Dynamic rays for transcendental holomorphic self-maps of $\mathbb{C}$ and $\mathbb{C}^*$

David Martí Pete

Master's thesis in the Research pathway directed by
Núria Fagella Rabionet
who was also my tutor

UNIVERSITAT DE BARCELONA

Màster de Matemàtica Avançada i Professional
Facultat de Matemàtiques
Universitat de Barcelona

2010 – 2011 Academic Year

Acknowledgements

First of all I want to thank my supervisor, Núria Fagella, not only for all her support with the mathematical stuff but for taking care of me all this time. During the months I have been doing this project I began attending to the *Complex Dynamics Working Seminar* at my faculty. I would like to express my gratitude to all the members of the group for being so welcoming. Specially to Xavier Jarque with who I had a couple of talks about this topic. The first time I gave a talk in a cientific event it was in the *Spring School: Topics in Complex Dynamics 2011* at the IMUB (organized by this group) and it was about this project.
I think that it is fair to thank also Sebastian Vogel, from Kiel. I met him in Göttingen during a summer school and it was a big surprise to find out that he did his masters thesis about R^3S too. He was so kind and gave me some helpful tips and advises. In that school I had the chance to attend to a mini-course on this topic given by Helena Mihaljević-Brandt which was very nice.
At this point, let me mention Jordi Taixés who introduced me to complex dynamics when I was a first year undergraduate. Thank you very much for discovering me such a wonderful world!
And, finally, I cannot forget to thank my family and friends. Without your support this could not be possible, I love you.

Contents

Introduction

This Master's thesis belongs to the field of complex dynamical systems, those generated by the iteration of self holomorphic maps of a Riemann surface. The theory is interesting in 3 cases: the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (rational maps), the complex plane (entire transcendental maps) and the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (maps with two essential singularities). In any other case the Julia set is empty and the dynamical study becomes trivial. By Montel's theorem, if we iterate a map omitting at least three points of $\widehat{\mathbb{C}}$ then every point must be normal.
The main goal of this project is twofold. On the one hand we study the recent article *Dynamic rays of bounded-type entire functions* by Günter Rottenfußer, Johannes Rückert, Lasse Rempe and Dierk Schleicher published in Annals of Mathematics (Second Series) in 2011, [RRRS11]. The results in this paper are a serious advance in the theory of iteration of entire transcendental maps, since they apply to a wide class of maps, setting the basis for further work in the field. The tools used in the paper are many and of varied nature, and we have made an effort to introduce them properly and fill in all details. On the other hand this project also contains original work by the author, namely the initial steps necessary to extend the above mentioned theory to self holomorphic maps of $\mathbb{C}^*$.
We start with a brief historical note about complex dynamics and afterwards we will motivate the project and present our main results. At the end there is a section to clarify the notation we are going to use throughout this work.

A bit of history

Given a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ and a seed $z_0 \in \mathbb{C}$ we are interested in the behaviour of the sequence

$$z_n = f(z_{n-1}) = f^n(z_0), \quad n \geqslant 1$$

called the (forward) orbit of z_0 under f. The origins of complex dynamics go back to the first studies of the Newton's method, one of the oldest root-finding algorithm but at the same time very efficient. Given a holomorphic function f and a seed z_0 close enough to a zero α of f then when we iterate the function

$$N_f(z) = z - \frac{f(z)}{f'(z)}$$

the orbit of z_0 converges to α. The first time that the iteration of holomorphic functions is mentioned is in 1870 in the studies of Ernst Schröder (1841-1902). A few years later, Arthur Cayley (1821-1895) also became interested in this topic. Both Cayley and

Schröder developed greatly the local study of the method. They were worried about questions like finding sufficient conditions for the local convergence or improving the speed of convergence. However, they also considered global questions (separating the plane into different attracting basins) but they only solved the polynomial case of degree 2. If you apply the Newton's method to $P(z) = (z - \alpha)(z - \beta)$, $\alpha \neq \beta$, you obtain two half planes of initial conditions converging respectively to α and β divided by the line bisecting the segment $\overline{\alpha\beta}$. These intermediate points, when used as initial conditions, produce orbits which do not converge to any of the roots. Cayley already noticed the difficulty of the degree 3 case. When we consider a polynomial with three roots there appear fractal structures as you can see in Figure 1, hence it was really difficult to solve it analytically with the tools that they had by that time.

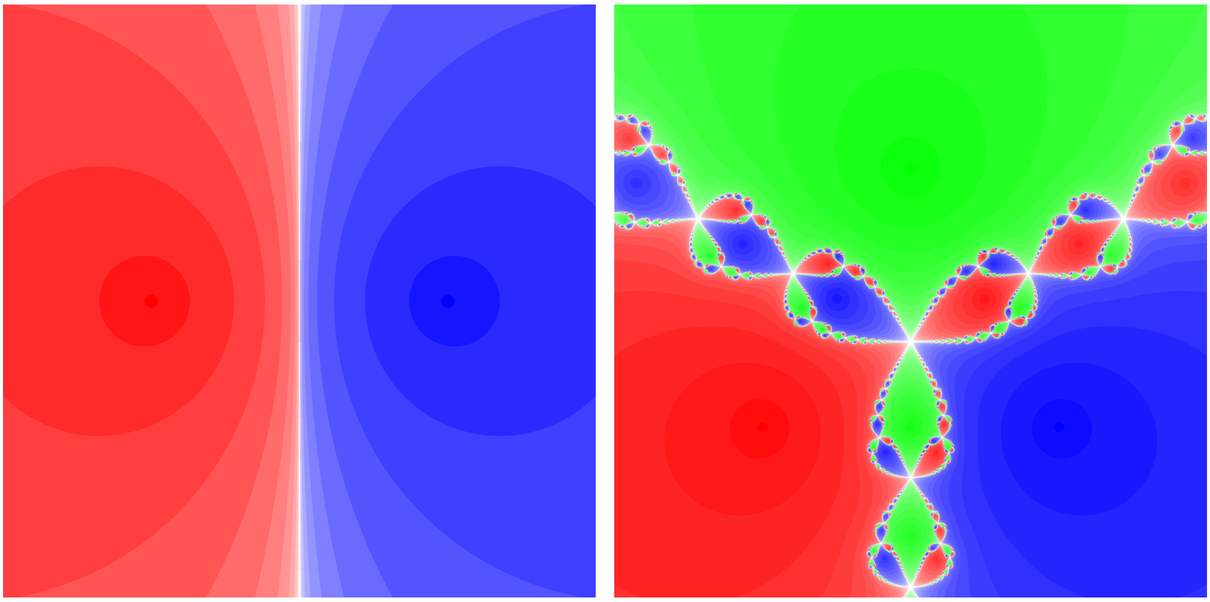

Figure 1: Phase space of the Newton's method applied to a polynomial. **L:** degree 2; **R:** degree 3. The color indicates which is the limit point of every seed and the number of iterates needed to enter a certain neighbourhood of the root, the white points belong to the Julia set.

There were no significant contributions to the global study until the beginning of the 20th century with the works by Pierre Fatou (1878-1929) and Gaston Julia (1893-1978) about the iteration of holomorphic functions on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. On 1915, the *Académie des sciences* from France announced that the *Grand Prix des Sciences Mathématiques* of 1918 was going to be awarded to the best work on iteration, specifying that it needed to be a global study. This choice could have been motivated by the works of Henri Poincaré. This contest led to a strong rivalry between Fatou and Julia. In the last moment Fatou decided not to participate in the contest and the *prix* was awarded to Julia. However, both produced excellent works which are basic to understand complex dynamics as we do nowadays.

They introduced the use of normal families to decompose the phase space. Every normal point has a neighbourhood of points which behave in a similar fashion when iterated: these points are in the stable set. Conversely, every point in the complement of this set has a chaotic behaviour. Today, the stable set is known as the Fatou set and its complement

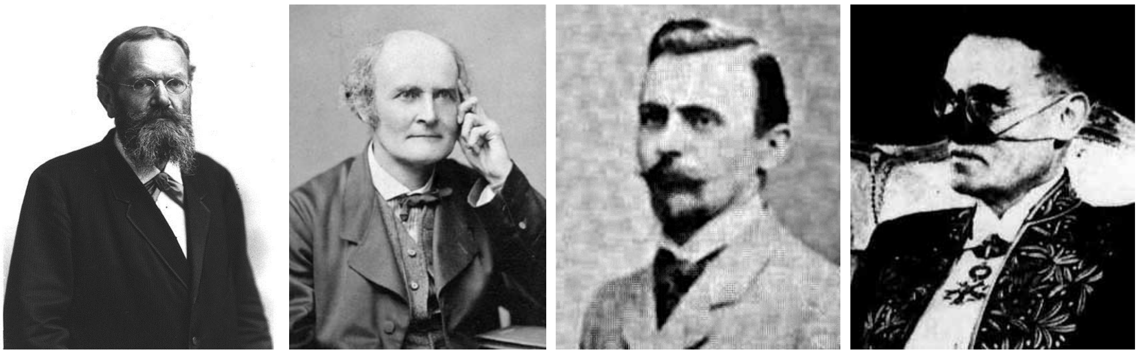

Figure 2: From left to right, Ernst Schröder, Arthur Cayley, Pierre Fatou and Gaston Julia. Pictures from the internet.

is called the Julia set. When they tried to study the Julia set they encountered the same difficulty as Cayley, it is a fractal set. In Figure 3 you can see how Julia tried to draw it.

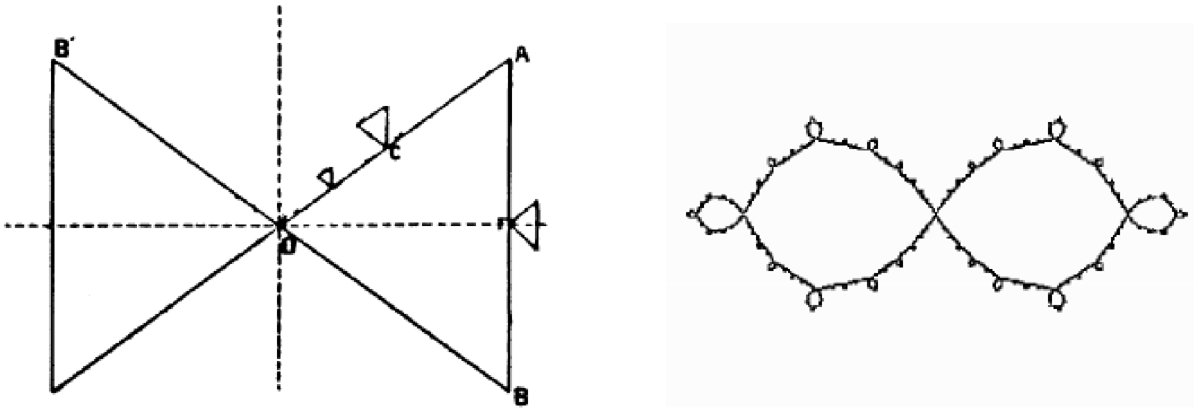

Figure 3: **L:** the sketch that Gaston Julia made of a Julia set; **R:** the actual Julia set. Picture from [Ale94].

Fortunately, the use of computers was a major breakthrough for complex dynamics. Now we can visualize the Fatou and Julia sets, and hence we have a much better intuition of what is happening. The phase space of polynomials and rational functions has been widely studied and is fairly well understood. Nevertheless there are still many relevant open problems, specially those related about parameter spaces.

In parallel, but with a lesser speed, the theory of transcendental maps has been developed. Maps with an essential singularity show a substantial increase of difficulty. For instance, Great Picard's theorem tells us that every neighbourhood of the essential singularity must be mapped to the whole plane with the exception, at most, of one point. This simle fact, adds plenty of chaos to the system producing unbounded Julia sets with very interesting topology. Fatou was the first to consider the dynamics of entire transcendental functions in his article [Fat26]. Since then, transcendental dynamics has become a very active area of research.

If you are interested in the history of complex dynamics, you may like the book [Ale94].

Transcendental maps: Motivation and contents

In his article [Fat26], Fatou already observed that the Julia set of certain entire transcendental functions contains curves of points that escape to infinity under iteration and wondered if this was a general property. Alexandre Eremenko introduced the notion of escaping set in his article [Ere89]. Given an entire transcendental function f,

$$I(f) := \{z \in \mathbb{C} \ : \ |f^n(z)| \to +\infty\}.$$

He proved that every component of $\overline{I(f)}$ is unbounded and conjectured that

- each component of $I(f)$ is unbounded (weak Eremenko's conjecture);
- every point in $I(f)$ can be joined with ∞ by a curve in $I(f)$ (strong Eremenko's conjecture).

Such curves are called dynamical rays (or hairs) in analogy to the dynamic rays of polynomials introduced by A. Douady and J. Hubbard [Mil06], These type of invariant objects, governed by symbolic dynamic, proved themselves to be a crucial tool in the development of the theory since they often can be used to define dynamicaly meaningful partitions of the phase space.

In contrast to the polynomial case where dynamic rays belong to the Fatou (or stable) set, dynamic rays for transcendental maps are part of the Julia (or chaotic) set. Studies about the existence and topological description of (transcendental) dynamic rays were developed in the 80's by Robert L. Devaney and his collaborators. They started with the exponential family as the simplest model for these types of maps, and they later moved to entire maps *of finite type* (i.e., with a finite number of singularities of the inverse map). The main reference is by Devaney and Tangerman [referencia] where they gave some conditions under which they could prove the existence of *Cantor Bouquets*, consisting of Cantor sets of dynamic rays. This seminal work gave rise to many intents of generalizations. The most successful one is the recent work of Rempe et al. in [RRRS11], which is partially the object of this thesis. In this paper, the authors set up a general theory which partially proves Eremenko's conjecture for a wide class of functions in class $\mathcal{B}$, i.e., maps with a bounded set of singularities of the inverse. More precisely, their main theorem reads as follows.

Theorem (Entire functions with dynamic rays)**.** *Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in I(f)$ can be connected to ∞ by a curve γ such that $f^n_{|\gamma} \to \infty$ uniformly.*

Chapter 4 of this project is devoted to prove this theorem. In the same paper they also showed that some assumptions will be necessary, giving in fact a counterexample to the strong Eremenko's conjecture.

Theorem (Entire functions without dynamic rays)**.** *There exists a hyperbolic entire function $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded.*

Furthermore we are interested in studying some structural properties of the Julia set of holomorphic self-maps of the complex punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. A reason which makes this class of self-maps very interesting is that they often arise as complexifications

of analytic maps of the circle, crucial for the study of rotation domains in real and in complex dynamical systems of all types. For instance, in Section 5.4 we will see that the complexification of the Arnol'd standard family is of this type.
If we lift a self-map of $\mathbb{C}^*$, say f, by $h(z) = \exp(z)$ we get an entire transcendental map, say F, satisfying a lift property, i.e. $F(z+2\pi i) = F(z)+2k\pi i$, where $k \in \mathbb{N}$. The relevant dynamical objects (singularities of the inverse like critical points, asymptotic values, etc.) and even the Julia sets for these two maps are in correspondence via the exponential map [Ber95]. A consequence of this fact, because of the periodicity of the exponential, is that such lift necessarily will have infinitely many of such elements, distributed in the whole complex plane. In particular, no such lift belongs to class $\mathcal{B}$.
The theory developed in [RRRS11] therefore cannot be applied directly to self-maps of $\mathbb{C}^*$. It would be desirable however to establish such results, specially having in mind further applications. This is the goal of a long term project as part of the author's future Ph.D. thesis. In the meantime, in this Master's thesis, we construct the right setup under which the results of [RRRS11] can be applied, providing some partial results in the right direction. More precisely we show the following. Let $I_0(f)$ and $I_\infty(f)$ be the sets of points that converge uniformly to 0 and ∞ respectively under iteration by a holomorphic self-map of $\mathbb{C}^*$ f. For this class of functions we must take into account two orders of growth, one at zero and the other one at infinity. We say that f has *finite order* if both

$$\rho_\infty(f) := \lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|}, \quad \rho_0(f) := \rho_\infty(h\circ f\circ h)$$

are finite, where $h(z) := 1/z$ conjugates a neighbourhood of ∞ to a neighbourhood of 0.

Theorem. *Let $f :\in \mathcal{B}^*$ be a function of finite order or a finite composition of such maps. Then every point $z \in I_\infty(f)$ can be connected to ∞ by a curve γ such that $f^n_{|\gamma} \to \infty$ uniformly. Similarly, every every point $z \in I_0(f)$ can be connected to 0 by a curve γ such that $f^n_{|\gamma} \to 0$ uniformly.*

Structure of the project

The first three chapters are preliminary sections which contain the tools used in the main chapters which are 4 and 5.
In the first chapter we introduce hyperbolic geometry. We begin by defining the Poincaré metric in the unit disc $\mathbb{D}$ and study the existence and uniqueness of geodesics, minimal length curves joining two points. Our purpose is to endow arbitrary domains with a hyperbolic metric and for this we introduce the notion of covering space. In the last section we prove Pick's theorem and the standard estimate, which will be used many times during the project.
Chapter 2 is dedicated to Continuum theory and the main goal is to prove the so-called *Non-cut point characterization of the arc* which will be the key point in the proof of the main result in Chapter 4. A *continuum* is a non-empty, compact, connected metric space and we call *cut point* of a topological space S to a point $p \in S$ such that $S \setminus \{p\}$ is disconnected. Otherwise, the point p is called a non-cut point of S. Using some *separation ordering* we show that a continuum is an arc if and only if it has exactly two non-cut

points. On the other hand, this is equivalent to the fact that the continuum has some topological properties and its topology agrees with the separation order topology.
In Chapter 3 we introduce the basic concepts in complex dynamics, stating with the notions of normality and the Fatou and Julia sets. We state their basic topological and dynamical properties and classify the Fatou components for rational and entire transcendental functions. A very characteristic property of transcendental functions is that their Julia set contains Cantor bouquets. At the end of the chapter we describe the general properties of the escaping set.
Chapter 4 is devoted to prove the main theorem in [RRRS11], as explained in the introduction. To do so, we define logarithmic tracts and introduce *logarithmic coordinates*, an exponential lift of the restriction of the function to the tracts. Using hyperbolic geometry, we prove some expansivity properties similar to the ones of the exponential function. We conclude this chapter with a discussion about the existence of Cantor bouquets. We would like to remark that we required the use of Zorn's lemma in the proof of Proposition 3.7.3. Therefore you may be aware that this construction depends on the Axiom of choice.
Finally in Chapter 5 we study self holomorphic maps of $\mathbb{C}^*$. We describe the geometry of the tracts and logarithmic coordinates and provide the setup to apply the results in [RRRS11] to this class of functions.
All the pictures in this project except a couple in the Introduction have been made by the author, either using standard software (Mathematica, Geogebra, Inkscape) or own programs written in C++ language.

Notation

Throughout this project, we will denote by $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the Riemann sphere and by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ the punctured plane. $\mathbb{D}$ will stand for the open unit disc, $\mathbb{H}$ the right half-plane and if $R > 0$,

$$B_R(z_0) := \{z \in \mathbb{C} \ : \ |z - z_0| < R\}, \quad \mathbb{H}_R := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > R\}.$$

If $A \subseteq \mathbb{C}$, then $\overline{A}$ and $\widehat{A}$ denote respectively the closures of A in $\mathbb{C}$ and $\widehat{\mathbb{C}}$, ∂A is the boundary of A in $\mathbb{C}$ and

$$\mathring{A} = \overline{A} \setminus \partial A$$

is the interior of A.
If $P, Q \in \mathbb{C}$, $\overline{PQ}$ is the straight segment joining these two points. Euclidean length and distance are denoted by l and dist. Let $X \subseteq \mathbb{C}$, then ρ_X, l_X and dist_X denote respectively the hyperbolic density, length and distance with respect to the domain X. See Section 1.5 for the precise definitions.
In the context of continuum theory, we introduce the notation $Y = P|Q$ to denote a partition into mutually separated sets (see Section 2.3) and $S(p, q)$ for the set of points separating p and q (see Section 2.4). We use $\prec$ to refer to non-standard orderings, like the separation ordering in Chapter 2 or the speed ordering in Chapter 4.
If f is a function and $n \in \mathbb{N}$, by f^n we will always mean the composition of f with itself n times,

$$f^n = f \circ \overset{n}{\cdots} \circ f,$$

e.g. $f^2(z) = f(f(z))$ which is different of $g(z) = (f(z))^2$. We understand that $\mathbb{N}$ begins with 0 and define $f^0(z) = z$. Some authors use the notation $f^{\circ n}$ to denote this, but we will avoid it because we think that there is no possibility of confusion. As usual, $J(f)$ and $F(f)$ are the Julia set and the Fatou set of f, the dynamical partition of the plane associated to f. $I(f)$ is the escaping set of f. Some related sets are $J^K(f)$ and $J_{\underline{s}}$, $I_{\underline{s}}$, $J_{\underline{s}}^K$. You can find out what all this terminology is by looking at Section 4.3. If z_0 is a fixed point for f, $A_f(z_0)$ denotes its attracting basin and $A_f^*(z_0)$ its immediate attracting basin (the component of $A_f(z_0)$ containing z_0).
The Eremenko-Lyubich class $\mathcal{B}$ is introduced in the beginning of Chapter 4. Along that chapter we define class $\mathcal{B}_{log}$ and all its variants $\mathcal{B}_{log}^n$ (normalized), $\mathcal{B}_{log}(\alpha, \beta)$ (bounded slope) and $\mathcal{B}_{log}^n(\alpha, \beta)$. In Chapter 5 we define the analogs of $\mathcal{B}$ and $\mathcal{B}_{log}$ for the punctured plane and we have chosen to call them $\mathcal{B}^*$ and $\mathcal{B}_{\text{log}}^*$, but this notation is not standard at all.

Chapter 1

Introduction to hyperbolic geometry and covering spaces

First of all let us remind some general notions that will become useful tools afterwards. We introduce the hyperbolic metric (also know as Poincaré metric) on the Poincaré disc model. In particular, we focus on the existence and uniqueness of geodesics and describe their shape. After this we move to the study of covering spaces, discussing their lifting properties and the existence of a universal covering. Then using the universal cover from the disc we define the Poincaré metric for arbitrary domains. The last sections are devoted to prove two properties that we will use many times.

1.1 Hyperbolic metric in the unit disc

In this section we are going to introduce the basic notions of hyperbolic geometry in dimension two. We are going to use the Poincaré disc model which consists of the unit disc and can be embedded in the complex plane $\mathbb{C}$,

$$\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}.$$

The Möbius transformations form a group of conformal maps of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A Möbius transformation preserving $\mathbb{D}$ can be written in the form

$$A(z) = e^{i\theta} \frac{z + a}{1 + \overline{a}z}$$

where $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. The group of all such automorphisms will be denoted by $\mathrm{Aut}(\mathbb{D})$.

