of the chapter we describe the general properties of the escaping set.
Chapter 4 is devoted to prove the main theorem in [RRRS11], as explained in the introduction. To do so, we define logarithmic tracts and introduce *logarithmic coordinates*, an exponential lift of the restriction of the function to the tracts. Using hyperbolic geometry, we prove some expansivity properties similar to the ones of the exponential function. We conclude this chapter with a discussion about the existence of Cantor bouquets. We would like to remark that we required the use of Zorn's lemma in the proof of Proposition 3.7.3. Therefore you may be aware that this construction depends on the Axiom of choice.
Finally in Chapter 5 we study self holomorphic maps of $\mathbb{C}^*$. We describe the geometry of the tracts and logarithmic coordinates and provide the setup to apply the results in [RRRS11] to this class of functions.
All the pictures in this project except a couple in the Introduction have been made by the author, either using standard software (Mathematica, Geogebra, Inkscape) or own programs written in C++ language.

Notation

Throughout this project, we will denote by $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the Riemann sphere and by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ the punctured plane. $\mathbb{D}$ will stand for the open unit disc, $\mathbb{H}$ the right half-plane and if $R > 0$,

$$B_R(z_0) := \{z \in \mathbb{C} \ : \ |z - z_0| < R\}, \quad \mathbb{H}_R := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > R\}.$$

If $A \subseteq \mathbb{C}$, then $\overline{A}$ and $\widehat{A}$ denote respectively the closures of A in $\mathbb{C}$ and $\widehat{\mathbb{C}}$, ∂A is the boundary of A in $\mathbb{C}$ and

$$\mathring{A} = \overline{A} \setminus \partial A$$

is the interior of A.
If $P, Q \in \mathbb{C}$, $\overline{PQ}$ is the straight segment joining these two points. Euclidean length and distance are denoted by l and dist. Let $X \subseteq \mathbb{C}$, then ρ_X, l_X and dist_X denote respectively the hyperbolic density, length and distance with respect to the domain X. See Section 1.5 for the precise definitions.
In the context of continuum theory, we introduce the notation $Y = P|Q$ to denote a partition into mutually separated sets (see Section 2.3) and $S(p, q)$ for the set of points separating p and q (see Section 2.4). We use $\prec$ to refer to non-standard orderings, like the separation ordering in Chapter 2 or the speed ordering in Chapter 4.
If f is a function and $n \in \mathbb{N}$, by f^n we will always mean the composition of f with itself n times,

$$f^n = f \circ \overset{n}{\cdots} \circ f,$$

e.g. $f^2(z) = f(f(z))$ which is different of $g(z) = (f(z))^2$. We understand that $\mathbb{N}$ begins with 0 and define $f^0(z) = z$. Some authors use the notation $f^{\circ n}$ to denote this, but we will avoid it because we think that there is no possibility of confusion. As usual, $J(f)$ and $F(f)$ are the Julia set and the Fatou set of f, the dynamical partition of the plane associated to f. $I(f)$ is the escaping set of f. Some related sets are $J^K(f)$ and $J_{\underline{s}}$, $I_{\underline{s}}$, $J_{\underline{s}}^K$. You can find out what all this terminology is by looking at Section 4.3. If z_0 is a fixed point for f, $A_f(z_0)$ denotes its attracting basin and $A_f^*(z_0)$ its immediate attracting basin (the component of $A_f(z_0)$ containing z_0).
The Eremenko-Lyubich class $\mathcal{B}$ is introduced in the beginning of Chapter 4. Along that chapter we define class $\mathcal{B}_{log}$ and all its variants $\mathcal{B}_{log}^n$ (normalized), $\mathcal{B}_{log}(\alpha, \beta)$ (bounded slope) and $\mathcal{B}_{log}^n(\alpha, \beta)$. In Chapter 5 we define the analogs of $\mathcal{B}$ and $\mathcal{B}_{log}$ for the punctured plane and we have chosen to call them $\mathcal{B}^*$ and $\mathcal{B}_{\log}^*$, but this notation is not standard at all.