Definition 1.1.1 (Hyperbolic density on $\mathbb{D}$). The *hyperbolic density* at a point $a \in \mathbb{D}$ is defined by

$$\rho_{\mathbb{D}}(a) = \frac{2}{1 - |a|^2}.$$

Observe that it only depends on the Euclidean distance of a to the origin. It takes its minimum value at $a = 0$ where it equals 2 and tends to ∞ as a approaches the boundary of $\mathbb{D}$. Some authors, like in [KL07], use the convention that $\rho_{\mathbb{D}}(0) = 1$ (i.e. they put a 1 in the numerator) but we prefer this one because then the metric has constant Gaussian curvature -1 instead of -4.

Lemma 1.1.1 (Invariance under automorphisms). *The hyperbolic density on $\mathbb{D}$ is invariant under automorphisms of $\mathbb{D}$.*

Proof. Let $A \in \operatorname{Aut}(\mathbb{D})$,

$$A(z) = e^{i\theta}\frac{z+a}{1+\overline{a}z}$$

where $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. We want to see that

$$\rho_{\mathbb{D}}(A(t))|dA(t)| = \rho_{\mathbb{D}}(A(t))|A'(t)||dt| = \rho_{\mathbb{D}}(t)|dt|,$$

this is

$$\rho_{\mathbb{D}}(A(t))|A'(t)| = \rho_{\mathbb{D}}(t).$$

We have

$$\begin{aligned}\rho_{\mathbb{D}}(A(t)) &= \frac{2}{1-\frac{|t+a|^2}{|1+\overline{a}t|^2}} = \frac{2|1+\overline{a}t|^2}{|1+\overline{a}t|^2-|t+a|^2} = \frac{2|1+\overline{a}t|^2}{1+2|\overline{a}t|+|\overline{a}t|^2-|t|^2-2|at|-|a|^2}\\ &= \frac{2|1+\overline{a}t|^2}{1+|\overline{a}t|^2-|t|^2-|a|^2} = \frac{2|1+\overline{a}t|^2}{(1-|t|^2)(1-|a|^2)}\end{aligned}$$

and on the other hand

$$|A'(t)| = \frac{|1+\overline{a}t-(t+a)\overline{a}|}{|1+\overline{a}t|^2} = \frac{|1-a\overline{a}|}{|1+\overline{a}t|^2} = \frac{1-|a|^2}{|1+\overline{a}t|^2}.$$

Putting this together we get

$$\rho_{\mathbb{D}}(A(t))|A'(t)| = \frac{2(1-|a|^2)}{(1-|t|^2)(1-|a|^2)} = \frac{2}{1-|t|^2} = \rho_{\mathbb{D}}(t).$$

■

Using the hyperbolic density we can define a metric on $\mathbb{D}$ called the hyperbolic metric or the Poincaré metric.

Definition 1.1.2 (Hyperbolic length on $\mathbb{D}$). Let γ be a path in $\mathbb{D}$ joining two points p and q. Then define the *hyperbolic length* of γ by

$$l_{\mathbb{D}}(\gamma) = \int_{\gamma}\rho_{\mathbb{D}}(t)|dt|.$$

Definition 1.1.3 (Hyperbolic distance on $\mathbb{D}$). Given two points $p, q \in \mathbb{D}$ we define the *hyperbolic distance* between them to be

$$\operatorname{dist}_{\mathbb{D}}(p,q) = \inf_{\gamma} l_{\mathbb{D}}(\gamma)$$

where the paths γ are contained in $\mathbb{D}$ and join p and q.

Let us check that the hyperbolic distance defines a metric on $\mathbb{D}$.

Lemma 1.1.2. *The unit disc $\mathbb{D}$ together with the hyperbolic distance $dist_{\mathbb{D}}$ is a metric space.*

Proof. It is clear that the definition is symmetric, $\text{dist}_{\mathbb{D}}(p,q) = \text{dist}_{\mathbb{D}}(q,p)$. Note that $\rho_{\mathbb{D}}(a) \geqslant \rho_{\mathbb{D}}(0) = 2$ for all $a \in \mathbb{D}$. Therefore, $l_{\mathbb{D}}(\gamma) \geqslant 0$ for any path γ in $\mathbb{D}$ and it vanishes if and only if the path is a point. This proves the non-negativity of the distance function and shows that $\text{dist}_{\mathbb{D}}(p,q) = 0$ if and only if $p = q$. Finally, the infimum in the definition ensures that the triangle inequality is satisfied. Indeed, if there existed $r \in \mathbb{D}$ such that

$$\text{dist}_{\mathbb{D}}(p,q) > \text{dist}_{\mathbb{D}}(p,r) + \text{dist}_{\mathbb{D}}(r,q)$$

then concatenating these two limit sets of paths from p to r and from r to q we would contradict the minimality of γ. ■

It is a direct consequence of Lemma 1.1.1 that if γ is a path in $\mathbb{D}$ and $A \in \text{Aut}(\mathbb{D})$ then $l_{\mathbb{D}}(\gamma) = l_{\mathbb{D}}(A(\gamma))$ and moreover if $p,q \in \mathbb{D}$,

$$\text{dist}_{\mathbb{D}}(p,q) = \text{dist}_{\mathbb{D}}(A(p), A(q)).$$

1.2 Hyperbolic geodesics

In general, the infimum in the definition of distance may not be realized by a curve. For instance, in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the Euclidean distance between -1 and 1 is 2 but none of the curves contained in $\mathbb{C}^*$ joining these points has Euclidean length 2. In the other extremum, in the sphere two antipodal points have an infinite number of curves joining them minimizing the spherical length. However this cannot happen in $\mathbb{D}$, we will prove the existence and uniqueness of such curves. First, let us introduce the notion of geodesic.

Definition 1.2.1 (Geodesic). A curve γ in $\mathbb{D}$ is called a *geodesic* if for all $t_1 < t_2 < t_3$ we have

$$\text{dist}_{\mathbb{D}}(\gamma(t_1), \gamma(t_3)) = \text{dist}_{\mathbb{D}}(\gamma(t_1), \gamma(t_2)) + \text{dist}_{\mathbb{D}}(\gamma(t_2), \gamma(t_3)).$$

If the extrempoints of γ belong to $\mathbb{D}$ we say that γ is a geodesic segment while if they belong to $\partial\mathbb{D}$ we say that γ is an infinite geodesic.

By definition, geodesics always realize the minimum distance between two points. The next lemma tells us that the converse is also true. Hence, geodesics can be characterized as being the shortest paths between the points of $\mathbb{D}$.

Lemma 1.2.1 (Characterization of geodesics). *If a curve γ joining two points $p, q \in \mathbb{D}$ realizes the infimum length among all such curves then it is a geodesic.*

Proof. By definition, $\text{dist}_{\mathbb{D}}(p,q) = l_{\mathbb{D}}(\gamma)$. Let $r \in \gamma$ be an intermediate point. Then if we split γ in two curves γ_1, γ_2 going from p to r and from r to q respectively we have

$$l_{\mathbb{D}}(\gamma) = l_{\mathbb{D}}(\gamma_1) + l_{\mathbb{D}}(\gamma_2)$$

by linearity of the integral operator. The minimality of γ implies that $l_{\mathbb{D}}(\gamma_1) = \text{dist}_{\mathbb{D}}(p,r)$ and $l_{\mathbb{D}}(\gamma_2) = \text{dist}_{\mathbb{D}}(r,q)$. Hence,

$$\text{dist}_{\mathbb{D}}(p,q) = l_{\mathbb{D}}(\gamma) = l_{\mathbb{D}}(\gamma_1) + l_{\mathbb{D}}(\gamma_2) = \text{dist}_{\mathbb{D}}(p,r) + \text{dist}_{\mathbb{D}}(r,q)$$

and γ is a geodesic. ■

Now let us prove their existence in a geometric way. The next lemma deals with geodesics of $\mathbb{D}$ containing the origin.

Proposition 1.2.2 (Existence of geodesics). *Consider $p \in \mathbb{D}$, $p \neq 0$. The straight segment between 0 and p realizes the shortest path with respect to the hyperbolic metric on $\mathbb{D}$ and has length*

$$dist_{\mathbb{D}}(0,p) = \log \frac{1+|p|}{1-|p|}.$$

Proof. We can parametrize the straight segment joining 0 to p by $\gamma_0(t) = tp$ for $t \in I$ and then its hyperbolic length is given by

$$l_{\mathbb{D}}(\gamma_0) = \int_{\gamma_0} \rho_{\mathbb{D}}(t)|dt| = \int_0^1 \frac{2}{1-t^2|p|^2}|p|dt.$$

Let us compute this integral,

$$\int \frac{2|p|}{1-t^2|p|^2}dt = \int \frac{2|p|}{(1+t|p|)(1-t|p|)}dt = \int \frac{\frac{2|p|}{(1-t|p|)^2}}{\frac{1+t|p|}{1-t|p|}}dt = \log \frac{1+t|p|}{1-t|p|} + C$$

and hence

$$l_{\mathbb{D}}(\gamma_0) = \left[\log \frac{1+t|p|}{1-t|p|}\right]_0^1 = \log \frac{1+|p|}{1-|p|}.$$

Consider now an arbitrary curve $\gamma : I \to \mathbb{D}$ connecting 0 to p. Take a partition $\mathcal{P}$ of I, $0 = t_0 < \cdots < t_n = 1$, then the Riemann sum

$$L_{\mathbb{D}}(\gamma; \mathcal{P}) = \sum_{i=0}^{n-1} \rho_{\mathbb{D}}(\gamma(t_i))|\gamma(t_{i+1}) - \gamma(t_i)|$$

approaches $l_{\mathbb{D}}(\gamma)$ as the mesh of $\mathcal{P}$ tends to 0. This can be thought as considering a piecewise linear curve with vertices in γ. These vertices can be transported radially to the straight line defined by 0 and p by the map

$$\gamma(t_i) \mapsto |\gamma(t_i)|\frac{p}{|p|} =: \gamma_0^i.$$

If we join the image points by straight segments we get a curve γ_0' that may fold many times over itself and which is contained in the line defined by γ_0. If we compare it with γ_0, γ_0' is clearly longer because it goes from 0 to p but we have to add the auto-intersections, hence

$$L_{\mathbb{D}}(\gamma_0'; \mathcal{P}) = \sum_{i=0}^{n-1} \rho_{\mathbb{D}}(\gamma_0^i)|\gamma_0^{i+1} - \gamma_0^i|$$

is an upper bound of $l_{\mathbb{D}}(\gamma_0)$. Observe that since the hyperbolic density is invariant under automorphisms of the disc and, in particular, under rotations about the origin,

$$\rho_{\mathbb{D}}(\gamma_0^i) = \rho_{\mathbb{D}}(\gamma(t_i)).$$

On the other hand, it is well known that given two concentric circumferences S_1, S_2 with centre C and a point $x \in S_1$, the minimum distance between x and S_2 is realized by the point in the intersection of S_2 and the line through C and x. Thus,

$$|\gamma_0^{i+1} - \gamma_0^i| \leqslant |\gamma(t_{i+1}) - \gamma(t_i)|.$$

Putting these together,

$$L_{\mathbb{D}}(\gamma_0'; \mathcal{P}) = \sum_{i=0}^{n-1} \rho_{\mathbb{D}}(\gamma_0^i)|\gamma_0^{i+1} - \gamma_0^i| \leqslant \sum_{i=0}^{n-1} \rho_{\mathbb{D}}(\gamma(t_i))|\gamma(t_{i+1}) - \gamma(t_i)| = L_{\mathbb{D}}(\gamma; \mathcal{P}).$$

For every partition $\mathcal{P}$,

$$l_{\mathbb{D}}(\gamma_0) \leqslant L_{\mathbb{D}}(\gamma_0'; \mathcal{P}) \leqslant L_{\mathbb{D}}(\gamma; \mathcal{P}).$$

Taking the limit as the mesh of $\mathcal{P}$ tends to 0 we get

$$l_{\mathbb{D}}(\gamma_0) \leqslant l_{\mathbb{D}}(\gamma)$$

and conclude that γ_0 realizes the minimum length path between 0 and p. ■

Corollary 1.2.3 (Existence of geodesics). *Given two points $p, q \in \mathbb{D}$, $p, q \neq 0$, there exists a geodesic joining them and its length equals*

$$dist_{\mathbb{D}}(p, q) = \log \frac{|1 - \overline{p}q| + |q - p|}{|1 - \overline{p}q| - |q - p|}.$$

This geodesic corresponds to an arc of circle orthogonal to $\partial\mathbb{D}$.

Proof. Recall that the hyperbolic distance is invariant under automorphisms of $\mathbb{D}$. Such transformations are of the form

$$A(z) = e^{i\theta} \frac{z + a}{1 + \overline{a}z}$$

for some θ and a. Let us impose that $A(p) = 0$:

$$A(p) = e^{i\theta} \frac{p + a}{1 + \overline{a}p} = 0 \quad \Leftrightarrow \quad a = -p.$$

Since θ is arbitrary, fix $\theta = 0$ for simplicity. We will denote this map by A_p. Then, $\operatorname{dist}_{\mathbb{D}}(p, q) = \operatorname{dist}_{\mathbb{D}}(0, A_p(q))$ where

$$A_p(q) = \frac{q - p}{1 - \overline{p}q}.$$

Denote by D the diameter of $\mathbb{D}$ that contains $A_p(q)$. Then $A_p^{-1}(D)$ is the geodesic connecting p and q. Using Proposition 1.2.2,

$$\operatorname{dist}_{\mathbb{D}}(0, A_p(q)) = \log \frac{1 + \frac{|q-p|}{|1-\overline{p}q|}}{1 - \frac{|q-p|}{|1-\overline{p}q|}} = \log \frac{|1 - \overline{p}q| + |q - p|}{|1 - \overline{p}q| - |q - p|}.$$

Recall that $\operatorname{Aut}(\mathbb{D})$ is a group and Möbius transformations are conformal and map circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$. Since D is orthogonal to $\partial\mathbb{D}$ and $\partial\mathbb{D}$ is invariant under $\operatorname{Aut}(\mathbb{D})$, $A_p^{-1}(D)$ must be an arc of a circle orthogonal to $\partial\mathbb{D}$. ■

To complete the proof of our claim, let us show that the geodesics of $\mathbb{D}$ are unique.

Proposition 1.2.4 (Uniqueness of geodesics). *For any point $p \neq 0$ in $\mathbb{D}$ there exists a unique geodesic joining 0 to p.*

Proof. Proposition 1.2.2 shows that the straight segment γ_0 from 0 to p is a geodesic. Assume to the contrary that there exists another geodesic γ connecting 0 and p,

$$\operatorname{dist}_{\mathbb{D}}(0,p) = l_{\mathbb{D}}(\gamma) = l_{\mathbb{D}}(\gamma_0).$$

Since they are different, there must exist one point $q \in \gamma$ such that $q \notin \gamma_0$. Let q_0 be the radial projection of q onto γ_0 and let C be the circle centred at 0 and containing q and q_0. The geodesic property gives

$$\operatorname{dist}_{\mathbb{D}}(0,p) = \operatorname{dist}_{\mathbb{D}}(0,q) + \operatorname{dist}_{\mathbb{D}}(q,p) = \operatorname{dist}_{\mathbb{D}}(0,q_0) + \operatorname{dist}_{\mathbb{D}}(q_0,p)$$

and since $|q| = |q_0|$, $\operatorname{dist}_{\mathbb{D}}(0,q) = \operatorname{dist}_{\mathbb{D}}(0,q_0)$ and hence we get $\operatorname{dist}_{\mathbb{D}}(q,p) = \operatorname{dist}_{\mathbb{D}}(q_0,p)$. Now, consider the Möbius transformation

$$A_p(z) = \frac{z-p}{q-\overline{p}z}$$

mapping p to 0 and 0 to $-p$. Let $\widetilde{q} = A_p(q)$ and $\widetilde{q_0} = A_p(q_0)$, by Lemma 1.1.1

$$\operatorname{dist}_{\mathbb{D}}(\widetilde{q},0) = \operatorname{dist}_{\mathbb{D}}(q,p) = \operatorname{dist}_{\mathbb{D}}(q_0,p) = \operatorname{dist}_{\mathbb{D}}(\widetilde{q_0},0)$$

and therefore $|\widetilde{q}| = |\widetilde{q_0}|$. A_p maps the segment $\overline{0p}$ to the segment $\overline{-p0}$ and the circle C to the circle with hyperbolic centre $-p$ containing $\widetilde{q}$ and $\widetilde{q_0}$. Then either $\widetilde{q} = \widetilde{q_0}$ (which is impossible if $q \neq q_0$, A_p is univalent) or they are at a different Euclidean distance from 0, $|\widetilde{q}| \neq |\widetilde{q_0}|$, raising a contradiction in both cases. The geodesic γ_0 must be unique. ∎

To conclude this section we will see how to construct hyperbolic geodesics with compass and straight-edge, see Figure 1.1. Let $P, Q \in \mathbb{D}$ such that P, Q, O are not collinear. If they were collinear, just trace the diameter through P and Q. You have to follow these steps:

- Trace the line through O and P.
- Take the perpendicular r to $\overline{OP}$ through P.
- Let A be the point in the intersection of r and $\partial\mathbb{D}$.
- Let s be the perpendicular to $\overline{OA}$ through A.
- Call B the intersection of s and $\overline{OA}$ (B is the inverse of P with respect to $\partial\mathbb{D}$).
- Draw the circle C passing through P, Q, B.

The arc of C lying inside $\mathbb{D}$ is the hyperbolic geodesic joining P and Q.

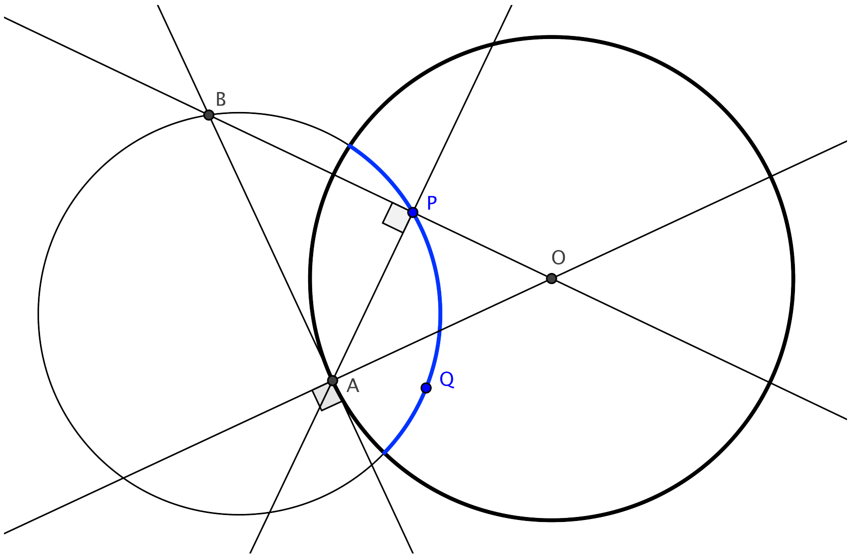

Figure 1.1: Construction of the hyperbolic geodesic with compass and straight-edge.

1.3 Covering spaces

In this section we introduce the notions of covering space and universal covering that are very useful for many reasons. Fist of all, as we will see in the next section, they allow us to put a hyperbolic metric in domains far more general than the Poincaré disc. On the other hand, they are a very important tool in the construction of the logarithmic coordinates. We will see that logarithmic tracts and logarithmic coordinates are both universal covers.

Definition 1.3.1 (**Covering space**)**.** Let X be a topological space. A topological space E together with a projection map $p : E \to X$ is a *covering space* of X if every $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets $S_i \subseteq E$, each of which is mapped homeomorphically onto U by p. The covering is holomorphic if π is holomorphic. We say that the sets U are *evenly covered* and S_i are *sheets over* U.

Note that for all $x \in X$, the fibre $p^{-1}(x)$ is a discrete set. Since p is a local homeomorphism, E and X share the same local properties, for instance E is locally connected if and only if X is so. Finally let us remark that we can endow X with the quotient topology from E.

Definition 1.3.2 (**Section**)**.** Let $p : E \to X$ be a covering space. A continuous map $s : X \to E$ such that $p \circ s = \mathrm{id}_X$ is called a *section* of the covering space.

$$\begin{tikzcd} E \arrow[r, shift left, "p"] & X \arrow[l, shift left, dashed, "s"] \end{tikzcd}$$

The main example that motivated the theory of covering spaces and that we should keep in mind is

$$\begin{array}{rccc} p: & \mathbb{R} & \to & \mathbb{S}^1 \\ & t & \mapsto & e^{it}. \end{array}$$

Any branch of the logarithm would be a section of this covering. According to the above definition, it is clear that $p : \mathbb{R} \to \mathbb{S}^1$ is a covering space. We can take two charts in S^1 consisting of overlapping open arcs U_1, U_2 and then the preimages of each of them by p will be and infinite set of disjoint open intervals. For example if we take $U_1 = S^1 \setminus \{1\}$ and $U_2 = S^1 \setminus \{-1\}$, then

$$p^{-1}(U_1) = \bigsqcup_{k\in\mathbb{Z}} (k,\ k+1), \quad p^{-1}(U_2) = \bigsqcup_{k\in\mathbb{Z}} (k-1/2,\ k+1/2).$$

Another interesting example would be the helicoid. Consider the surface

$$S = \{(s\cos(2\pi t), s\sin(2\pi t), t) \in \mathbb{R}^3 \ :\ (s,t) \in (0,\infty) \times \mathbb{R}\},$$

then

$$\begin{array}{rccc} p_{12}: & S \subseteq \mathbb{R}^3 & \to & \mathbb{R}^2 \setminus \{0\} \\ & (x,y,z) & \mapsto & (x,y) \end{array}$$

is a covering space. See Figure 1.2.

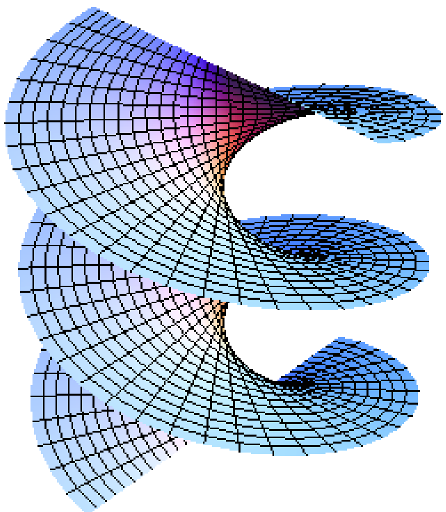

Figure 1.2: The helicoid.

This example is closely related to the following one. The complex exponential

$$\begin{array}{rccc} \exp: & \mathbb{C} & \to & \mathbb{C}^* \\ & z & \mapsto & e^z \end{array}$$

is a covering of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Every point in $\mathbb{C}^*$ is covered infinitely many times by the exponential map. This is very different of what happens with the complex powers defined on $\mathbb{C}^*$,

$$\begin{array}{rccc} p_n: & \mathbb{C}^* & \to & \mathbb{C}^* \\ & z & \mapsto & z^n \end{array}$$

is a covering space for all $n \in \mathbb{N}$, but in this case over each point there are exactly n sheets.

Definition 1.3.3 (Critical point). Let $f : Y \to X$ be a holomorphic function. We say that $c \in Y$ is a *critical point* of f if $f'(c) = 0$.

Lemma 1.3.1. *A holomorphic function f defined on a domain Ω is a holomorphic cover of its image $f(\Omega)$ if and only if f has no critical points in Ω.*

According to this lemma, since $\exp(z)$ has no critical points it is a holomorphic cover of $\mathbb{C}^*$. In the case of $p_n(z) = z^n$, the only critical point is the origin and since it has been removed from its domain it is also a holomorphic cover of $\mathbb{C}^*$ for every $n \in \mathbb{N}$.

Definition 1.3.4 (Lift). Let $p : E \to X$ be a covering space. A lift of a map $f : Y \to X$ is a map $\widetilde{f} : Y \to E$ such that $p\widetilde{f} = f$.

The following diagram illustrates this situation:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & X. \end{tikzcd}$$

We will devote the rest of this section to discuss the existence and uniqueness of lifts. We are going to use the notation $f : (Y, y_0) \to (X, x_0)$ for a map (or a covering) with specified basepoints. This only means that $y_0 \in Y$ is a point in the preimage of $x_0 \in X$ by f, i.e. $f(y_0) = x_0$.

Theorem 1.3.2 (Unique lifting theorem). *Given a covering space $p : E \to X$ and a map $f : Y \to X$ with two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to E$ that agree at one point of Y, then if Y is connected, these two lifts must agree on all of Y.*

Proof. This can be restated in terms of maps with specified basepoints saying that if $p : (E, e_0) \to (X, x_0)$ and $f : (Y, y_0) \to (X, x_0)$ then if there is a lift $\widetilde{f} : (Y, y_0) \to (E, e_0)$ it is unique. Assume to the contrary that there were two of them, say $\widetilde{f}_1, \widetilde{f}_2 : (Y, y_0) \to (E, e_0)$. We can consider the set of points where they coincide

$$A = \{y \in Y \ : \ \widetilde{f}_1(y) = \widetilde{f}_2(y)\}$$

and also $D = Y \setminus A$ where they differ. Observe that the specified basepoint y_0 belongs to A and hence A is not empty by assumption.
Take a point $y_1 \in Y$ and let U be an evenly covered neighbourhood of $f(y_1)$ in X. Consider S_1, S_2 the connected components of $p^{-1}(U)$ containing respectively $\widetilde{f}_1(y), \widetilde{f}_2(y)$. In the case that $y_1 \in A$ we these two sheets will be equal, $S_1 = S_2$, while if $y_1 \in D$ they will be different. In any case, by the continuity of the lifts, the set

$$\widetilde{f}_1(S_1) \cap \widetilde{f}_2(S_2)$$

will be open and entirely contained in A or D. Therefore, we conclude that both A and D are open sets. But since $Y = A \sqcup D$ is connected, D must be empty reaching a contradiction. ■

Note that lifts may not exist. In the special case of paths, the next result guarantees the existence of lifts. The general case will be discussed later when we state the Lifting criterion.

Theorem 1.3.3 (Path lifting theorem). *For $p : (E, e_0) \to (X, x_0)$ as above, if σ is a path in X with initial point x_0, there is a unique path $\widetilde{\sigma}_{e_0}$ in E with initial point e_0 such that $p\widetilde{\sigma}_{e_0} = \sigma$.*

Proof. The uniqueness is guaranteed by Theorem 1.3.2. If the whole space X is evenly covered and $e_0 \in S \subseteq E$, the projection p restricted to the sheet S, $p_{|S} : S \to X$, is a homeomorphism. Consider its inverse $\psi = (p_{|S})^{-1}$ which is also a continuous map. The curve defined by $\psi \circ \sigma$ is a lift of σ with initial point e_0.
In general X may not be evenly covered, there might be more than one evenly covered chart. Since I is compact, there is a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that for all i, $\sigma([t_i, t_{i+1}])$ is contained in an evenly covered neighbourhood U_i of $\sigma(t_i)$. Using the above argument we can lift the path $\sigma_{|[0,t_1]}$ to a curve $\widetilde{\sigma}_1 : [0, t_1] \to E$ with initial point e_0. We will proceed by induction. Assume we have a lift $\widetilde{\sigma}_i$ of $\sigma_{|[0,t_i]}$ with initial point e_0. Again, since we are inside an evenly covered chart, by the same argument as before we can lift $\sigma_{|[t_i,t_{i+1}]}$ to a path mapping t_i to the endpoint of the previous curve, $\widetilde{\sigma}_i(t_i)$. Thus, $\widetilde{\sigma}_i$ can be extended to $\widetilde{\sigma}_{i+1}$ in a continuous way. In a finite number of steps we get the desired lift $\widetilde{\sigma}_{e_0} = \widetilde{\sigma}_n$. ■

Now we are going to prove a much more general result concerning homotopies, but before let us recall the definition of homotopy.

Definition 1.3.5 (Homotopy). A *homotopy* between two continuous functions $f, g : Y \to X$ is a continuous function $H : Y \times I \to X$ where $I = [0, 1]$ such that for all $y \in Y$, $H(y, 0) = f(y)$ and $H(y, 1) = g(y)$.

Theorem 1.3.4 (Covering homotopy theorem). *Let $p : (E, e_0) \to (X, x_0)$ be a covering with base points and let $f : (Y, y_0) \to (X, x_0)$ be a map which has lifting $\widetilde{f} : (Y, y_0) \to (E, e_0)$. Then, any homotopy $F : Y \times I \to X$ with $F(y, 0) = f(y)$ for all $y \in Y$ can be lifted to a homotopy $\widetilde{F} : Y \times I \to E$ with $\widetilde{F}(y, 0) = \widetilde{f}(y)$.*

Proof. If X is evenly covered, we can use the homeomorphism to lift the homotopy from X to E. Otherwise, we will proceed like in the proof of Theorem 1.3.3. By compactness of I, for every $y \in Y$ there exists an open neighbourhood N_y of y and a partition of I

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

possibly depending on y, such that $F(N_y \times [t_i, t_{i+1}])$ is contained in some evenly covered neighbourhood of $F(y, t_i)$. Since for each subinterval $[t_i, t_{i+1}]$ we are inside an evenly covered chart, we can construct a lift $\widetilde{F}_y : N_y \times I \to E$ of the homotopy F restricted to $N_y \times I$ satisfying that $\widetilde{F}_y(y', 0) = \widetilde{f}(y')$ for all $y' \in N_y$.
Finally, the lifts of two different neighbourhoods $N_{y_1} \times I$ and $N_{y_2} \times I$ must agree on the intersection

$$(N_{y_1} \times I) \cap (N_{y_2} \times I) = (N_{y_1} \cap N_{y_2}) \times I.$$

If $y_0 \in N_{y_1} \cap N_{y_2}$, then we have two lifts of $F_{|\{y_0\}\times I}$ that coincide at the initial point $(y_0, 0)$. Therefore, since $\{y_0\} \times I$ is connected, by Theorem 1.3.2 these two lifts must be the same one. ■

In fact, Theorem 1.3.3 is a particular case of this theorem taking $Y = \{*\}$. If we take $Y = I$ we obtain the following corollary.

Corollary 1.3.5 (Monodromy theorem). *If σ, τ are two homotopic paths in X with initial point x_0, then $\widetilde{\sigma}_{e_0}$ is homotopic to $\widetilde{\tau}_{e_0}$ in E. In particular, they have the same end point.*

Corollary 1.3.6. *The map $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ induced by a covering space $p : (E, e_0) \to (X, x_0)$ is a monomorphism. The image subgroup $p_*\pi_1(E, e_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to E starting at e_0 are loops.*

Proof. We will prove that the kernel of p_* is trivial. Let $\widetilde{f}_0 \in \operatorname{Ker} p_*$, this is a loop in E such that its projection $f_0 = p\widetilde{f}_0$ is a loop in X homotopic to a trivial loop f_1. Denote by $F : I \times I \to X$ the homotopy between f_0 and f_1. By Corollary 1.3.5, there is a lifted homotopy $\widetilde{F} : I \times I \to E$ such that $\widetilde{F}(t, 0) = \widetilde{f}_0(t)$ and $\widetilde{F}(t, 1) = \widetilde{f}_1$, with $\widetilde{f}_1$ the lift of the constant loop f_1 and hence trivial in E too. ■

Note that if σ is a loop at x_0, its lift $\widetilde{\sigma}_{e_0}$ needs not to be a loop in E. We only can say that its end point will belong to $p^{-1}(x_0)$.

Proposition 1.3.7. *The number of sheets of a covering space $p : (E, e_0) \to (X, x_0)$ with X and E path-connected equals the index of $p_*(\pi_1(E, e_0))$.*

You can find the proof of this result in [Hat02, Proposition 1.32]. In particular, in this situation all the fibres have the same cardinality, something that we cannot assert in general. To conclude this section let us study the existence of lifts.

Theorem 1.3.8 (Lifting criterion). *Suppose given a covering space $p : (E, e_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then, a lift $\widetilde{f} : (Y, y_0) \to (E, e_0)$ of f exists if, and only if,*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0)).$$

Proof. π_1 is a covariant functor from the category of pointed topological spaces to the category of groups. If there exists a lift $\widetilde{f}$ of f, $p\widetilde{f} = f$, then by the functoriality of π_1, $\pi_1(f) = \pi_1(p\widetilde{f}) = \pi_1(p)\pi_1(\widetilde{f})$ or

$$f_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0)))$$

and hence, since $\widetilde{f}_*(\pi_1(Y, y_0)) \subseteq \pi_1(E, e_0)$,

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0)).$$

Conversely, assuming that this property is true, we can construct a lift of f. Choose for any $y \in Y$ a path σ joining y to y_0. Then $f\sigma$ is a path from x_0 to $x = f(y)$. Define

$$\widetilde{f}(y) = \widetilde{(f\sigma)}_{e_0}(1).$$

then

$$p(\widetilde{f}(y)) = p((\widetilde{f\sigma})_{e_0})(1) = f(\sigma(1)) = f(y)$$

and thus $\widetilde{f}$ is a map lifting f. ■

Corollary 1.3.9. *In the assumptions of the theorem, if Y is simply connected, the lifting $\widetilde{f}$ always exists.*

1.4 Universal coverings

Definition 1.4.1 (Semilocally simply-connected). We say that X is *semilocally simply-connected* if each point $x \in X$ has a neighbourhood U such that the inclusion-induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

This is a necessary condition for X to have a simply-connected covering space. Observe that both simply-connected and locally simply-connected spaces are semilocally simply-connected, but none of the converses is true.

Theorem 1.4.1 (Covering classification theorem). *Let X be path-connected, locally path-connected and semilocally simply connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (E, e_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(E, e_0))$ to the covering space (E, e_0). If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : E \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

The proof of this statement can be found in [Hat02, Theorem 1.38].

Definition 1.4.2 (Universal covering). A covering space of X, $p : E \to X$, is called *universal* if E is simply-connected.

Proposition 1.4.2. *A universal cover $p : E \to X$ of a path-connected, locally path-connected space X is a covering space of every other path-connected covering space $p' : E' \to X$, i.e.*

$$\begin{tikzcd} E \arrow[r, dashed] \arrow[d, "p"'] & E' \arrow[dl, "p'"] \\ X. & \end{tikzcd}$$

It is unique up to isomorphism.

This is a consequence of Theorem 1.3.8. After this result, it makes sense to introduce an equivalence relation between covering spaces of the same base space.

Definition 1.4.3 (Equivalent covering spaces). We say that two covering spaces $p : (E, e_0) \to (X, x_0)$ and $p' : (E', e'_0) \to (X, x_0)$ are *equivalent* if there is a homeomorphism $\phi : (E', e'_0) \to (E, e_0)$ such that $p\phi = p'$.

The Classification theorem induces a partial ordering on the set of covering spaces of a given space X, being the universal covering on the top. If we go back to the initial examples, we see that the complex plane (with $\exp z$) and the helicoid S (with p_{12}) were both universal covers of the punctured plane $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{C} \setminus \{0\}$. Thus, there must be a homeomorphism between them. For this, if we set $z = x + iy$, we only have to solve the following system of equations:

$$\left\{ \begin{array}{l} e^x \cos y = s\cos(2\pi t) \\ e^x \sin y = s\sin(2\pi t) \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} s = e^x \\ 2\pi t = y. \end{array} \right.$$

So, we get a homeomorphism from the complex plane to the helicoid S

$$\begin{array}{ccccc} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 & \longleftrightarrow & S \subseteq \mathbb{R}^3 \\ (\mathrm{Re}z,\ \mathrm{Im}z) & \to & (s = e^{\mathrm{Re}z},\ t = \mathrm{Im}\, z/(2\pi)) & \to & (s\cos(2\pi t),\ s\sin(2\pi t),\ t) \\ (x = \ln s,\ y = 2\pi t) & \leftarrow & (s = \sqrt{x^2+y^2},\ t = z) & \leftarrow & (x,\ y,\ z). \end{array}$$

Theorem 1.4.3 (Existence of the universal cover). *Every path-connected, locally path-connected, and semilocally simply-connected space X has a universal covering.*

Let us give a brief sketch of the construction of the universal covering. Given a path-connected, locally path-connected, semilocally simply-connected space X with a base-point $x_0 \in X$ we can define

$$\widetilde{X} := \{[\gamma] \ : \ \gamma : I \to X,\ \gamma(0) = x_0\}$$

where $[\gamma]$ is the homotopy class of all the curves in X fixing $\gamma(0) = x_0$ and $\gamma(1)$. Consider the map $p : \widetilde{X} \to X$ given by

$$p([\gamma]) = \gamma(1).$$

We claim that $p : \widetilde{X} \to X$ is the universal covering of X. See [Hat02, p. 64-65] for a complete and detailed construction.
Now we would like to know which domains admit $\mathbb{D}$ as a universal covering. The motivation for this question is that afterwards we will use these coverings to transport the Poincaré metric to that domains.

Theorem 1.4.4 (Riemann mapping theorem). *Let U be a non-empty simply connected proper domain of the complex plane. Then there exists a bijective holomorphic mapping from U onto the open unit disk $\mathbb{D}$. Moreover, if $z_0 \in X$ is chosen and ϕ is normalized so that $\phi(z_0) = 0$ and $\phi'(z_0) > 0$, then ϕ is unique.*

Lemma 1.4.5. *If $f : U \to V$ is a bijective holomorphic map then $f^{-1} : V \to U$ is also holomorphic.*

Therefore, $\mathbb{D}$ is a holomorphic covering space of every simply connected plane domain. Using the notion of universal covering, this theorem can be generalized to Riemann surfaces, complex analytic manifolds of dimension 1, and is known as the Uniformization theorem.

Theorem 1.4.6 (Uniformization theorem). *The universal covering space $\widetilde{X}$ of an arbitrary Riemann surface X is homeomorphic, by a conformal map, to either the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$.*

The proof of this theorem is complicated and can be found in any standard reference about Riemann surfaces, for instance [FK92].
Given a universal covering, let us discuss what is the relationship between all the points in the fibre of a given point.

Definition 1.4.4 (Deck transformation). A *deck transformation* or *automorphism of a covering space* $p : E \to X$ is a homeomorphism $f : E \to E$ such that $p \circ f = p$.

The set of all deck transformations of a given covering space forms a group with composition. Every deck transformation (also known as covering transformation) permutes the points in each fibre.

Theorem 1.4.7. *If $p : E \to X$ is a universal covering map and $t, s \in E$ are such that $p(t) = p(s)$, then there exists a deck transformation $f : E \to E$ such that $f(t) = s$.*

Corollary 1.4.8. *If a deck transformation has a fixed point then it must be the identity.*

1.5 Hyperbolic metric for arbitrary domains

We want to use what we have learnt about covering spaces to put a hyperbolic metric to more general domains.

Definition 1.5.1 (Hyperbolic domain). A plane domain X is called *hyperbolic* if it has at least two boundary points.

Corollary 1.5.1. *$\mathbb{D}$ is the universal covering space of every hyperbolic plane domain X.*

This is a consequence of the Uniformization theorem. A plane domain with more than two boundary points cannot be conformally isomorphic to the Riemann sphere nor the complex plane.

Definition 1.5.2 (Hyperbolic density). Let X be a hyperbolic domain and let $\pi : \mathbb{D} \to X$ be its universal covering. The *hyperbolic density with respect to X* at a point $z \in X$ is defined by

$$\rho_X(z) = \frac{\rho_{\mathbb{D}}(t)}{|\pi'(t)|} = \rho_{\mathbb{D}}(s(z))|s'(z)| = \frac{2|s'(z)|}{1 - |s(z)|^2}$$

where s denotes a section of π.

Note that, in particular, the hyperbolic density function is a positive continuous function. We define the hyperbolic length and the hyperbolic distance on an hyperbolic domain X in the same fashion we did for $\mathbb{D}$.

Definition 1.5.3 (Hyperbolic length). Let γ be a curve in a hyperbolic domain X. We define the *hyperbolic length* of γ with respect to X as

$$l_X(\gamma) = \int_\gamma \rho_X(t)|dt|.$$

Definition 1.5.4 (Hyperbolic distance). Let X be a hyperbolic domain. If $p, q \in X$ we define its *hyperbolic distance* with respect to X as

$$\operatorname{dist}_X(p,q) = \inf_{\gamma} l_X(\gamma)$$

where the infimum is taken over all the paths γ in X joining p to q.

Theorem 1.5.2. *Suppose that g is holomorphic covering map from a hyperbolic plane domain X onto a plane domain Ω. Then g is an infinitesimal isometry, that is*

$$\rho_\Omega(g(t))|g'(t)| = \rho_X(t)$$

for all $t \in X$.

Proof. Let π be the universal covering map from $\mathbb{D}$ to X. Then the composition $g \circ \pi : \mathbb{D} \to \Omega$ a universal covering. Let $z \in \Omega$ and $s \in \mathbb{D}$ such that $\pi(s) = t$ and $g(t) = z$, then

$$\rho_\Omega(z) = \frac{\rho_\mathbb{D}(s)}{|(g \circ \pi)'(s)|} = \frac{\rho_\mathbb{D}(s)}{|g'(\pi(s))| \cdot |\pi'(x)|} = \frac{\rho_X(t)}{|g'(t)|}$$

because $\rho_X(t) = \rho(s)/|\pi'(s)|$. ∎

Corollary 1.5.3. *Let X, Ω be plane domains, X being hyperbolic. Any holomorphic covering map g from X onto Ω preserves the hyperbolic length of curves,*

$$l_X(\gamma) = l_\Omega(g(\gamma)).$$

Theorem 1.5.4. *Let π be a universal cover from $\mathbb{D}$ onto a plane domain Ω. If $z, w \in X$ and $t \in \mathbb{D}$ is any pre-image of z, then*

$$dist_\Omega(z,w) = \min\{dist_\mathbb{D}(t,s) \ : \ s \in \mathbb{D},\ \pi(s) = w\}.$$

Proof. By definition, $\operatorname{dist}_\Omega(z,w)$ is the infimum of $l_\Omega(\gamma)$ over all curves γ connecting z and w in Ω. Corollary 1.5.3 tells us that when we take the preimage of each of these curves we have the same hyperbolic length. Then,

$$\operatorname{dist}_\Omega(z,w) = \inf\{\operatorname{dist}_\mathbb{D}(t,s) \ : \ t, s \in \mathbb{D},\ \pi(t) = z,\ \pi(s) = w\}$$

and since the map π is continuous, if we fix a preimage t of z,

$$\operatorname{dist}_\Omega(z,w) = \min\{\operatorname{dist}_\mathbb{D}(t,s) \ : \ s \in \mathbb{D},\ \pi(s) = w\}.$$

∎

Proposition 1.5.5. *Every hyperbolic plane domain X endowed with its hyperbolic distance $dist_X$ is a metric space.*

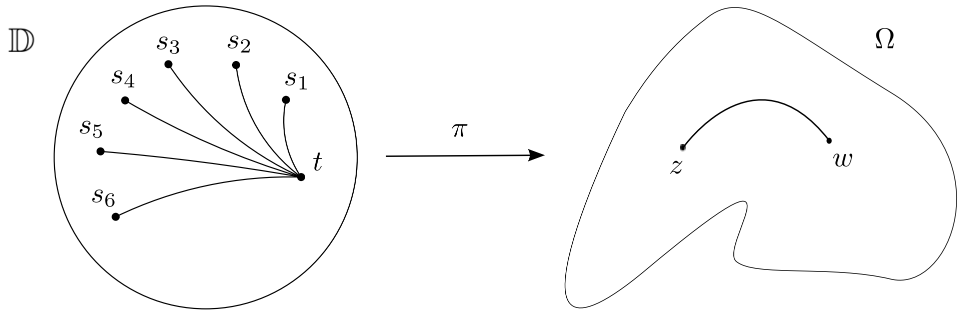

Figure 1.3: Scheme from Theorem 1.5.4.

Proof. The facts that dist_X is symmetric and satisfies the triangle inequality follow directly from the definition. We only need to show that $\text{dist}_X(z, w) = 0$ if and only if $z = w$. Assume that $z \neq w$ and let $\pi : \mathbb{D} \to X$ be a universal cover, there exist $s, t \in \mathbb{D}$ such that $\pi(s) = z$ and $\pi(t) = w$. Note that $s \neq t$. By Theorem 1.5.4,

$$\text{dist}_X(z, w) = \text{dist}_{\mathbb{D}}(t, s) > 0.$$

Conversely, if $z = w$ clearly $\text{dist}_X(z, z) = \text{dist}_{\mathbb{D}}(s, s) = 0$. Finally, as usual, the other three properties imply the non-negativity of dist_X:

$$2\text{dist}_X(x, y) = \text{dist}_X(x, y) + \text{dist}_X(x, y) \geqslant \text{dist}_X(x, x) = 0$$

thus $\text{dist}_X(x, y) \geqslant 0$. ■

It can be shown that the metric space (X, dist_X) is complete. Let us turn our attention to the hyperbolic geodesics of an arbitrary domain. We will use the projection of the universal covering to transport the geodesics of $\mathbb{D}$ to X.

Definition 1.5.5 (Hyperbolic geodesic). Let X be a hyperbolic domain and let $\pi : \mathbb{D} \to X$ be a universal covering map. A curve $\gamma \subseteq X$ is called *hyperboli geodesic* of X if and only if every lift $\pi^{-1}(\gamma)$ is a geodesic in $\mathbb{D}$.

Proposition 1.5.6. *If γ is a curve in the hyperbolic domain X such that for every $t_1 < t_2 < t_3$*

$$dist_X(\gamma(t_1), \gamma(t_3)) = dist_X(\gamma(t_1), \gamma(t_2)) + dist_X(\gamma(t_2), \gamma(t_3))$$

then γ is a geodesic on X.

Theorem 1.5.7 (Existence of geodesics). *For every two distinct points z and w in the domain X, there exists at least one shortest path γ joining z to w. Furthermore γ is a geodesic.*

Proof. Let z and w be distinct points in X. Theorem 1.5.4 guarantees the existence of two points $t, s \in \mathbb{D}$ such that $\pi(t) = z$, $\pi(s) = w$ and $\text{dist}_X(z, w) = \text{dist}_{\mathbb{D}}(s, t)$. By Lemma 1.2.1, there exists a geodesic γ joining t and s in $\mathbb{D}$. The curve $\pi(\gamma)$ connects the points z and w in X and by Theorem 1.4.7 must be a geodesic. Since covering maps preserve lengths of curves,

$$\text{dist}_X(z, w) = \text{dist}_{\mathbb{D}}(t, s) = l_{\mathbb{D}}(\gamma) = l_X(\pi(\gamma)).$$

■

Theorem 1.5.8 (Uniqueness of geodesics). *Let z and w be any two distinct points in X. Then there is a unique geodesic in every homotopy class of curves joining z and w.*

Proof. Let $z, w \in X$ with $z \neq w$ and let δ be any curve in a given homotopy class of curves joining z and w in X. Consider the universal covering $\pi : \mathbb{D} \to X$ and take $s \in \mathbb{D}$ be a point in the fibre over z. By the Path lifting theorem, Theorem 1.3.3, there exists a unique lift $\widetilde{\delta}$ of δ beginning at s. Let t be the endpoint of $\widetilde{\delta}$. The Monodromy theorem, Corollary 1.3.5, the lifts of all homotopic paths have the same endpoint, thus t is determined by the homotopy class of δ. There is a unique geodesic in $\mathbb{D}$ joining s and t and its projection is a geodesic of X in the homotopy class of δ. If there were two different geodesics γ and γ' in the same homotopy class, then Monodromy theorem tells us that their lifts would be two homotopic geodesics in $\mathbb{D}$ joining the same pair of points and hence equal. ■

Example 1.5.1. We are going to study an important example: the upper half plane

$$\mathbb{H}' := \{z \in \mathbb{C} \ : \ \operatorname{Im} z > 0\}$$

together with the Möbius transformations as isometries give the Poincaré half-plane model for the hyperbolic geometry. Using the results of this section, let us find out an expression for the hyperbolic distance on $\mathbb{H}'$. We need a conformal map $h : \mathbb{H}' \to \mathbb{D}$. Möbius transformations are determined by the image of three points. Let

$$h(z) = \frac{az + b}{cz + d},$$

we want to map $0 \mapsto -1$ ($b/d = -1$), $\infty \mapsto 1$ ($a/c = 1$) and $i \mapsto 0$ ($ai + b = 0$). Therefore, if we set $a = 1$,

$$h(z) = \frac{z - i}{z + i}.$$

Then, by definition,

$$\rho_{\mathbb{H}'}(z) = \frac{2|h'(z)|}{1 - |h(z)|^2},$$

and since

$$|h'(z)| = \frac{2}{(z + i)^2}$$

we have

$$\rho_{\mathbb{H}'}(z) = \frac{4}{|z + i|^2 - |z - i|^2}.$$

Then, using a bit of trigonometry,

$$\begin{aligned}
|z + i|^2 - |z - i|^2 &= |\operatorname{Re} z|^2 + |\operatorname{Im} z + 1|^2 - |\operatorname{Re} z|^2 - |\operatorname{Im} z - 1|^2 \\
&= |\operatorname{Im} z + 1|^2 - |\operatorname{Im} z - 1|^2 \\
&= |\operatorname{Im} z|^2 + 2\,|\operatorname{Im} z| + 1 - |\operatorname{Im} z|^2 + 2\,|\operatorname{Im} z| - 1 \\
&= 4\,|\operatorname{Im} z|
\end{aligned}$$

and hence

$$\rho_{\mathbb{H}'}(z) = \frac{4}{4\,|\operatorname{Im} z|} = \frac{1}{\operatorname{Im} z}.$$

The isometries of this model are the group of Möbius transformations leaving $\mathbb{H}'$ invariant, i.e. maps of the form

$$A(z) = \frac{az+b}{cz+d}, \ ad-bc=1, \ a,b,c,d \in \mathbb{R}.$$

Geodesics are the image under h of geodesics of $\mathbb{D}$, they are vertical lines and arcs of circles orthogonal to the real axis.

Example 1.5.2. We will be interested in a slight modification of this model, we will consider

$$\mathbb{H} = \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 0\}.$$

You can pass from one model to the other by multiplying by i or $-i$, thus

$$\rho_{\mathbb{H}}(z) = \frac{1}{|i|}\rho_{\mathbb{H}'}(iz) = \frac{1}{\operatorname{Im}(iz)} = \frac{1}{\operatorname{Re} z}.$$

Geodesics of $\mathbb{H}$ are horizontal lines and circles orthogonal to the imaginary axis.

1.6 Pick's theorem

In this section we introduce one of the most powerful tools of hyperbolic geometry. Pick's theorem, named after Georg Pick, tells us that holomorphic maps are contractions with respect to the hyperbolic metrics.

Theorem 1.6.1 (Schwarz lemma). *If f is a holomorphic self-map of $\mathbb{D}$ such that $f(0)=0$, then $|f(z)| \leqslant |z|$ and $|f'(0)| \leqslant 1$. Equality holds if and only if f is a rotation, $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$.*

Proof. Consider the function $g(z) := f(z)/z$ which is holomorphic in $\mathbb{D} \setminus \{0\}$. Since

$$\lim_{z\to 0} g(z) = \lim_{z \to 0} \frac{f(z)-f(0)}{z-0} = f'(0)$$

the origin is a removable singularity for g and hence it can be extended to a map $\widetilde{g}$ holomorphic in the whole disc. We have

$$|\widetilde{g}(z)| = |g(z)| \leqslant \frac{1}{r}$$

for all z such that $|z| = r$ and by the Maximum modulus principle this holds in $D(0,r)$. Since $r \leqslant 1$ is arbitrary, taking the limit we can conclude that

$$|\widetilde{g}(z)| = \frac{|f(z)|}{|z|} \leqslant 1$$

and hence $|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$. In particular, since

$$f'(0) = \lim_{z\to 0} g(z) = \widetilde{g}(0)$$

then $|f'(0)| \leqslant 1$. Let us prove the last claim. If f is a rotation these properties are clearly satisfied,

$$\frac{|f(z)|}{|z|} = \frac{|e^{i\theta}z|}{|z|} = \frac{|z|}{|z|} = 1, \quad |f'(0)| = |e^{i\theta}| = 1.$$

Conversely, if there is a $z_0 \in \mathbb{D}$, $z_0 \neq 0$, such that $|f(z_0)| = |z_0|$ then this means that $\widetilde{g}$ attains its maximum value in an interior point and hence, by the Maximum modulus principle, it is a constant function. Thus, $|f(z)| = |z|$ for all $z \in \mathbb{D}$, a characteristic property of rotations. On the other hand, if $|f'(0)| = 1$, then $|\widetilde{g}(0)| = |f'(0)| = 1$ and the same argument applies. ∎

The Schwarz-Pick lemma is an extension of this classical result to hyperbolic metrics.

Theorem 1.6.2 (Schwarz-Pick lemma). *If f is a holomorphic self-map of $\mathbb{D}$, then f is both an infinitesimal and a global contraction with respect to the hyperbolic metric on $\mathbb{D}$. That is,*

$$\rho_{\mathbb{D}}(f(t))|f'(t)| \leqslant \rho_{\mathbb{D}}(t)$$

for all $t \in \mathbb{D}$, and

$$dist_{\mathbb{D}}(f(z), f(w)) \leqslant dist_{\mathbb{D}}(z, w)$$

for all $z, w \in \mathbb{D}$. In particular, if f is a holomorphic self-map of $\mathbb{D}$ fixing 0 then $|f(z)| \leqslant |z|$ and $|f'(0)| \leqslant 1$.

Proof. Suppose that f is an arbitrary holomorphic function from $\mathbb{D}$ into $\mathbb{D}$. For every $t \in \mathbb{D}$, consider the self-maps of $\mathbb{D}$

$$h_t(z) := \frac{z - f(t)}{1 - \overline{f(t)}z}, \quad k_t(z) := \frac{z + t}{1 + \bar{t}z}.$$

and define $g_t(z) := h_t \circ f \circ k_t(z)$ which is another map from $\mathbb{D}$ into itself. We have

$$g(0) = h_t \circ f \circ k_t(0) = h\big(f(t)\big) = \frac{f(t) - f(t)}{1 - |f(t)|^2} = 0$$

for all $t \in \mathbb{D}$ because $|f(t)|^2 < 1$. Thus, for every value $t \in \mathbb{D}$ we can apply the Schwarz lemma to g_t and we get $|g_t'(0)| \leqslant 1$. Let us compute $g'(0)$,

$$g_t'(0) = h_t'(f(k_t(0))) \cdot f'(k_t(0)) \cdot k_t'(0)$$

we have

$$h_t'(z) = \frac{1 - |f(t)|^2}{(1 - \overline{f(t)}z)^2}, \quad k_t'(z) = \frac{1 - |t|^2}{(1 + \bar{t}z)^2}$$

and hence

$$|g_t'(0)| = \frac{(1 - |f(t)|^2) \cdot |f'(t)| \cdot (1 - |t|^2)}{(1 - |f(t)|^2)^2} = |f'(t)| \frac{1 - |t|^2}{1 - |f(t)|^2}.$$

Thus,

$$|f'(t)|\rho_{\mathbb{D}}(f(t)) = |f'(t)| \frac{2}{1 - |f(t)|^2} \leqslant \frac{2}{1 - |t|^2} = \rho_{\mathbb{D}}(t)$$

proving that it is an infinitesimal contraction. Let us see now that it is also a global contraction. Suppose that $z, w \in \mathbb{D}$ and γ is the geodesic joining them. Then

$$\operatorname{dist}_{\mathbb{D}}(f(z), f(w)) \leqslant l_{\mathbb{D}}(f(\gamma)) = \int_{f(\gamma)} \rho_{\mathbb{D}}(t)|dt| = \int_{\gamma} \rho_{\mathbb{D}}(f(t))|f'(t)||dt|.$$

Using that it is an infinitesimal contraction and that the density functions are always positive,

$$\operatorname{dist}_{\mathbb{D}}(f(z), f(w)) \leqslant \int_{\gamma} \rho_{\mathbb{D}}(t)|dt| = l_{\mathbb{D}}(\gamma) = \operatorname{dist}_{\mathbb{D}}(z, w).$$

■

Let us prove a more general version of this theorem that applies to hyperbolic metrics in arbitrary domains. It is called the Schwarz-Ahlfors-Pick theorem or simply Pick's theorem.

Theorem 1.6.3 (Pick's theorem). *If f is a holomorphic map from a domain Y into a domain X, then f is both an infinitesimal and a global contraction with respect to the corresponding hyperbolic metrics on Y and X. That is,*

$$\rho_X(f(t))|f'(t)| \leqslant \rho_Y(t)$$

for all $t \in Y$, and

$$dist_X(f(t), f(s)) \leqslant dist_Y(t, s)$$

for all $t, s \in Y$.

Proof. Consider the universal coverings $\pi_Y : \mathbb{D} \to Y$ and $\pi_X : \mathbb{D} \to X$. We are going to construct a holomorphic lift $\widetilde{f}$ of f making the diagram

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_Y"'] & \mathbb{D} \arrow[d, "\pi_X"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

commutative. Fix an arbitrary point $s \in Y$ and let $p, q \in \mathbb{D}$ be such that $\pi_Y(p) = s$ and $\pi_X(q) = f(s)$. To find the image of a point $a \in \mathbb{D}$, join it to p by a curve γ in $\mathbb{D}$, project it by π_Y, take the image under f and lift it up by π_X so that the initial point is q. We define $\widetilde{f}(a)$ to be the endpoint of the resulting curve in $\mathbb{D}$. Since all curves joining a and p are homotopic in $\mathbb{D}$, the Monodromy theorem (Corollary 1.3.5) tells us that $\widetilde{f}$ is well-defined. Note that, in particular,

$$f \circ \pi_Y(a) = \pi_X \circ \widetilde{f}(a)$$

for all $a \in \mathbb{D}$ and since f, π_Y, π_X are holomorphic, $\widetilde{f}$ is holomorphic too. Differentiating the above equality we get

$$f'(\pi_Y(a))\pi_Y'(a) = \pi_X'(\widetilde{f}(a))\widetilde{f}'(a).$$

Applying the disc version of Pick's theorem, Theorem 1.6.2, we have

$$\rho_{\mathbb{D}}(\widetilde{f}(a))|\widetilde{f}'(a)| \leqslant \rho_{\mathbb{D}}(a).$$

which can be rewritten as

$$\rho_{\mathbb{D}}(\widetilde{f}(a))\frac{\rho_{\mathbb{D}}(\widetilde{f}(a))|f'(\pi_Y(a))||\pi_Y'(a)|}{|\pi_X'(\widetilde{f}(a))|} \leqslant \rho_{\mathbb{D}}(a)$$

or equivalently

$$\rho_X(\pi_X(\widetilde{f}(a)))|f'(\pi_Y(a))| = \frac{\rho_{\mathbb{D}}(\widetilde{f}(a))}{|\pi_X'(\widetilde{f}(a))|}|f'(\pi_Y(a))| \leqslant \frac{\rho_{\mathbb{D}}(a)}{|\pi_Y'(a)|} = \rho_Y(\pi_Y(a)).$$

for all $a \in \mathbb{D}$. Using that $f\pi_Y = \pi_X\widetilde{f}$,

$$\rho_X(f(\pi_Y(a)))|f'(\pi_Y(a))| = \rho_X(\pi_X(\widetilde{f}(a)))|f'(\pi_Y(a))| \leqslant \rho_Y(\pi_Y(a))$$

Finally, since π_Y is surjective, every $t \in Y$ is equal to $\pi_Y(a)$ for some $a \in \mathbb{D}$, thus

$$\rho_X(f(t))|f'(t)| \leqslant \rho_Y(t)$$

for all $t \in Y$. To prove the global part, take two points $s, t \in Y$. Theorem 1.5.7 ensures the existence of a unique geodesic γ in Y going from s to t. Thus,

$$\operatorname{dist}_X(f(s), f(t)) \leqslant l_X(f(\gamma)) = \int_{f(\gamma)} \rho_X(\tau)|d\tau| = \int_\gamma \rho_X(f(\sigma))|f'(\sigma)||d\sigma| \leqslant$$

$$\leqslant \int_\gamma \rho_Y(\sigma)|d\sigma| = l_Y(\gamma) = \operatorname{dist}_Y(s,t).$$

■

Corollary 1.6.4. *If f is a holomorphic one-to-one map from Y onto X, then f is both an infinitesimal and a global isometry with respect to the corresponding hyperbolic metrics on Y and X. That is,*

$$\rho_X(f(t))|f'(t)| = \rho_Y(t)$$

for all $t \in Y$, and

$$dist_X(f(t), f(s)) = dist_Y(t,s)$$

for all $t, s \in Y$.

Proof. Let g be the inverse of f and set $s = f(t)$. Applying Pick's theorem to g we have

$$\rho_Y(g(s))|g'(s)| \leqslant \rho_X(s)$$

and therefore

$$\rho_X(f(t))|f'(t)| = \rho_X(s)|f'(g(s))| = \rho_X(s)\frac{1}{|g'(s)|} \geqslant \rho_Y(g(s)) = \rho_Y(t).$$

On the other hand, for all $t, s \in Y$

$$\rho_Y(t,s) = \rho_Y(g(f(t)), g(f(s))) \leqslant \rho_X(f(t), f(s)).$$

Since these are the converse inequalities in Pick's theorem we conclude that both are equalities. ■

1.7 Hyperbolic vs Euclidean distance

In this section we compare the hyperbolic and Euclidean distances. Following the discussion in [Mil06], we devote most of the section to prove the Koebe-Bieberbach quarter theorem and then we get the standard estimate as a corollary. Let us begin with a theorem that will be needed in the proof of the Bieberbach theorem, which afterwards will be used to prove the quarter theorem.

Theorem 1.7.1 (Gronwall area inequality). *Let $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ be a conformal isomorphism where K is a compact connected set and let*

$$\phi(w) = b_1 w + b_0 + \frac{b_{-1}}{w} + \frac{b_{-2}}{w^2} + O(|w^{-3}|)$$

be its Laurent expansion. Then $|b_1| \geqslant |b_{-1}|$, with equality if and only if K is a straight line segment.

You can find the proof of this theorem in [Mil06, A.4].

Theorem 1.7.2 (Bieberbach theorem). *Let $\psi : \mathbb{D} \to U$ be a conformal isomorphism with power series expansion*

$$\psi(\eta) = \sum_{n \geqslant 1} a_n \eta^n.$$

Then $|a_2| \leqslant 2|a_1|$, with equality if and only if $\mathbb{C} \setminus U$ is a closed half-line pointing towards the origin.

Proof. By composing ψ with a linear transformation, we can assume without loss of generality that $a_1 = 1$. Let $h : \mathbb{C} \setminus \overline{D} \to \mathbb{D}$ defined by $h(z) = 1/z^2$. Consider φ the map conjugated to ψ by h,

$$\begin{tikzcd} \mathbb{C} \setminus \overline{\mathbb{D}} \arrow[r, "\varphi"] \arrow[d, "h"'] & N \arrow[d, "h"] \\ \mathbb{D} \arrow[r, "\psi"'] & U \end{tikzcd}$$

where N is some neighbourhood of ∞. Then,

$$\varphi(w) = h^{-1}\psi h(w) = h^{-1}\psi\left(\frac{1}{w^2}\right) = \sqrt{\frac{1}{\psi\left(\frac{1}{w^2}\right)}} = \frac{1}{\sqrt{\psi\left(\frac{1}{w^2}\right)}}.$$

We want to obtain an expression for the Taylor expansion of φ. We have

$$\psi\left(\frac{1}{w^2}\right) = \frac{1}{w^2} + \frac{a_2}{w^4} + O(|w^{-5}|)$$

and since this has no term with a positive power of w, there cannot be such term in its square root. In general,

$$\left(\frac{b_1}{w} + \frac{b_2}{w^2} + \frac{b_3}{w^3} + O(|w^{-4}|)\right)^2 = b_1^2\frac{1}{w^2} + 2b_1b_2\frac{1}{w^3} + (2b_1b_3 + b_2^2)\frac{1}{w^4} + O(|w^{-5}|) = \psi\left(\frac{1}{w^2}\right)$$

thus,

$$b_1^2 = 1, \quad 2b_1b_2 = 0, \quad 2b_1b_3 + b_2^2 = a_2$$

and choosing $b_1 = 1$ we get $b_2 = 0$ and $b_3 = a_2/2$. Since φ is conjugated to ψ, the reciprocal of

$$\sqrt{\psi\left(\frac{1}{w^2}\right)} = \frac{1}{w} + \frac{a_2}{2}\frac{1}{w^3} + O(|w^{-4}|)$$

must be of the form

$$\varphi(w) = c_{-1}w + c_0 + \frac{c_1}{w} + O(|w^{-2}|).$$

We have to solve

$$\left(\frac{1}{w} + \frac{a_2}{2}\frac{1}{w^3} + O(|w^{-4}|)\right)\left(c_{-1}w + c_0 + \frac{c_1}{w} + O(|w^{-2}|)\right) = 1,$$

thus,

$$c_{-1} = 1, \quad c_0 = 0, \quad c_1 + \frac{a_2}{2} = 0$$

and hence

$$\varphi(w) = w - \frac{a_2}{2}\frac{1}{w} + O(|w^{-2}|).$$

By Theorem 1.7.1,

$$1 \geqslant |c_1| = \frac{|a_2|}{2} \quad \Leftrightarrow \quad |a_2| \leqslant 2$$

and we have equality if and only if N is the complement of a straight line segment K that can be assumed to contain the origin. Thus, equality holds if and only if $U = h(\mathbb{C} \setminus K)$ which is a half line pointing to the origin. ■

In fact, Ludwig Bieberbach conjectured in 1916 that $|a_n| \leqslant n|a_1|$ for all $n \geqslant 2$, having equality in the same situation as above. After many partial results due to different mathematicians, this was proved in 1984 by Louis de Branges.

Theorem 1.7.3 (Koebe-Bieberbach quarter theorem). *Let $\psi : \mathbb{D} \to U \subseteq \mathbb{C}$ be a univalent analytic function. Then*

$$\frac{1}{4}|\psi'(0)| \leqslant dist(\psi(0), \partial U) \leqslant |\psi'(0)|$$

where the first equality holds if and only if $\mathbb{C} \setminus U$ is a half-line pointing towards the origin, and the second inequality holds if and only if U is a disk centred at the origin.

Proof. Suppose by now that $\psi(0) = 0$. Then the power expansion of ψ centred at the origin is of the form

$$\psi(z) = \sum_{n \geqslant 1} a_n z^n$$

and, again, without loss of generality we can assume that $a_1 = 1$. Let $z_0 \in \partial U$ be a point of minimal distance to the origin. Consider them Möbius map

$$A(z) = \frac{z}{1 - \frac{z}{z_0}}$$

mapping z_0 to ∞. Let $\varphi = A \circ \psi$,

$$\varphi(z) = \frac{\psi(z)}{1 - \frac{\psi(z)}{z_0}} = z + \left(a_2 + \frac{1}{z_0}\right) z^2 + O(|z|^3).$$

To get the formal power expansion of φ we have to solve

$$\left(z + a_2 z^2 + O(|z^3|)\right) = \left(1 - \frac{a_1}{z_0} z - \frac{a_2}{z_0} z^2 + O(|z^3|)\right) \left(z + b_2 z^2 + O(|z^3|)\right)$$

which gives

$$b_2 = a_2 + \frac{1}{z_0}.$$

Applying Theorem 1.7.2 to ψ and φ we get respectively

$$|a_2| \leqslant 2, \quad |b_2| = \left|a_2 - \frac{1}{z_0}\right| \leqslant 2.$$

Hence, by triangle inequality

$$\left|\frac{1}{z_0}\right| = \left|a_2 + \frac{1}{z_0} - a_2\right| \leqslant \left|a_2 + \frac{1}{z_0}\right| + |a_2| \leqslant 2 + 2 = 4$$

and thus

$$\operatorname{dist}(0, \partial U) = |z_0| \geqslant \frac{1}{4}.$$

We have equality if and only if

$$|a_2| = \left|a_2 - \frac{1}{z_0}\right| = 2$$

and then by Theorem 1.7.2 this is equivalent to the fact that $\mathbb{C} \setminus U$ is a half-line pointing towards the origin.
On the other hand, assume to the contrary that $\operatorname{dist}(0, \partial U) > 1$. Then, the inverse function ψ^{-1} maps $\mathbb{D}$ into $\mathbb{D}$ and by the Schwarz lemma (Theorem 1.6.1), since

$$|(\psi^{-1})'(0)| = \frac{1}{|\psi'(0)|} = 1,$$

ψ^{-1} is a rotation. Then, ψ must be a rotation too, being $\mathbb{D} = \psi(\mathbb{D})$ but this contradicts the fact that $\operatorname{dist}(0, \partial U) > 1$. Hence, $\operatorname{dist}(0, \partial U) \leqslant 1$. By Schwarz lemma, equality holds if and only if ψ is a rotation, therefore the domain must be a round disc.
Finally, the general case follows from the argument above taking $\widetilde{\psi}(z) = \psi(z) - \psi(0)$. ∎

This theorem was conjectured in 1907 by Paul Koebe, a PhD student of Hermann Schwarz, and proved in 1914 by Ludwig Bieberbach. Using this classic result of complex analysis we can obtain both a higher and a lower estimate for the hyperbolic density.

Corollary 1.7.4 (**Standard estimate**). *If* $V \subseteq \mathbb{C}$ *is simply connected, then the Poincaré metric on* V *agrees with the metric* $|dz|/dist(z, \partial V)$ *up to a factor of two in either direction. That is*

$$\frac{1}{2dist(z, \partial V)} \leqslant \rho_V(z) \leqslant \frac{2}{dist(z, \partial V)}$$

for all $z \in V$. *The left equality holds if and only if* $\mathbb{C} \setminus V$ *is a half-line pointing towards the point* $z \in V$, *while the right equality holds if and only if* V *is a round disk centred at* z.

Proof. Let $z_0 \in V$. By the Riemann mapping theorem (Theorem 1.4.4), there is a conformal isomorphism $\varphi : V \to \mathbb{D}$ such that $\varphi(z_0) = 0$. Recall that by definition

$$\rho_V(z_0) = \frac{1}{|\varphi'(0)|} \frac{2}{1 - |0|^2} = \frac{2}{|\varphi'(0)|}.$$

Then applying Theorem 1.7.3 to φ we get

$$\frac{1}{4}|\varphi'(0)| \leqslant \mathrm{dist}\,(z_0, \partial U) \leqslant |\varphi'(0)| \quad \Leftrightarrow \quad \mathrm{dist}\,(z_0, \partial U) \leqslant |\varphi'(0)| \leqslant 4\ \mathrm{dist}\,(z_0, \partial U).$$

Combining these two facts,

$$\frac{1}{2\ \mathrm{dist}\,(z_0, \partial U)} = \frac{2}{4\ \mathrm{dist}\,(z_0, \partial U)} \leqslant \rho_V(z_0) \leqslant \frac{2}{\mathrm{dist}\,(z_0, \partial U)}$$

as we wanted to show. ■

Chapter 2

Introduction to continuum theory

In this section we review the main results of *Chapter V: The Boundary Bumping Theorems* and *Chapter VI: Existence of Non-Cut Points* from [Nad92]. The non-cut point characterization of the arc will be a key point in the proof of the main theorem.

2.1 Basic properties of continua

Definition 2.1.1 (Continuum). A *(metric) continuum* is a non-empty, compact, connected metric space. More generally, a *Hausdorff continuum* is a non-empty, compact, connected Hausdorff space. A Hausdorff continuum is said to be *non-degenerate* if it has more than one point.

Note that every metric space is Hausdorff. Therefore the notion of Hausdorff continuum is weaker, Hausdorff continua might not be continua.

Definition 2.1.2 (Irreducible continuum). Let Y be a continuum and let $A \subseteq Y$. Then, Y is said to be *irreducible about* A provided that no proper subcontinuum of Y contains A. A continuum Y is said to be *irreducible* provided that Y is irreducible about $\{p, q\}$ for some $p, q \in Y$.

Let us introduce here the notion of indecomposable continuum. For instance, this kind of continua appear in the phase space of some functions of the exponential family.

Definition 2.1.3 (Indecomposable continuum). A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. A continuum which is not decomposable is said to be *indecomposable*.

One of the most important techniques for obtaining interesting examples of continua is the use of nested intersections. Let us state a couple of results about them which will be used later on.

Proposition 2.1.1. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of compact metric spaces such that $X_i \supseteq X_{i+1}$ for each $i = 1, 2, \ldots$, and let*

$$X = \bigcap_{i=1}^{\infty} X_i.$$

If U is an open subset of X_1 such that $U \supseteq X$, then there exists $N \in \mathbb{N}$ such that $U \supseteq X_i$ for all $i \geqslant N$. In particular, if each $X_i \neq \emptyset$, then $X \neq \emptyset$ (and, clearly, compact metric).

Proof. Assume to the contrary that for all $i \in \mathbb{N}$ there exists $x_i \in X_i \setminus U \subseteq X_1 \setminus U$. Since $X_1 \setminus U$ is a compact metric space, the sequence $\{x_i\}_{i=1}^{\infty}$ has limit $p \in X_1 \setminus U$. For every $k \in \mathbb{N}$, X_k contains infinitely many x_i, $x_i \in X_k$ for all $i \geqslant k$. Therefore, the limit point p belongs to every set X_k and hence belongs to their intersection, $p \in X$. But since $p \notin U$, we get a contradiction with the assumption that $X \subseteq U$. Thus, it must exists $N \in \mathbb{N}$ such that $X_i \setminus U = \emptyset$, or in other words $X_i \subseteq U$. Observe that once this happens for some N, since $X_{i+1} \subseteq X_i$, the same must happen for all $i \geqslant N$.
Suppose now that $X = \emptyset$ and $X_i \neq \emptyset$ for all $i \in \mathbb{N}$. Then we could choose $U = \emptyset$ and from the first part there would exist $N \in \mathbb{N}$ such that $X_i = \emptyset$ for all $i \geqslant N$, contradicting our assumption. ∎

Theorem 2.1.2. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of continua such that $X_i \supseteq X_{i+1}$ for each $i \in \mathbb{N}$. Then*

$$X = \bigcap_{i=1}^{\infty} X_i$$

is a continuum.

To conclude this section we state a theorem that will be used in the proof of the first Boundary bumping theorem. You can find a proof of it in [Nad92, Theorem 5.2].

Theorem 2.1.3 (Cut wire theorem). *Let X be a compact metric space and let A, B be closed subsets of X. If no connected subset of X intersects both A and B (equivalently, no connected component of X does), then $X = X_1 \cup X_2$ where X_1 and X_2 are disjoint closed subsets of X with $A \subseteq X_1$ and $B \subseteq X_2$.*

2.2 Boundary bumping theorems

Given a topological space S and a subspace $H \subseteq S$ recall that the boundary of H with respect to S can be defined as

$$\partial H = \overline{H} \cap (\overline{S \setminus H}).$$

When we consider the boundary of a space H without specifying what is the space S that we consider the boundary relative to, we will assume that S is the largest space under consideration.

Theorem 2.2.1 (Boundary bumping theorem I). *Let X be a continuum and let U be a non-empty, proper, open subset of X. If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$ (equivalently, since $K \subseteq \overline{U}$ and U is open, $K \cap (X \setminus U) \neq \emptyset$).*

Proof. Assume, by way of contradiction, that $K \cap \partial U = \emptyset$. K and ∂U are closed subsets of $\overline{U}$ and note that no connected component of $\overline{U}$ intersects K and ∂U at the same time. Indeed, all connected components are disjoint, therefore the only connected component intersecting K is K itself and, by assumption, $K \cap \partial U = \emptyset$. Applying Theorem 2.1.3, there are $M_1, M_2 \subseteq \overline{U}$ closed and such that $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 = \overline{U}$, $K \subseteq M_1$ and $\partial U \subseteq M_2$. Let $M_3 := M_2 \cup (X \setminus U)$, a closed subset of X. Since $U \subseteq \overline{U} = M_1 \cup M_2$,

$$X = U \sqcup (X \setminus U) \subseteq M_1 \cup M_2 \cup (X \setminus U) = M_1 \cup M_3$$

and since $M_1, M_3 \subseteq X$ we have $X = M_1 \cup M_3$. Now we have

$$\emptyset \neq K \subset M_1, \quad \emptyset \neq X \setminus U \subseteq M_3$$

and hence $M_1, M_3 \neq \emptyset$. On the other hand,

$$M_1 \cap M_3 = M_1 \cap \big(M_2 \cup (X \setminus U)\big) = (M_1 \cap M_2) \cup \big(M_1 \cap (X \setminus U)\big) = M_1 \cap (X \setminus U)$$

because $M_1 \cap M_2 = \emptyset$. Then

$$M_1 \cap M_3 \subseteq \overline{U} \cap (X \setminus U) = \partial U \subseteq M_2$$

and therefore $M_1 \cap M_3 \subseteq (M_1 \cap M_2) \cap M_3 = \emptyset$. Thus, X is the union of two non-empty, disjoint closed subsets, contradicting the assumption that X was connected. Hence, $K \cap \partial U \neq \emptyset$. ■

Corollary 2.2.2. *Let X be a non-degenerate continuum. If A is a proper subcontinuum of X and U is an open subset of X such that $A \subseteq U$, then there is a subcontinuum B of X such that*

$$A \subseteq B \neq A, \quad B \subseteq U.$$

In particular, every non-degenerate continuum X contains a non-degenerate proper subcontinuum.

Proof. Let V be a proper open subset of X compactly contained in U, $\overline{V} \subseteq U$, containing A. Since A is connected, let B be the component of $\overline{V}$ containing A. We have $A \subseteq B$ and $B \subseteq \overline{V} \subseteq U$. By Theorem 2.2.1,

$$B \cap (X \setminus V) \neq \emptyset$$

and since $A \subseteq V$, this implies that $B \neq A$.
The last claims follows from this taking $A = \{p\}$ with p any point in X and U an open set such that $p \in U$ and $U \neq X$. $B \subseteq U$ is a non-degenerate proper subcontinuum of X. ■

Theorem 2.2.3 (Boundary bumping theorem II). *Let X be a continuum, and let E be a non-empty proper subset of X. If K is a component of E, then*

$$\overline{K} \cap \partial E \neq \emptyset$$

or, equivalently, since $\overline{K} \subseteq \overline{E}$, $\overline{K} \cap (\overline{X \setminus E}) = \partial K \cap \partial E \neq \emptyset$.

Proof. Suppose that $\overline{K} \cap (\overline{X \setminus E}) = \emptyset$. $\overline{K} \neq \emptyset$ by definition and since $\overline{K} \subseteq \overset{\circ}{E} = X \setminus (\overline{X \setminus E})$ and $\overline{X \setminus E} \neq \emptyset$, $\overline{K} \neq X$. Therefore, $\overline{K}$ is a proper subcontinuum of X. Since $\overset{\circ}{E}$ is open in X and $\overline{K} \subseteq \overset{\circ}{E} \subseteq E$, by Corollary 2.2.2 there is a continuum B such that

$$\overline{K} \subseteq B \neq \overline{K}, \quad B \subseteq \overset{\circ}{E}.$$

Then K is a proper subset of a connected set $B \subseteq U \subseteq E$. This contradicts the fact that K is a component of E, hence $\overline{K} \cap (\overline{X \setminus E}) \neq \emptyset$. ■

We will use this theorem in the following situation.

Corollary 2.2.4. *Let $X \subseteq \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a compact connected set containing ∞ and $E = X \cap \mathbb{C}$. Then every component of E is unbounded.*

2.3 Existence of non-cut points

Definition 2.3.1 (Non-cut point). Let S be a connected topological space, and let $p \in S$. If $S \setminus \{p\}$ is connected, then p is called a *non-cut point* of S. If $S \setminus \{p\}$ is not connected, then p is called a *cut point* of S.

If Y is a topological space, we will write

$$Y = P|Q$$

to mean that $\{P, Q\}$ is a partition of Y (i.e. $Y = P \cup Q$, $P \cap Q = \emptyset$, $P, Q \neq \emptyset$) and P, Q are both open in Y. P and Q are open in Y if and only if they are mutually separated in Y, i.e.

$$P \cap \overline{Q} = \overline{P} \cap Q = \emptyset$$

and in this situation the condition $P \cap Q = \emptyset$ becomes trivial. It is very important to remark that P and Q need not be connected and hence the expression $Y = P|Q$ may not to be unique as it is shown in the next example.

Example 2.3.1. Let $Y = [-1, 1] \cup [-1, 1]i \subseteq \mathbb{C}$. Then $1, -1, i, -i$ are the only non-cut points of this space. $Y \setminus \{0\}$ is the disjoint union of four disjoint half-open intervals I_1, I_2, I_3, I_4. For instance, we can write $Y = (I_1 \cup I_2)|(I_3 \cup I_4)$ as well as $Y = I_1|(I_2 \cup I_3 \cup I_4)$.

Proposition 2.3.1. *Let S be a connected topological space and let C be a connected subset of S such that*

$$S \setminus C = A|B.$$

Then, $A \cup C$ and $B \cup C$ are connected. Hence, if S and C are continua, $A \cup C$ and $B \cup C$ are continua.

Proof. Suppose that $A \cup C = K|L$. Since C is connected, either $C \subseteq K$ or $C \subseteq L$. Assume, for instance, that $C \subseteq K$. Then, $L = (A \cup C) \setminus K \subseteq A$. Since A and B are open in $S \setminus C$,

$$\overline{L} \cap B \subseteq \overline{A} \cap B = \emptyset, \quad L \cap \overline{B} \subseteq A \cap \overline{B} = \emptyset,$$

thus $\overline{L} \cap B = L \cap \overline{B} = \emptyset$. We claim that

$$S = L|(B \cup K)$$

contradicting the fact that S is connected. Indeed, $L, B, K \neq \emptyset$ by assumption and hence also $B \cup K \neq \emptyset$. We have

$$L \cup (B \cup K) = (K \cup L) \cup B = (C \cup A) \cup B = S$$

and L and $B \cup K$ are mutually separated in Y: since $A \cup C = K|L$, K and L are mutually separated

$$L \cap \overline{K} = \overline{L} \cap K = \emptyset$$

and hence

$$L \cap (\overline{B \cup K}) = (L \cap \overline{B}) \cup (L \cap \overline{K}) = \emptyset, \quad \overline{K} \cap (B \cup K) = (\overline{L} \cap B) \cup (\overline{L} \cap K) = \emptyset.$$

Thus, this contradiction tells us that $A \cup C$ is connected and, by symmetry, $B \cup C$ too. Finally, suppose that S and C are compact. Then C must be closed in S and therefore both A, B must be open. Thus,

$$A \cup C = S \setminus B, \quad B \cup C = S \setminus A$$

are closed subsets of a compact set S, hence they are compact and continua. ∎

Lemma 2.3.2. *Let S be a connected topological space. Assume that $x, y \in S$ are such that*

$$S \setminus \{x\} = K|L, \quad S \setminus \{y\} = M|N.$$

If $x \in M$ and $y \in K$, then $N \cup \{y\} \subseteq K$.

Proof. By Proposition 2.3.1, $N \cup \{y\}$ is connected. Since $x \in M$, $x \notin N \cup \{y\}$ and therefore either $N \cup \{y\} \subseteq K$ or $N \cup \{y\} \subseteq L$. If $y \in K$,

$$(N \cup \{y\}) \cap K \neq \emptyset$$

and hence $(N \cup \{y\}) \subseteq K$. ∎

Theorem 2.3.3 (Non-cut point existence theorem). *Let S be a non-degenerate continuum. Assume that S has a cut point c and $S \setminus \{c\} = U|V$. Then, there is a non-cut point of S in U and there is a non-cut point of S in V. Hence, S has at least two non-cut points.*

Proof. Let N denote the set of all non-cut point of S which, of course, may be empty. Suppose to the contrary that $N \subseteq V$, thus every point in U is a cut point of S. Let D be a countable dense subset of S,

$$D = \{p_n \ : \ n \in \mathbb{N}\}.$$

Define $n(1) := \min\{n \in \mathbb{N} \ : \ p_n \in U\}$. Since U is non-empty and open in S, $n(1)$ is well defined. We have $p_{n(1)} \in U$ and hence is a cut point of S, let

$$S \setminus \{p_{n(1)}\} = E_1|F_1$$

so that $c \in E_1$. Lemma 2.3.2 implies

$$F_1 \cup \{p_{n(1)}\} \subseteq U.$$

Now we can define $n(2) := \min\{n \in \mathbb{N} \ : \ p_n \in F_1\}$ and since $F_1 \subseteq U \setminus \{p_{n(1)}\}$ we have $n(2) > n(1)$ and $p_{n(2)} \in U$ is a cut point,

$$S \setminus \{p_{n(2)}\} = E_2|F_2$$

with $p_{n(1)} \in E_2$. Again, by Lemma 2.3.2 $F_2 \cup \{p_{n(2)}\} \subseteq F_1$. We can repeat this process systematically. Given a partition

$$S \setminus \{p_{n(k)}\} = E_k|F_k$$

with $F_k \cup \{p_{n(k)}\} \subseteq F_{k-1}$ we can consider

$$n(k+1) := \min\{n \in \mathbb{N} \ : \ p_n \in F_k\} > n(k)$$

and since $p_{n(k+1)} \in F_k \subseteq F_{k-1} \subseteq \cdots \subseteq F_1 \subseteq U$, it is a cut point, let

$$S \setminus \{p_{n(k+1)}\} = E_{k+1}|F_{k+1}$$

with $p_{n(k)} \in E_{k+1}$. Finally, Lemma 2.3.2 gives $F_{k+1} \cup \{p_{n(k+1)}\} \subseteq F_k$ and therefore $E_k \subseteq E_{k+1}$. At every step, Proposition 2.3.1 ensures that

$$F_{k+1} \cup \{p_{n(k+1)}\}, \ E_{k+1} \cup \{p_{n(k+1)}\}$$

are continua. By induction we get two infinite collections of subcontinua of S,

$$\{F_{k+1} \cup \{p_{n(k+1)}\}\}_{k=1}^{\infty}, \ \{E_{k+1} \cup \{p_{n(k+1)}\}\}_{k=1}^{\infty}$$

the first one being decreasing and the second one increasing. Consider

$$F := \bigcap_{k=1}^{\infty} (F_k \cup \{p_{n(k)}\})$$

which by Proposition 2.1.1 is not empty, let $p \in F$. Then, using the De Morgan laws

$$\begin{aligned} S \setminus F &= \bigcup_{k=1}^{\infty} S \setminus (F_k \cup \{p_{n(k)}\}) = \bigcup_{k=1}^{\infty} \big((S \setminus F_k) \cap (S \setminus \{p_{n(k)}\})\big) \\ &= \bigcup_{k=1}^{\infty} \big((E_k \cup \{p_{n(k)}\}) \cap (E_k \cup F_k)\big) = \bigcup_{k=1}^{\infty} E_k =: E. \end{aligned}$$

E is a increasing union of nested connected sets, therefore it must be connected. Note that the dense set D is entirely contained in E. We have

$$S \setminus \{p\} \supseteq S \setminus F = E \supseteq D.$$

Since $p \in F \subseteq U$ it must be a cut point, but if this was true we could write

$$S \setminus \{p\} = A|B$$

where A and B are open. Since E is connected, $E \subseteq A$ or $E \subseteq B$. Assume, for instance, that $E \subseteq A$, then $B \cap D = \emptyset$, contradicting the fact that D is a dense set in S. ■

The next corollary tells us that continua are irreducible about its set of non-cut points.

Corollary 2.3.4. *Let S be a non-degenerate continuum. Let N denote the set of all non-cut points of S. Then, no proper connected subset of S contains N.*

The Non-cut point existence theorem is true for Hausdorff continua but you shall use the Hausdorff maximal principle to prove it, which is equivalent to the Axiom of choice. Using this, you can also obtain a version of Corollary 2.3.4 for Hausdorff continua.

2.4 Separation ordering

Definition 2.4.1 (Separating point). Let Z be a topological space and let $p, q \in Z$ with $p \neq q$. A point $z \in Z$ is said to *separate* p and q in Z provided that $Z \setminus \{z\} = A|B$ with $p \in A$ and $q \in B$. We will denote by $S(p, q)$ the set of such points plus p and q.

Proposition 2.4.1. *Let Z be a non-degenerate continuum. Then, Z has exactly two non-cut points if and only if $Z = S(p, q)$ for some $p, q \in Z$.*

Proof. Suppose that Z has exactly two non-cut points p and q. Let $c \in Z \setminus \{p, q\}$ which must be a cut point of Z, thus $Z \setminus \{c\} = U|V$. By the Non-cut point existence theorem (Theorem 2.3.3), one of the non-cut points must belong to U and the other one to V. Hence, $c \in S(p, q)$ and $Z = S(p, q)$.
Conversely, if $Z = S(p, q)$ for some $p, q \in Z$, then the only possible non-cut points of Z are p and q. But since by Theorem 2.3.3 there must be two non-cut points in Z, p, q are cut-points of Z. ∎

Lemma 2.4.2. *Let Z be a connected topological space, and let $p, q \in Z$ with $p \neq q$. Let $x, y \in S(p, q) \setminus \{p, q\}$ with*

$$Z \setminus \{x\} = A_1|B_1 = A_2|B_2, \qquad p \in A_1 \cap A_2,\ q \in B_1 \cap B_2;$$
$$Z \setminus \{y\} = C|D, \qquad p \in C,\ q \in D.$$

Then, (1) and (2) below hold:

(1) If $y \in A_1 \cup A_2$, then $C \cup \{y\} \subseteq A_1 \cap A_2$.

(2) If $y \in B_1$, then $A_1 \cup \{x\} \subseteq C$.

Proof. Assume without loss of generality that $y \in A_1$. Since C and D are disjoint, then either $x \in C$ or $x \in D$ (or, in other words, either $x \notin D \cup \{y\}$ or $x \notin C \cup \{y\}$), thus

$$C \cup \{y\} \subseteq A_1 \cup B_1 \quad \text{or} \quad D \cup \{y\} \subseteq A_1 \cup B_1.$$

Moreover, since by Proposition 2.3.1 $C \cup \{y\}$ and $D \cup \{y\}$ are connected and $y \in A_1$ we have

$$C \cup \{y\} \subseteq A_1 \quad \text{or} \quad D \cup \{y\} \subseteq A_1$$

and since $q \in B_1 \cap B_2 \subseteq X \setminus A_1$ and $q \in D$

$$C \cup \{y\} \subseteq A_1.$$

On the other hand, since $x \notin A_1$, $x \notin C \cup \{y\}$ and hence

$$C \cup \{y\} \subseteq A_2 \cup B_2$$

but since $p \in A_2 \cap C$ and we already noted that $C \cup \{y\}$ is connected,

$$C \cup \{y\} \subseteq A_2$$

and this together with the previous inclusion gives $C \cup \{y\} \subseteq A_1 \cap A_2$. The proof is analogous if you assume from the beginning that $y \in A_2$.
To prove the second part, assume now that $y \in B_1 = Z \setminus (A_1 \cup \{x\})$. Then,

$$A_1 \cup \{x\} \subseteq C \cup D$$

and, since $A_1 \cup \{x\}$ is connected by Proposition 2.3.1,

$$A_1 \cup \{x\} \subseteq C \quad \text{or} \quad A_1 \cup \{x\} \cup D$$

but as $p \in A_1 \cap C$, only the first one can hold. ■

Corollary 2.4.3. *Let Z be a connected topological space, and let $p, q \in Z$ with $p \neq q$.*

(1) For each $x \in S(p,q) \setminus \{p,q\}$, there exist unique sets P_x and Q_x such that

$$S(p,q) \setminus \{x\} = P_x | Q_x, \quad p \in P_x,\ q \in Q_x.$$

(2) If Z is a Fréchet space (or T_1) then P_x and Q_x are open in the subspace topology for $S(p,q)$.

Furthermore, if $x, y \in S(p,q) \setminus \{p,q\}$ with $x \neq y$, then:

(3) $y \in P_x \cup Q_x$;

(4) if $y \in P_x$, then $P_y \cup \{y\} \subseteq P_x$;

(5) if $y \in Q_x$, then $P_x \cup \{x\} \subseteq P_y$ and, thus, $x \in P_y$.

Proof. Let us prove each one of these items:

(1) Suppose that $Z \setminus \{x\} = A_1 | B_1 = A_2 | B_2$. By Lemma 2.4.2 (1), given $y \in S(p,q) \setminus \{p,q\}$, if $y \in A_i$ then $y \in A_1 \cap A_2$, thus

$$A_1 \cap S(p,q) = A_2 \cap S(p,q)$$

and hence also $B_1 \cap S(p,q) = B_2 \cap S(p,q)$. Therefore the sets $P_x := A_1 \cap S(p,q)$ and $Q_x := B_1 \cap S(p,q)$ must be unique.

(2) By definition, P_x and Q_x are open in $S(p,q) \setminus \{x\}$. Since Z is Fréchet, every single point is a closed subset and hence $S(p,q) \setminus \{x\}$ is open in $S(p,q)$. Hence, P_x and Q_x are open in $S(p,q)$.

(3) It is just because $x \neq y$.

(4) This is a direct consequence of Lemma 2.4.2 (1). Using the notation from there, if $y \in P_x = A_1 \cap S(p,q)$, then

$$P_y \cup \{y\} = \big(C \cap S(p,q)\big) \cup \{y\} = (C \cup \{y\}) \cap S(p,q) \subseteq (A_1 \cap A_2) \cap S(p,q) = P_x.$$

(5) This follows from Lemma 2.4.2 (2). If $y \in Q_x = B_1 \cap S(p,q)$,

$$P_x \cup \{x\} = \big(A_1 \cap S(p,q)\big) \cup \{x\} = (A_1 \cup \{x\}) \cap S(p,q) \subseteq C \cap S(p,q) = P_y.$$

■

Definition 2.4.2 (**Separation ordering**). Let Z be a connected topological space and let $p, q \in Z$ with $p \neq q$. We denote by $\prec_s$ the *separation ordering* for $S(p,q)$ defined by:

- for any $z \in S(p,q) \setminus \{p\}$, $p \prec_s z$;
- for any $z \in S(p,q) \setminus \{q\}$, $z \prec_s q$;
- for any $x, y \in S(p,q) \setminus \{p,q\}$,

$$x \prec_s y \quad \Leftrightarrow \quad x \in P_y$$

  where P_y is such that $S(p,q) \setminus \{y\} = P_y | Q_y$ and $q \in Q_y$.

Let us recall what are the properties that must satisfy a simple ordering. In Lemma 2.4.5 we check that indeed $\prec_s$ is a simple ordering on $S(p,q)$.

Definition 2.4.3 (**Simple ordering**). A binary relation $\prec$ for a set Y is called a *simple ordering* (or *strict total ordering*) provided that

(i) $\prec$ is irreflexive: no element is related to itself, $x \prec y \Rightarrow x \neq y$;

(ii) $\prec$ is transitive: if $x \prec y$ and $y \prec z$ then $x \prec z$;

(iii) $\prec$ is total: if $x \neq y$ then either $x \prec y$ or $y \prec x$.

Lemma 2.4.4. *A totally ordered set X with the order topology is a completely normal Hausdorff space (usually called T_5 or completely T_4), this is a completely normal space which is Fréchet. In particular, it is a Hausdorff space.*

Lemma 2.4.5. *The separation ordering $\prec_s$ for $S(p,q)$ is a simple ordering.*

Proof. Let us verify that $\prec_s$ satisfies the three properties above:

(i) Irreflexivity. By definition, $p \not\prec_s p$ and $q \not\prec_s q$. If $z \in S(p,q) \setminus \{p,q\}$, by Corollary 2.4.3 (1), $x \notin P_x$ and thus $z \not\prec_s z$.

(ii) Transitivity. Let $x \prec_s y$ and $y \prec_s z$. If $x = p$ or $z = q$ it is clear because they are respectively smaller and greater than any other point. Otherwise, we have

$$x \in P_y \subseteq P_y \cup \{y\} \subseteq P_z$$

by Corollary 2.4.3 (4). Hence $x \in P_z$ and $x \prec_x z$.

(iii) Totality. By definition, the extrempoints p and q are comparable to any other point. Since $x \neq y$, as remarked in Corollary 2.4.3 (3), $y \in P_x \cup Q_x$. If $y \in P_x$ then $y \prec_x x$ and we are done. Otherwise, if $y \in Q_x$ then, by Corollary 2.4.3 (5), $x \in P_y$ and therefore $x \prec_s y$.

Thus, $\prec_s$ is a strict total ordering on $S(p,q)$. ■

We have defined a total ordering on $S(p,q)$ using only its topological properties. Conversely, every totally ordered set admits a topology coming from the order. This notion generalizes the Euclidean topology of $\mathbb{R}$.

Definition 2.4.4 (Order topology). Given a totally ordered set $(X, \prec)$, the open intervals

$$(a,b)_\prec := \{x \in X \ : \ a \prec b \prec c\}$$

form a basis of the *order topology* in X. Equivalently, the set of all unbounded open intervals

$$(a,\infty)_\prec := \{x \in X \ : \ a \prec x\}, \quad (-\infty, b)_\prec := \{x \in X \ : \ x \prec b\}$$

form a subbasis of this topology.

Proposition 2.4.6 (Separation order topology vs subspace topology). *If Z is Fréchet, the separation order topology for $S(p,q)$ is contained in the subspace topology for $S(p,q)$.*

Proof. Let T_S be the subspace topology for $S(p,q)$ induced by the one of Z. For every $x \in S(p,q)$ consider

$$U_x := (-\infty, x)_{\prec_s} = \{y \in S(p,q) \ : \ y \prec_s x\}, \quad V_x := (x, \infty)_{\prec_s} = \{y \in S(p,q) \ : \ x \prec_s y\}.$$

Since these sets form a subbasis of the topology of $S(p,q)$, it is enough to check it for them. By irreflexivity, no point is related to itself and, in particular, $p \not\prec_s p$ and $q \not\prec_s q$. By (1) in the definition of separation ordering, every point $y \in S(p,q) \setminus \{p\}$ satisfies $p \prec_s y$ therefore $V_p = S(p,q) \setminus \{p\}$. If $y \prec_s p$ then by transitivity we would have $p \prec_s p$ which is not possible by irreflexivity, hence $U_p = \emptyset$. Similarly, by (2) every point $y \in S(p,q) \setminus \{q\}$ satisfies $y \prec_s q$, $U_q = S(p,q) \setminus \{q\}$, and $q \not\prec_s y$ in order not to contradict irreflexivity, $V_q = \emptyset$. Thus, $U_p = V_q = \emptyset \in T_S$ and, since Z is Fréchet, $U_q, V_p \in T_S$ too.
On the other hand, if $z \in S(p,q) \setminus \{p,q\}$ by (3)

$$U_z \setminus \{p,q\} = P_z \setminus \{p,q\}.$$

Like before, $p \in U_z$ and $q \notin U_z$ by definition and, by (1) of Corollary 2.4.3, $p \in P_z$ and $q \notin P_z$. We have $U_z = P_z$ and (2) of the same corollary tells us that P_z and Q_z are both open in the subspace topology. Hence, $U_z \in T_S$. Since $S(p,q) \setminus \{z\} = P_z | Q_z$ and the separation ordering is total,

$$V_z \setminus \{p,q\} = Q_z \setminus \{p,q\}.$$

Similarly, $p \notin V_z$ and $q \in V_z$ by definition and, by (1) of Corollary 2.4.3, $q \in Q_z$ and $p \notin Q_z$ because Q_z is disjoint of P_z and $p \in P_z$. ■

2.5 Non-cut point characterization of the arc

The goal of this section is to prove an order characterization of the arc. In the next chapter, we will define an ordering for some components of the Julia set in terms of the dynamics of the function and then thanks to this we will be able to ensure that this components are arcs.

Theorem 2.5.1. *Let Z be a non-degenerate continuum. If Z has exactly two non-cut points, then $Z = S(p,q)$ for some $p,q \in Z$ and the topology of Z is equal to the separation order topology in Z. Conversely, if Z is endowed with the order topology then Z has exactly two non-cut points.*

Proof. Let T be the topology of Z. Suppose that Z has exactly two non-cut points. Then, by Proposition 2.4.1, $Z = S(p,q)$ for some $p,q \in Z$ and, by Proposition 2.4.6, if T' denotes the separation order topology of Z then $T' \subseteq T$. Therefore, the identity map from (Z,T) to (Z,T') is continuous. Since (Z,T) is a compact space and by Corollary 2.4.3 (2) (Z,T) is Hausdorff, the identity provides a homeomorphism between (Z,T) and (Z,T') and hence both topologies are equal.
Conversely, let T be the order topology coming from some total order $\prec$ on Z. Since Z is compact, there exist $p,q \in Z$ such that $p \prec z$ for all $z \in Z \setminus \{p\}$ and $z \prec z$ for all $z \in Z \setminus \{q\}$. Explicitly, consider a cover $\mathcal{A}$ of Z by open sets of the form

$$U_a := \{z \in Z \ : \ a \prec z\}$$

and since Z is compact there will be a finite subcover of $\mathcal{A}$, take p to be the minimum of the finite number of values of a from the subcover. You can construct q in an analogue way. Since T is the order topology, every $z \in Z \setminus \{p,q\}$ is a cut point, namely

$$Z \setminus \{z\} = (-\infty, z)_{\prec} | (z, \infty)_{\prec}$$

and hence, the only possible non-cut points are p and q. By Theorem 2.3.3, Z must have at least two non-cut points, thus p and q must be cut points of Z. ■

We would like to remark that Proposition 2.4.1 and Theorem 2.5.1 can also be stated for Hausdorff continua if you use the more general version of the Non-cut point existence theorem which requires the use of the Axiom of choice in its proof. But we will apply this to subsets of $\mathbb{C}$, which is a metric space.

Theorem 2.5.2 (Non-cut point characterization of the arc). *A continuum X is an arc if and only if X has exactly two non-cut points.*

Proof. Suppose that a continuum X has two non-cup points. Then, by Theorem 2.5.1, $X = S(p,q)$ for some $p,q \in X$ and has the topology induced by the separation ordering $\prec_s$. Let C be a countable dense subset of $X \setminus \{p,q\}$,

$$C = \{c_i \ : \ n \geqslant 1\}$$

and let D be the set of all dyadic rationals in $(0,1)$,

$$D = \{k/2^m \ : \ k < 2^m, \ k \geqslant 1, \ m \geqslant 1\} \subseteq (0,1) \cap \mathbb{Q}.$$

We are going to construct an order isomorphism between C and D. Let $f(c_1) = 1/2$. Since C is a dense set, there are $n(1,1), n(1,2) \in \mathbb{N}$ such that

$$p \prec_s c_{n(1,1)} \prec_s c_1 \prec_s c_{n(1,2)} \prec_s q.$$

Note that $S(p,c_1)$ and $S(c_1,q)$ are open subsets with the separation topology of $S(p,q)$. Let $f(c_{n(1,1)}) = 1/4$ and $f(c_{n(1,2)}) = 3/4$. In the next step we can find four middle points and assign them to $1/8,\ 3/8,\ 5/8$ and $7/8$. Observe that this covers all the points with denominator 2^3 because $2/8 = 1/4$, $4/8 = 1/2$ and $6/8 = 3/4$ were already assigned. We can proceed inductively. In the kth step we will have constructed

$$\sum_{k=0}^{k-1} 2^k = \frac{1-2^k}{1-2} = 2^k - 1$$

points. Let us call them $c_\sigma(k)$, $k \in \{1, \dots, 2^k - 1\}$. By the same argument before, we can find 2^k middle points between them

$$p \prec_s c_{n(k,1)} \prec_s c_{\sigma(1)} \prec_s \cdots \prec_s c_{\sigma(2^k-1)} \prec_s c_{n(k,2^k)} \prec_s q.$$

and assign them to the points in D of the form

$$\frac{a}{2^{k+1}}, \quad \gcd(a, 2^{k+1}) = 1.$$

Now we have defined the image of $2^k - 1 + 2^k = 2^{k+1} - 1$ points. In the limit, all points in C have an image in D. f is an order isomorphism,

$$c_i \prec_s c_j \quad \Rightarrow \quad f(c_i) < f(c_j).$$

This is satisfied by construction, if a point lies between two points then its image will be defined between the images of these points. Define now a function $h : S(p,q) \to I$ extending f: $h(p) := 0$, $h(q) := 1$, $h(c) = f(c)$ if $c \in C$ and

$$h(x) := \sup\{f(c_i) \ : \ c_i \in C,\ c_i \prec_s x\} = \inf\{f(c_i) \ : \ c_i \in C,\ x \prec_s c_i\}$$

for all $x \in X \setminus (C \cup \{p,q\})$. Since every point in X can be approximated by points in C and every point in I can be approximated by points in D we claim that h is a homeomorphism from X to I. Furthermore, h is order-preserving: let $z, y \in X \setminus (C \cup \{p,q\})$ such that $z \prec_s y$, then there is $w \in C$ such that $z \prec_s w \prec_s y$, thus

$$h(z) = \inf\{f(c_i) \ : \ c_i \in C,\ z \prec_s c_i\} \leqslant f(w) \leqslant \sup\{f(c_i) \ : \ c_i \in C,\ c_i \prec_s y\} = f(y)$$

and the claim follows from the injectivity of h.
Conversely, it is clear that an arc has exactly two non-cut points. If h is a homeomorphism from I to X then $h(0)$ and $h(1)$ must be non-cut points and every other $h(x)$ must be a cut point for $x \in (0,1)$. ■

Putting these two theorems together we obtain a characterization of the arc in terms of its ordering.

Corollary 2.5.3 (Order characterization of the arc). *Let X be a continuum. Suppose that there is a total ordering $\prec$ on X such that the order topology of $(X, \prec)$ agrees with the metric topology of X. Then either X consists of a single point or there is an order-preserving homeomorphism from X to the unit interval.*

Chapter 3

Introduction to transcendental dynamics

Complex dynamics studies the iteration of holomorphic functions on some domain of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In this chapter we introduce the basic notion of this area. For a complete proof of all these facts we refer to [Mil06] for the rational case and to[Sch10] for the entire transcendental case.

3.1 Iteration of holomorphic functions

By *iteration* of a point $z_0 \in \widehat{\mathbb{C}}$ by a holomorphic function f we mean the image of z_0 under the self-composition of the function f, $f^n = f \circ \overset{n}{\cdots} \circ f$. The point z_0 is usually called the *seed* of the iteration.

Definition 3.1.1 (Orbit). Given a point $z \in \Omega \subseteq \widehat{\mathbb{C}}$ and a function f analytic on Ω, we can consider the sequence

$$O^+(z_0) := \{f^n(z_0)\}_{n \geqslant 0}$$

called the *positive orbit of* z_0 and the set

$$\mathcal{O}(z_0) := \{z \in \widehat{\mathbb{C}} \ : \ \exists m, n \in \mathbb{N}, \ f^m(z_0) = f^n(z)\}$$

called the *grand orbit of* z_0.

Note that grand orbits are completely invariant sets, $f(\mathcal{O}(z_0)) = \mathcal{O}(z_0)$. The goal of complex dynamics is to understand what is the structure and the behaviour of these sets for every analytic function. Let us introduce now a very special kind of orbits.

Definition 3.1.2 (Periodic orbit). Let $z_0 \in \widehat{\mathbb{C}}$ and f be a holomorphic function. We say that z_0 is a *periodic point* under f if there is $k \in \mathbb{N}$ such that $f^k(z_0) = z_0$. The *period* of a point is the minimum integer with this property. If the period is $k = 1$, then we say that z_0 is a *fixed point*.

Observe that if z_0 is a p-periodic point of f then it is a fixed point of the map f^p.

Definition 3.1.3 (**Preperiodic point**). A point $z_0 \in \widehat{\mathbb{C}}$ is said to be preperiodic under f if there is some $n \in \mathbb{N}$ such that $f^n(z_0)$ is a periodic point, i.e.

$$f^{k+n}(z_0) = f^n(z_0)$$

for some $k \in \mathbb{N}$.

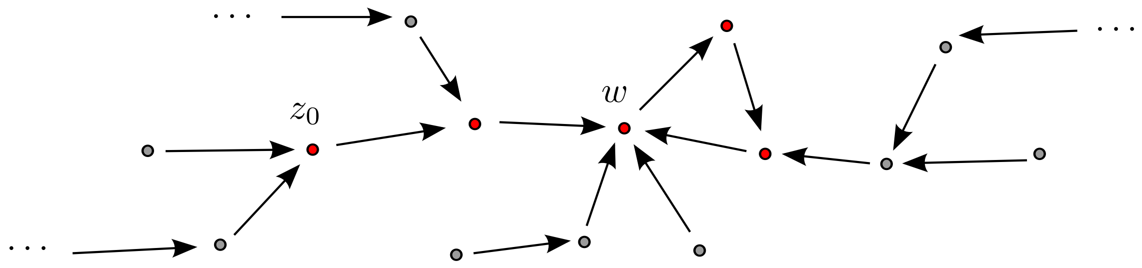

Figure 3.1: Scheme of the grand orbit of a preperiodic point z_0. Observe that w has period 3. The red points are the positive orbit of z_0.

As a curiosity, let us say that a transcendental entire map does not need to have a fixed point (e.g. $f(z) = e^z + z \neq z$ for every $z \in \mathbb{C}$) but f^2 has to have a fixed point. Berweiler showed in [Ber91] that entire transcendental functions have infinitely many periodic points of every period $n \geqslant 2$, proving a conjecture of Baker. Now we want to study the behaviour of the points nearby a periodic orbit.

Definition 3.1.4 (**Stability of a periodic orbit**). Let z_0 be a p-periodic point of a function f and denote $z_k := f^k(z_0)$, $k = 1, \dots, p-1$. Consider

$$\lambda_f(z_0) := (f^p)'(z_0) = f'(z_0) \cdot f'(z_1) \cdots f'(z_{p-1})$$

the *multiplier* of the orbit of z_0 under f, then

- if $\lambda_f(z_0) = 0$ we say that z_0 is a *superattracting* periodic point;
- if $|\lambda_f(z_0)| < 1$ we say that z_0 is an *attracting* periodic point;
- if $|\lambda_f(z_0)| > 1$ we say that z_0 is a *repelling* periodic point.

Every (super)attracting periodic point z_0 is equipped with a neighbourhood of points that converge to it under iteration by f. Similarly, every repelling point has a neighbourhood of points that are mapped outside of it eventually.

Definition 3.1.5 (**Basin of attraction**). Let z_0 be a (super)attracting periodic point of a function f. The *attracting basin* of z_0 is defined as

$$A_f(z_0) := \{z \in \widehat{\mathbb{C}} \ : \ f^n(z) \to z_0\}.$$

Note that it may not be a connected set. We denote by $A_f^*(z_0)$ the connected component of $A_f(z_0)$ containing z_0 and we call it the *immediate attracting basin* of z_0.

Definition 3.1.6 (Parabolic point). Let z_0 be a p-periodic point for some function f. Then we say that z_0 is a *parabolic point* if

$$\lambda_f(z_0) = e^{2\pi i p/q}$$

for some $p, q \in \mathbb{Z}$ such that $(p, q) = 1$. We call *parabolic basin of attraction* to the set of points converging to a parabolic point.

The major difference between a regular attracting basin and a parabolic basin of attraction is that if the periodic point is parabolic then it is located in the boundary of the basin while otherwise it lies in its interior.
Observe that for a polynomial ∞ is always a superattracting fixed point, therefore we always have an attracting basin $A_P(\infty)$ in the phase space. If you consider a rational function, still analytic on $\widehat{\mathbb{C}}$, ∞ is no longer a special point: it has a well-defined image and some preimages (poles) as well. Conversely, there is a substantial increase of difficulty when you study *entire transcendental functions*, i.e. entire functions for which ∞ is an essential singularity.

Definition 3.1.7 (Essential singularity). Let $U \subseteq \widehat{\mathbb{C}}$ be a domain with $\alpha \in U$ and let $f : U \setminus \{\alpha\} \to \widehat{\mathbb{C}}$ be a holomorphic function. Then α is said to be an *essential singularity* if the Laurent series expansion of f arround α has infinitely many terms with negative powers of $(z - \alpha)$.

Lemma 3.1.1. *A point α is an essential singularity for f if and only if the limit of $f(z)$ as $z \to \alpha$ does not exitst.*

Definition 3.1.8 (Transcendental function). A function is said to be transcendental if it has at least one essential singularity.

Example 3.1.1. The point at ∞ is an essential singularity for the exponential map or, equivalently, 0 is an essential singularity for $\exp(1/z)$. Observe that

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

is the Laurent series expansion of $\exp(1/z)$ around the origin. On the other hand, the directional real limits

$$\lim_{x\to 0^-} e^{1/x} = \lim_{y\to -\infty} e^y = 0, \quad \lim_{x\to 0^+} e^{1/x} = \lim_{y\to +\infty} e^y = +\infty$$

show that the limit of $\exp(1/z)$ at $z = 0$ does not exist. Other examples of entire transcendental functions are $\sin(z)$ and $\cos(z)$.

The following theorem illustrates the chaos in the dynamics of a function introduced by an essential singularity.

Theorem 3.1.2 (Great Picard theorem). *If an analytic function f has an essential singularity at a point w, then on any open set containing w, the function f takes on all possible complex values, with at most a single exception, infinitely often.*

Therefore, if f is an entire transcendental function, when you map close to ∞ in the next iterate you can be mapped everywhere. This injects a lot of chaos to the dynamics of the function and makes it more interesting.

3.2 Domains of normality

All the points in the attracting basin of some periodic point behave in a similar way. In this sense we say that they have a stable behaviour. On the other hand, the points in the boundary of an attracting basin behave in a more chaotic way. To formalize this, let us introduce the notion of normality.

Definition 3.2.1 (Normality). Let $\mathcal{F}$ be a family of analytic functions on some domain Ω. We say that $\mathcal{F}$ is a *normal family* (in the sense of Montel) if every infinite sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

Definition 3.2.2 (Fatou and Julia sets). Let f be an analytic self-map of a domain $D \subseteq \widehat{\mathbb{C}}$. We call the *Fatou set* of f, denoted by $F(f)$, to the set of points that have a neighbourhood $U \subseteq D$ where the family of iterates $\mathcal{F} = \{f^n_{|U}\}_n$ is normal in the sense of Montel. The complement $J(f) = D \setminus F(f)$ is called the *Julia set* of f and has chaotic behaviour.

Throughout this chapter and the next one the domain D will equal $\mathbb{C}$ or $\widehat{\mathbb{C}}$ but we have stated the definition in a more general context to fit the setting $D = \mathbb{C}^*$ in the last chapter as well.

Theorem 3.2.1 (Montel's theorem). *If $U \subseteq \mathbb{C}$ and there are $a, b, c \in \widehat{\mathbb{C}}$ pairwise distinct such that $f_n : U \to \widehat{\mathbb{C}} \setminus \{a, b, c\}$ for all $n \in \mathbb{N}$, then $\{f^n_{|U}\}_n$ is a normal family.*

Corollary 3.2.2. *If $z \in J(f)$ and U is any neighbourhood of z, then*

$$\bigcup_{n \in \mathbb{N}} f^n(U)$$

covers the whole Riemann sphere $\widehat{\mathbb{C}}$ with the exception of, at most, two points.

Lemma 3.2.3 (Characterization of the Julia set). *Let f be an entire function. Then $J(f)$ is the closure of the set of repelling periodic points of f.*

Proposition 3.2.4 (Properties of the Fatou and Julia sets). *Let f be an entire transcendental function or a polynomial of degree greater than 2. Then,*

- *$J(f)$ and $F(f)$ are forward invariant;*
- *$J(f)$ is closed and $F(f)$ is open;*
- *$J(f)$ is non-empty, unbounded and has not isolated points;*
- *for all $n \geqslant 1$, $J(f^n) = J(f)$ and $F(f^n) = F(f)$.*

Lemma 3.2.5 (Filled Julia set). *Let P be a polynomial. Then $K(P) := \mathbb{C} \setminus A_P(\infty)$ is called the filled Julia set and $J(P) = \partial K(P)$.*

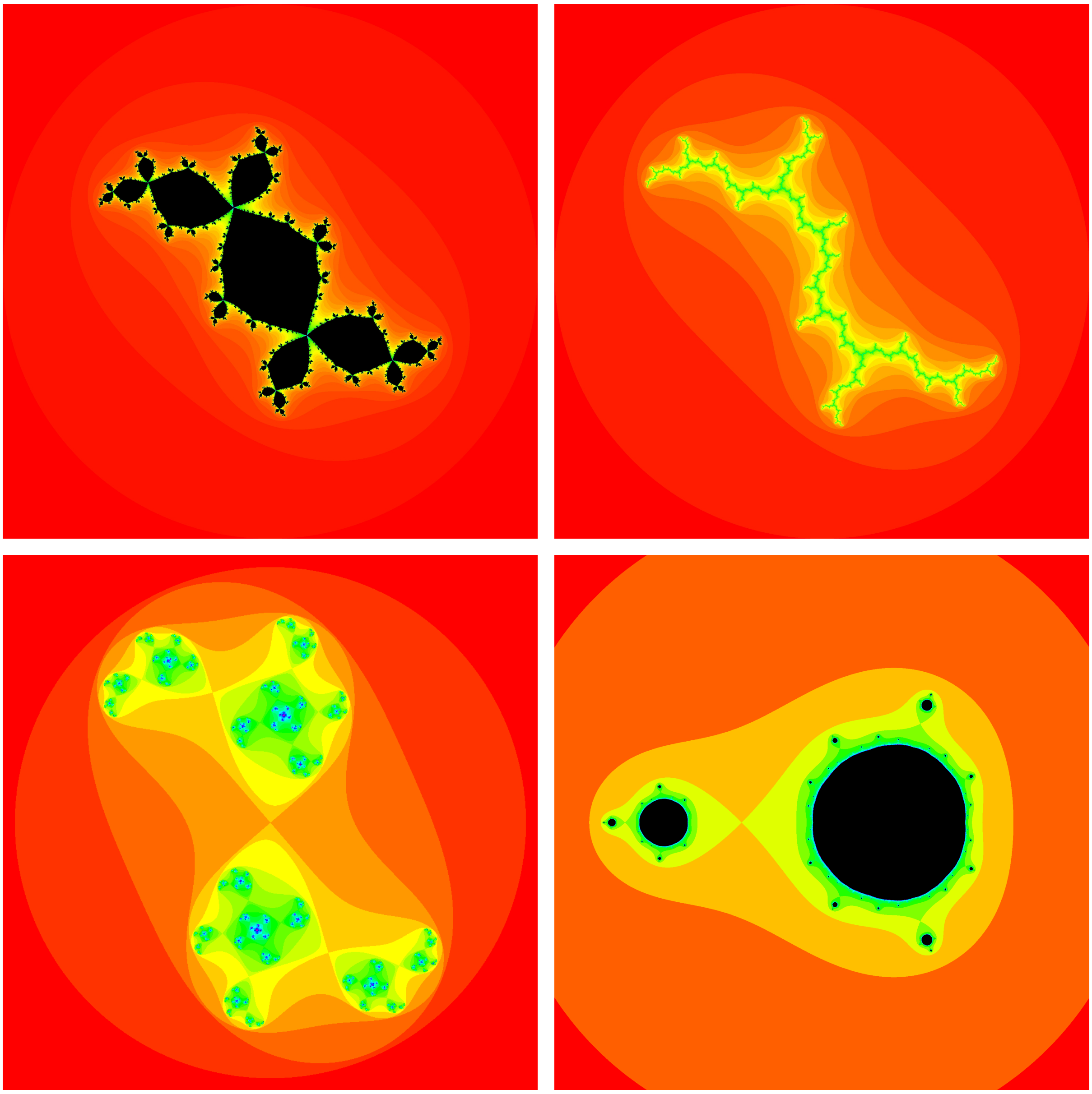

Figure 3.2: Polynomial Julia sets. The colors indicate the number of iterates needed to escape some bound. In black, the filled Julia set $K(P)$. **TL:** $P(z) = z^2 - 0.12 + 0.74i$ (the *Douady rabbit*); **TR:** $P(z) = z^2 + i$ (dendrite); **BL:** $P(z) = z^2 + 0.486 + 0.54i$ (Cantor dust); **BR:** $P(z) = z^3 - 1.08z - 0.161$ (a disconnected Julia set). $z \in [-2, 2] + i[-2, 2]$.

Figure 3.2 shows four examples of polynomial Julia sets which are different from the topological point of view. In the first one, 0 belongs to a superattracting 3-cycle and the Julia set is the boundary between $A_P(\infty)$ and $A_P(0)$. On the right hand side, $F(P) = A_P(\infty)$ and hence $J(P) = K(P)$. When this happens, we usually say that $J(P)$ is a *dendrite*. Downstairs we have two disconnected Julia sets. On the left, we have what is called a *Cantor dust*, i.e. a totally disconnected, compact, perfect set. On the right, the Julia set is disconnected but not totally disconnected. This last case cannot occur for polynomials of degree two because it requires to have two critical points, one of them escaping to ∞ and the other one being periodic, we will discuss this in the next section.

3.3 Classification of the Fatou components

Now we focus in the study of the Fatou set. We want to understand what are the possible dynamics for the connected components of $F(f)$ for any entire function f. We have already seen two examples of such components: attracting basins and parabolic basins of attraction are Fatou components.

Definition 3.3.1 (Siegel disc). A Siegel disc (or rotation disc) is an open set conformally equivalent to $\mathbb{D}$ containing a fixed point z_0 and such that the dynamics there are conformally conjugated to an irrational rotation $R_\theta(z) = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, on $\mathbb{D}$.

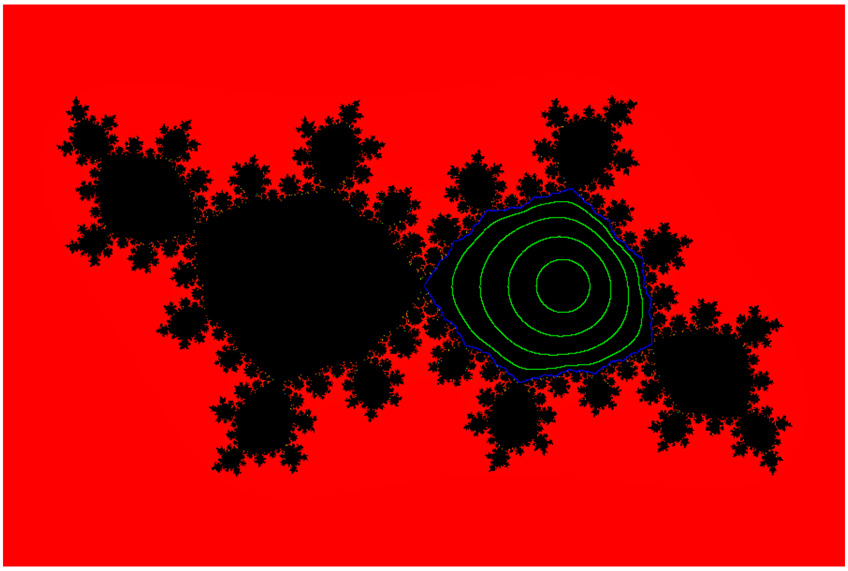

Figure 3.3: Siegel disc. Phase space of the function $P_\lambda(z) = \lambda z(1+z)$ with $\lambda = e^{2\pi i\theta}$ and $\theta = \frac{\sqrt{5}-1}{2}$, $z \in [-2,1] + i[-1,1]$. The origin is a fixed point with multiplier λ. The boundary of the Siegel disc has been drawn in blue and there are some invariant curves in green containing the points $-0.1, -0.2, -0.3, -0.4$. All the other components of the Fatou set are preimages of it.

Definition 3.3.2 (Herman ring). A Herman ring (or rotation ring) is an open set conformally equivalent to an anulus A such that the dynamics there are conformally conjugated to an irrational rotation $R_\theta(z) = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, on A.

Lemma 3.3.1. *Every Herman ring requires the existence of a pole inside the bounded component of its complement. Thus, entire maps cannot have Herman rings.*

Proof. Let γ be an invariant curve inside the Herman ring. On γ the dynamics are conjugated to an irrational rotation. Thus, for every point $z_0 \in \gamma$,

$$\sup_{n\in\mathbb{N}} |f^n(z_0)| \leqslant \max_{z\in\gamma} |z| =: M.$$

Call U the bounded component defined by γ. U contains the inner boundary of the Herman ring, therefore it contains points of the Julia set. Let $V \subseteq U$ be a neighbourhood

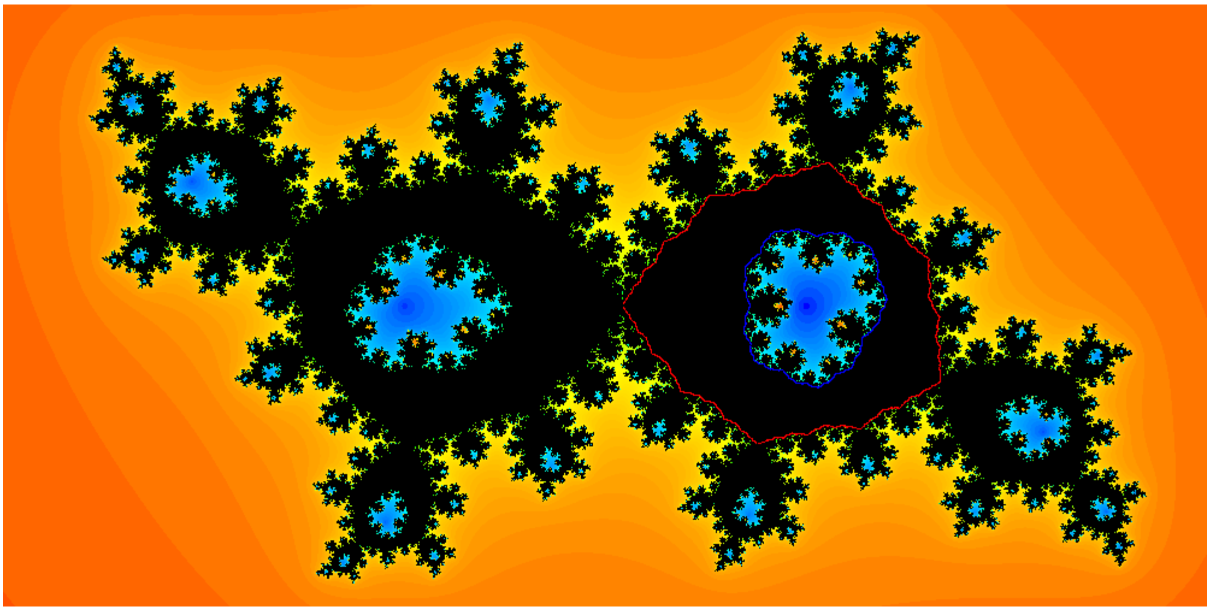

Figure 3.4: Herman ring. Phase space of the function $f(z) = e^{2\pi it} z^2 \frac{z-4}{1-4z}$ with $t = 0.6151732$ and $z \in [-4, 8] + i[-3, 3]$. In orange, $A_f(\infty)$ and, in blue, $A_f(4)$. The boundaries of the Herman ring have been drawn in blue and red. All the other Fatou components are preimages of it.

of one of these points. By Corollary 3.2.2, the iterates of V must cover the whole $\widehat{\mathbb{C}}$ with the exception of, at most, two points. Therefore, there is some point in $z_0 \in V$ such that $|f^n(z_0)| > M$ for some $n \in \mathbb{N}$. Applying the Maximum modulus principle to f^n on U, since the maximum of $|f^n(z)|$ lies not in $\gamma = \partial U$ but inside U and f^n is not constant, we conclude that f^n cannot be holomorphic, it must have a pole in U. Hence, there is a pole of f in $f^m(U)$ for some $m < n$. ∎

These sets are named after Carl Ludwig Siegel (1896–1981) and Michael Robert Herman (1942–2000). In 1918, Pierre Fatou proved the following classification theorem for the periodic Fatou components of a rational function.

Theorem 3.3.2 (**Fatou classification theorem**). *Let f be a rational function. If U is a periodic component of $F(f)$ then either*

- *U is a (super)attracting basin, or*
- *U is a parabolic basin of attraction, or*
- *U is a Siegel disc, or*
- *U is a Herman ring.*

The proof of this theorem is out of the scope of this project, you can find it in [Mil06, Theorem 13.1]. A priori, some components of the Fatou set may not be periodic nor preperiodic, this type of components are called wandering domains.

Definition 3.3.3 (**Wandering domain**). A wandering domain is a domain U such that $f^n(U) \cap f^m(U) = \emptyset$ for all $n, m \in \mathbb{N}$, $n \neq m$.

In 1985, Dennis Sullivan discarted the possibility of existence of wandering domains for rational functions, thus completing the classification of the Fatou components for rational functions.

Theorem 3.3.3 (**No wandering domain theorem**). *Let f be a rational function. Then every component of the Fatou set must be either periodic or preperiodic, there cannot be wandering components.*

The situation is different for transcendental functions. If the function has an essential singularity there can be another type of Fatou component.

Definition 3.3.4 (**Baker domain**). A Baker domain (or parabolic domain at infinity) is a forward invariant domain on which the sequence of iterates converges uniformly on compact subsets to the constant limit function ∞, which must be an essential singularity.

They received their name in honour of Irvine Noel Baker (1932–2001). The proof of the classification theorem for the Fatou components of entire transcendental functions goes back to the early 90's. We will continue this discussion in Section 5.3 studying the Fatou components of the self-maps of the punctured plane, functions with two essential singularities.

3.4 Singular values

The study of singular values is important for two reasons. First of all, we will see that nearly every Fatou component has associated a singular value and therefore computing the orbits of the singular values of a function gives us information about the composition of its Fatou set. On the other hand, they are problematic points in the sense that prevent the function to be conformal or to be a regular covering.

Definition 3.4.1 (**Singular value**). Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. We call a point $a \in \widehat{\mathbb{C}}$ a *singular value* of f if for every open neighbourhood U of a there exists a component V of $f^{-1}(U)$ such that $f : V \to U$ is not bijective. We will denote by $S(f)$ the set of all finite singular values of f.

Definition 3.4.2 (**Critical value**). We say that c is a *critical point* of order n if $0 = f'(c) = \cdots = f^{(n)}(c)$ and $f^{(n+1)}(c) \neq 0$. At a small neighbourhood of these points, the function f behaves like z^n and therefore it is non injective. Thus, the image of a critical point $v = f(c)$, what is called a *critical value*, is a singular value.

If the function we are considering has essential singularities, there is another kind of singular values to take into account.

Definition 3.4.3 (**Asymptotic value**). A point $a \in \widehat{\mathbb{C}}$ is an *asymptotic value* of f if there exists a curve $\gamma : [0, +\infty) \to \mathbb{C}$ such that

$$\lim_{t\to\infty} |\gamma(t)| = \infty \quad \text{and} \quad a = \lim_{t\to\infty} f(\gamma(t)).$$

Since the function f is not defined at ∞, every asymptotic value is a singular value.

Lemma 3.4.1. *$S(f)$ is the closure of the set sing(f^{-1}), the set of all finite critical and asymptotic values.*

The next theorem explains what is the relation between the singular values and the Fatou components. Check [Mil06] for a proof of it.

Theorem 3.4.2 (Singular values and Fatou components). *Every cycle of attracting and parabolic Fatou components contains a singular value. Every Siegel disc and Herman ring requires that the orbit of respectively one and two singular values accumulates in their boundary.*

This is a powerful tool to detect quickly the presence of these Fatou components. For instance, a rational function of degree d has at most $2d-2$ critical points, therefore this gives us an upper bound for the number of Fatou components.
Now let us introduce some special classes of functions depending on the geometry of its singular set. Many of the theorems in transcendental dynamics are restricted to these classes.

Definition 3.4.4 (Eremenko-Lyubich class $\mathcal{B}$). We say that an entire function f belongs to the *class* $\mathcal{B}$ if $S(f)$ is a bounded set.

Example 3.4.1. Every non-trivial function in the exponential family $E_\lambda(z) = \lambda e^z$ is of class $\mathcal{B}$ because the only singular value is 0. If $\lambda \neq 0$, E_λ has no critical points and 0 is the only omitted point, hence it must be an asymptotic value. Recall that by the Big Picard theorem, an entire transcendental function omits at most one point. See Section 3.6.

The exponential family is the main example that motivated all this theory. In some sense, the maps that we study here share the global behaviour with exponential maps. Once this family was better understood, Devaney and Tangerman moved to study a more general class of functions.

Definition 3.4.5 (Critically finite function). An entire transcendental function is said to be *critically finite* if $S(f)$ is a finite set.

In their article [DT86] they considered critically finite entire functions satisfying certain growth conditions. They proved that this functions have 'Cantor bouquets' in their Julia sets, see Section 3.5 for a disambiguation about the term *Cantor bouquet*. For instance, their results apply to $s(z) = \sin(z)$ and $c(z) = \cos(z)$. Note that these functions are all in $\mathcal{B}$. The setting considered by Rempe et al. in [RRRS11] is much more general, they study finite composition of functions $f \in \mathcal{B}$ of finite order.

3.5 Cantor bouquets

Cantor bouquets are a very interesting object from the topological point of view, for instance see Theorem 3.5.2 for a really surprising property. In the next section we will show that they are very related to the iteration of transcendental functions. You can find non-equivalent definitions of what is a Cantor bouquet in the literature. We will use the one introduced by Aarts and Oversteegen in [AO93] in terms of straight brushes.

Definition 3.5.1 (Straight brush). A subset B of $[0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

(a) Hairiness: for every $(x, y) \in B$ there exists $t_y \geqslant 0$ such that

$$\{x \ : \ (x, y) \in B\} = [t_y, +\infty).$$

(b) Density: the set $\{y \ : \ \exists x, \ (x, y) \in B\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n$, $\beta_n, \gamma_n \to y$ and $t_{\beta_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.

(c) Compact sections: the set B is a closed subset of $\mathbb{R}^2$.

The set $[t_y, +\infty) \times \{y\}$ is called the *hair* attached at y and the point (t_y, y) is called its *endpoint*. The set of endpoints is usually called the *crown* of the straight brush.

Proposition 3.5.1 (Accessible points of a straight brush). *Let B be an straigh brush. If $(x, y) \in B$ is not an endpoint, then (x, y) is not accessible from $\mathbb{R}^2 \setminus B$ in the sense that there is no continuous curve $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(t) \notin B$ for $0 \leqslant t < 1$ and $\gamma(1) = (x, y)$. On the other hand, the endpoint (t_y, y) is accessible from $\mathbb{R}^2 \setminus B$.*

Theorem 3.5.2 (Connectivity of the crown). *Let $\mathcal{E}$ be the crown of some straight brush. Then $\widehat{\mathcal{E}} = \mathcal{E} \cup \{\infty\}$ is a connected set, but $\mathcal{E}$ is totally disconnected.*

Definition 3.5.2 (Cantor bouquet). A *Cantor bouquet* is any subset of the plane that is ambiently homeomorphic to a straight brush.

In [DT86], they define a Cantor N-bouquet as a set homeomorphic to the Cartesian product of a Cantor set and a closed interval I. Then they say that a Cantor bouquet is a limit of such sets. Observe that this is not equivalent to the above definition.

Theorem 3.5.3. *Any two straight brushes $B_1, B_2 \subset \mathbb{R}^2$ are ambiently homeomorphic. That is, there is a homeomorphism $\varphi : B_1 \to B_2$ that can be extended to a homeomorphism $\varphi' : \mathbb{R}^2 \to \mathbb{R}^2$.*

3.6 The exponential family

In this section we will sketch the construction of the Julia set of a transcendental function explicitly. After this we will describe the dynamics of the exponential family for positive real values of the parameter. For a complete proof of these results we refer to Devaney's articles [DT86], [Dev94] and [Dev99].

Definition 3.6.1 (Exponential family). The *exponential family* is given by

$$E_\lambda(z) = \lambda \exp(z)$$

where $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Sometimes it is formulated as

$$\widetilde{E}_a(z) = \exp(z) + a$$

with $a \in \mathbb{C}$. Both families are conformally conjugated through the exponential map. Explicitly, the map $\widetilde{E}_a$ is conjugated to the map E_λ such that $\lambda = e^a$.

Usually the function $E_{1/e}(z) = e^{-1}e^z$ is among the transcendental functions with a simpler Julia set.

Theorem 3.6.1. *$J(E_{1/e})$ is an uncountable union of simple closed curves, each of them connecting a point in $\mathbb{C}$ to ∞. The Fatou set is connected and consists of the attraction basin of the attracting fixed point of $E_{1/e}$.*

Sketch of the proof. Let us construct this set explicitly. For simplicity denote $E := E_{1/e}$. It is easy to see that as a real function $E_{|\mathbb{R}}$ only has one fixed point, $E(1) = 1$ and $E'(1) = 1$. This neutral fixed point is attracting on the left and repelling on the right. Denote by B the parabolic basin of attraction of this point, $B = A_E(1)$. Consider the left half-plane

$$H = \{z \in \mathbb{C} \ : \ \operatorname{Re} z < 1\}.$$

Observe that E maps H to $\mathbb{D} \subseteq H$. The map f is a contraction on H,

$$\forall z \in H, \quad |E'(z)| = \frac{1}{e}\exp(\operatorname{Re} z) < 1$$

and hence $H \subseteq B$. Let

$$J = \{z \in \mathbb{C} \ : \ \forall n \geqslant 0, \ f^n(z) \in \mathbb{C} \setminus H\}.$$

Observe that the horizontal rays R_k with $\operatorname{Im} z = (2k+1)\pi$ and $\operatorname{Re} z \geqslant 1$ are mapped to the negative real axis $\mathbb{R}_- \subseteq H$. Indeed,

$$E(z) = \frac{1}{e}e^{\operatorname{Re} z}e^{i\operatorname{Im} z} = \frac{1}{e}e^{\operatorname{Re} z}\big(\cos(\operatorname{Im} z) + i\sin(\operatorname{Im} z)\big)$$

and therefore

$$\operatorname{Re} E(R_k) = \frac{1}{e}e^{\operatorname{Re} z}\cos(\operatorname{Im} z) = -\frac{1}{e}e^{\operatorname{Re} z} < 0, \quad \operatorname{Im} E(R_k) = \frac{1}{e}e^{\operatorname{Re} z}\sin(\operatorname{Im} z) = 0.$$

Thus, for every $k \in \mathbb{Z}$, $R_k \subseteq \mathbb{C} \setminus J$. Moreover, the same is true for an open neighbourhood of each curve. The preimage of H consists of $\mathbb{C}$ except a collection of unbounded Jordan domains containing each one a ray R_k. Each of these domains is mapped to the right half-plane $\mathbb{C} \setminus H$ bijectively. Therefore, every domain contains a preimage of all the domains. This leads to another collection of unbounded jordan domains inside every previous one. Continuing inductively we get a sequence of nested unbounded Jordan domains, which in the limit is an infinite set of injective curves tending to ∞. Namely, a Cantor bouquet.

□

You have a picture of the Julia set of this function in Figure 3.5. Below we describe the phase space of E_λ for parameters $\lambda \in \mathbb{R}_+$. The following theorem is proved in [Dev99, Theorems 3.4 and 5.4].

Theorem 3.6.2. *Let $E_\lambda(z) = \lambda \exp(z)$. If $0 < \lambda < 1/e$ then $J(E_\lambda)$ is a Cantor bouquet. On the other hand, if $\lambda > 1/e$ then $J(E_\lambda) = \mathbb{C}$.*

For $0 < \lambda < 1/e$ the Fatou set consists of an attracting basin. When $\lambda > 1/e$ this is replaced by an invariant set which is an indecomposable continua.

Example 3.6.1. The Julia sets of $S_\lambda(z) = \lambda \sin z$ for $\lambda \in (0, 1)$ are Cantor bouquets as well.

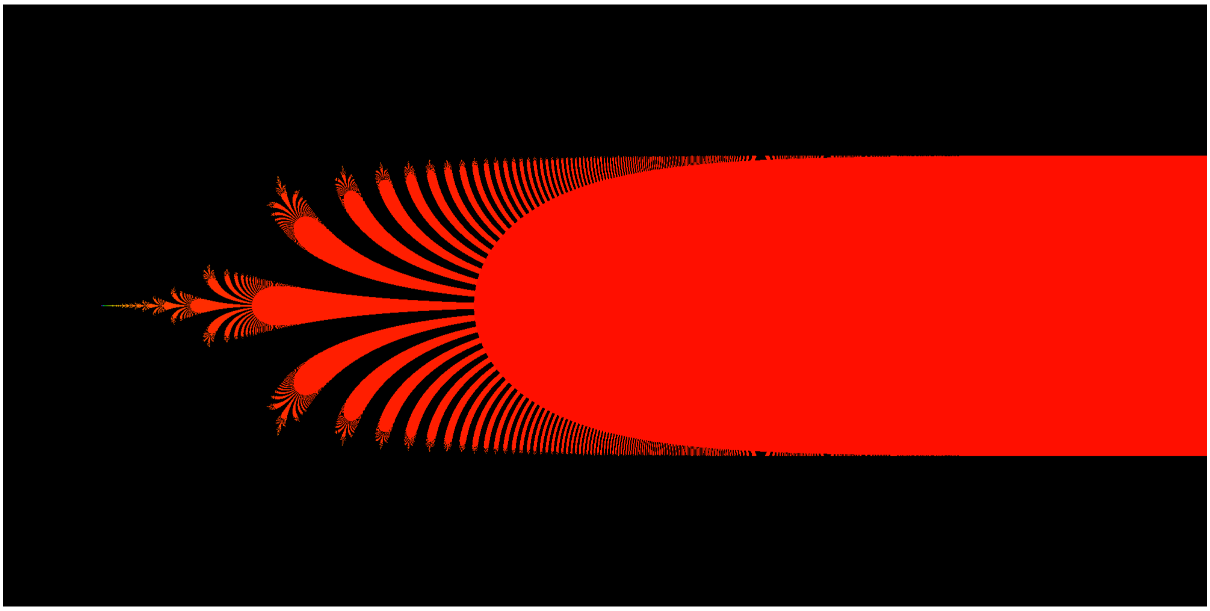

Figure 3.5: Cantor bouquet. Phase space of $E_\lambda(z) = \lambda \exp z$ with $\lambda = 0.367879441 \lesssim 1/e$, $z \in [0, 4\pi] + i[-\pi, \pi]$.

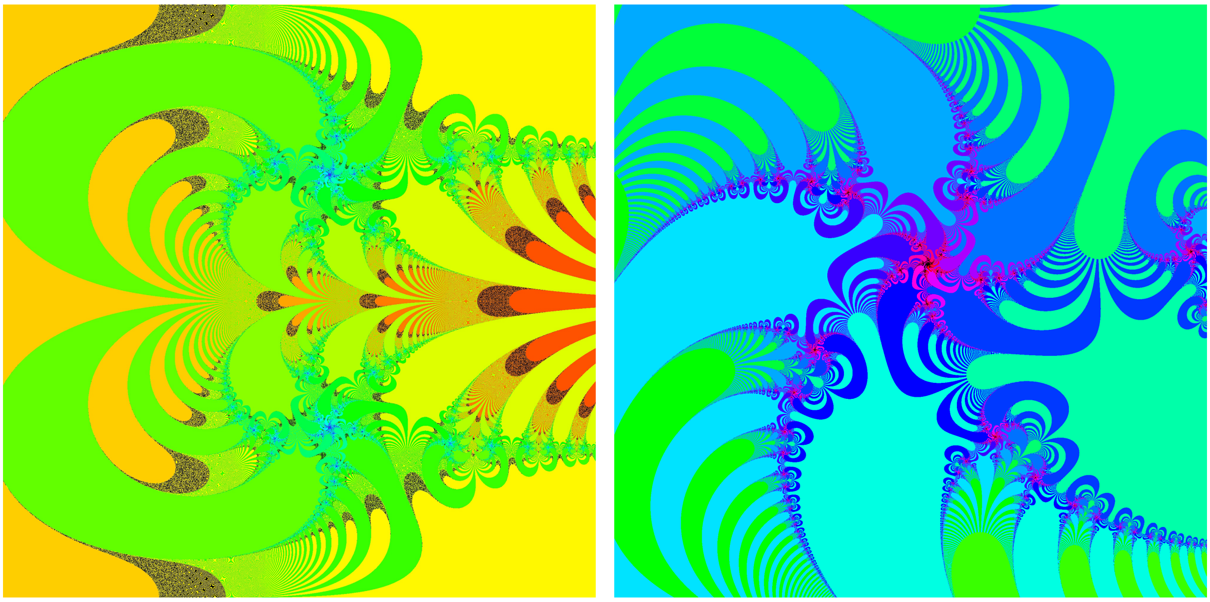

Figure 3.6: Undecomposable continua. Phase space of $E_1(z) = \exp z$ ($\lambda = 1 > 1/e$). **L:** $z \in ?$; **R:** a zoom of it.

3.7 Escaping set and dynamic rays

Definition 3.7.1 (Escaping set). Let $f : \mathbb{C} \to \mathbb{C}$, we call *escaping set* to

$$I(f) := \{z \in \mathbb{C} \ : \ |f^n(z)| \to \infty\}.$$

By definition, $I(f)$ is forward invariant. Eremenko proved the following properties for the transcendental case, see [Ere89].

Proposition 3.7.1 (Properties of $I(f)$). *Let f be an entire transcendental function. Then,*

- $I(f) \cap J(f) \neq \emptyset$;
- $J(f) = \partial I(f)$;
- *all the components of $I(f)$ are unbounded.*

Note that the second property is a third characterization of $J(f)$ for the transcendental case. It generalizes what happens for polynomials. If P is a polynomial of degree greater or equal than two, then $J(P) = \partial K(P)$ where $K(P) = \mathbb{C} \setminus A_P(\infty)$, the filled Julia set of P.

Proposition 3.7.2. *If $f \in \mathcal{B}$ then $I(f) \subseteq J(f)$. In particular, $I(f)$ has no interior.*

Definition 3.7.2 (Ray tail). Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function. A *ray tail* of f is an injective curve

$$\gamma : [0, \infty) \to I(f)$$

such that $|F^n(\gamma(t))| \to +\infty$ as $t \to \infty$ for all $n \geqslant 0$ and such that $|F^n(\gamma(t))| \to +\infty$ uniformly in t as $n \to \infty$.

Definition 3.7.3 (Dynamic ray). Let f be an entire function. A *dynamic ray* of f is a maximal injective curve $\gamma : (0, \infty) \to I(f)$ such that $\gamma|_{[t,\infty)}$ is a ray tail for every $t > 0$.

Proposition 3.7.3 (Escaping points on rays). *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $z \in I(f)$. Suppose that some iterate $f^k(z)$ is on a ray tail γ_k of f. Then either z is on a ray tail, or there is some $n \leqslant k$ such that $f^n(z)$ belongs to a ray tail that contains an asymptotic value of f.*
In particular, there is a curve γ_0 connecting z to ∞ such that $f^j|_{\gamma_0}$ tends to ∞ uniformly (in fact, $f^k(\gamma_0) \subseteq \gamma_k$).

Proof. Suppose that $\gamma_k : [0, \infty) \to \mathbb{C}$ is a parametrization of such ray tail and $\gamma_k(0) = f^k(z)$. We call $\gamma : [0, T) \to \mathbb{C}$ a lift of γ_k starting at $f^{(k-1)}(z)$ if

- $\gamma(0) = f^{(k-1)}(z)$ (i.e. $\gamma(0)$ is a preimage of $f^k(z)$);
- $\forall t \in [0, T),\ f(\gamma(t)) = \gamma_k(t)$;

The set of all these curves is a partially ordered set: if $\gamma_1 : [0, T_1) \to \mathbb{C}$ and $\gamma_2 : [0, T_2) \to \mathbb{C}$ are two of such lifts, we say that $\gamma_1 \leqslant \gamma_2$ if $T_1 \leqslant T_2$ and $\forall t \leqslant T_1,\ \gamma_1(t) = \gamma_2(t)$. Note that this set is not empty, $f^{(k-1)}(z)$ is always a preimage of $f^k(z)$ and a single point can also be considered a lift. In this set, every chain $\{\gamma_\alpha : [0, T_\alpha) \to \mathbb{C}\}_\alpha$ has an upper bound: let T_* be the supremum of $\{T_\alpha\}_\alpha$ and take $\gamma_* : [0, T_*) \to \mathbb{C}$ given by $\gamma_*(t) = \gamma_{\alpha_0}(t)$ for any $\alpha_0 > t$. Then by Zorn's Lemma there exists a maximal curve $\gamma_{k-1} : [0, T) \to \mathbb{C}$ satisfying the two properties above.
Now we have two possibilities, either $T = \infty$ or not. In the first case, $\gamma_{k-1}(t)$ must tend to ∞ as $t \to \infty$, otherwise we would have

$$f(z_0) = f\left(\lim_{t\to\infty} \gamma_{k-1}(t)\right) = \lim_{t\to\infty} f(\gamma_{k-1}(t)) = \lim_{t\to\infty} \gamma_k(t) = \infty$$

so z_0 would be a pole of f contradicting the assumption that f is entire. Thus, $f^{(k-1)}(z)$ is on a ray tail. Now consider the case $T < \infty$ and let

$$w = \lim_{t \to T} \gamma_{k-1}(t) \in \widehat{\mathbb{C}}.$$

Again, it cannot happen that $f(w) = \infty$, so $f(w) = \gamma_k(t_0)$ for some $t_0 \in [0, \infty)$. In this case, γ_{k-1} could be prolonged, contradicting its maximality. Note that if w was a critical point we would need to choose a branch of the inverse. Thus, $w = \infty$ and $\gamma_k(T)$ is an asymptotic value of f (possibly ∞). Then either we have found a ray tail $\gamma_{k-1} \subseteq f^{-1}(\gamma_k) \subseteq I(f)$ connecting $f^{(k-1)}(z)$ to ∞ or γ_k contains an asymptotic value.
Finally we will proceed by induction. At each step, due to the above reasoning, we can either construct a new rail tail connecting $f^j(z)$ to ∞ or we find an asymptotic value of f. After k steps, if we have not found an asymptotic value, we are going to have a ray tail γ_0 connecting z to ∞. ■

Regarding the escaping set and dynamical rays Alexandre Eremenko conjectured the following:

- each component of $I(f)$ is unbounded (weak Eremenko's conjecture);
- every point in $I(f)$ can be joined with ∞ by a curve in $I(f)$ (strong Eremenko's conjecture).

As we have explained in the Introduction, the article [RRRS11] gives a negative answer to strong Eremenko's conjecture even when we restrict to functions in the Eremenko-Lyubich class $\mathcal{B}$. However, they also give a partial positive result showing that this holds for a large class of functions in $\mathcal{B}$. This will be proved in the next chapter.

Chapter 4

Dynamic rays of bounded-type entire functions

In this chapter we are going to introduce our main tool, the logarithmic coordinates to study functions of the class $\mathcal{B}$.

4.1 Logarithmic coordinates

Definition 4.1.1 (Logarithmic singularity). Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental entire or meromorphic function and let $a \in \widehat{\mathbb{C}}$. Suppose there is some simply-connected open neighbourhood $D \subseteq \mathbb{C}$ of a and a component U of $f^{-1}(D \setminus \{a\})$ such that $f : U \to \mathbb{D} \setminus \{a\}$ is a universal covering map. Then we say that f has a *logarithmic singularity* over a.

Definition 4.1.2 (Tract). Let $f \in \mathcal{B}$ and let $D \subseteq \mathbb{C}$ be a bounded Jordan domain containing $S(f) \cup \{0, f(0)\}$. We call *tract* to each connected component V of $\mathcal{V} = f^{-1}(W)$ where $W = \mathbb{C} \setminus \overline{D}$.

Lemma 4.1.1. *A tract V is an unbounded Jordan domain (i.e. a disk whose closure contains ∞) and $f : V \to W$ is a universal covering.*

Proof. First of all we will prove that $f : V \to W$ is a covering map. It is clear that this map is continuous (f is entire) and surjective (by construction). Let $z \in W$ and $U \subseteq W$ be a small enough neighbourhood of z. Consider X a connected component of $f^{-1}(U)$. A priori there are two possibilities: either $\overline{X}$ is compact or not. In the first case, since $\overline{U}$ contains no critical values, $\overline{X}$ contains no critical points and $f : X \to U$ is a diffeomorphism. Suppose now that $\overline{X}$ was not compact. Then we can choose an exhaustion of $\overline{U}$ by simple closed curves γ_t, $0 \leqslant t \leqslant 1$, such that $\gamma_0 = \{q\}$ with $q \in U$ and $\gamma_1 = \partial\overline{U}$. See Figure 4.2. For small values of t, $f^{-1}(\gamma_t)$ is a simple closed curve but there will be $t_0 \leqslant 1$ such that $f^{-1}(\gamma_{t_0})$ is not. But since there are no critical points in $\overline{X}$, this curve is a submanifold of $\mathbb{C}$ and can be extended to ∞. This leads to the existence of an asymptotic value of f in the curve γ_{t_0} contradicting the assumption that all asymptotic values are in $\mathbb{C} \setminus W$. We conclude that $\overline{X}$ needs to be compact and $f : V \to W$ is a covering map.

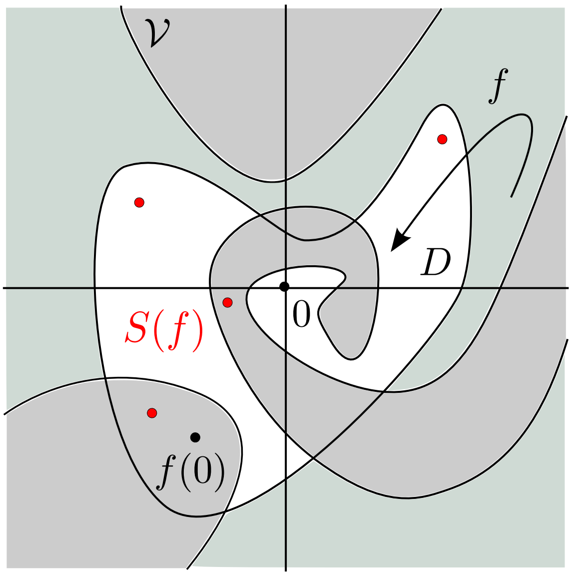

Figure 4.1: Tracts of a function $f \in \mathcal{B}$.

The preimage of W has to be either a disk or a punctured disk. If it was a disk, we would have a universal covering. Note that since W has points as close of ∞ as we please, if this disk was bounded then it would need to contain a pole but f is entire. On the other hand, if $f^{-1}(W)$ was a punctured disk then we can distinguish two cases depending on the nature of the puncture point a. If $a = \infty$ this means that f is an entire function that fixes the point at infinity and hence it must be a polynomial, not a transcendental function. If $a \neq \infty$ then a must be a pole of f, but we're assuming f to be entire. ■

This proof follows the one that Devaney and Tangermann made for critically finite entire transcendental functions in [DT86].

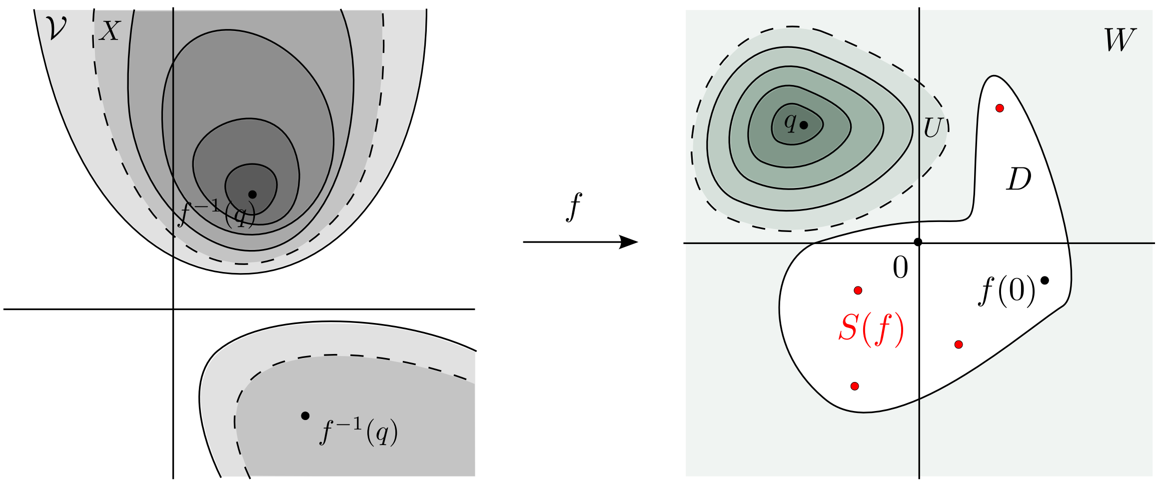

Figure 4.2: Sketch in the proof of Lemma 4.1.1.

Definition 4.1.3 (Logarithmic transform). Let $f \in \mathcal{B}$ and consider $\mathcal{T} := \exp^{-1}(\mathcal{V})$ and $H := \exp^{-1}(W)$. We will call *logarithmic transform* of f to the continuous function $F : \mathcal{T} \to H$ that makes the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "\exp"'] & H \arrow[d, "\exp"] \\ \mathcal{V} \arrow[r, "f"'] & W \end{tikzcd}$$

commute. The connected components of $\mathcal{T}$ are called *tracts* of F.

Example 4.1.1. Taking $f(z) = \exp z$ leads to a very special situation. If we choose $W = D(0,\ e)$ then $\mathcal{V}$ is the right half plane $\mathbb{H}_1$. Here $H = \mathcal{V}$ and $F = \exp z$. $\mathcal{T}$ is the second preimage of W by f and consists of some *finger shaped* sets contained in $\mathbb{H}_0$, you can see a picture of them in Figure 4.3.

Figure 4.3: The set $\exp^{-2}(D(0,\ e))$.

Proposition 4.1.2. *If $f \in \mathcal{B}$, then its logarithmic transform $F : \mathcal{T} \to H$ satisfies the following properties:*

a) H is a $2\pi i$-periodic Jordan domain that contains a right half plane;

b) every component T of $\mathcal{T}$ is an unbounded Jordan domain with real parts bounded below, but unbounded from above;

c) the components of $\overline{\mathcal{T}}$ have disjoint closures and accumulate only at infinity;

d) for every component T of $\mathcal{T}$, $F : T \to H$ is a conformal isomorphism;

e) for every component T of $\mathcal{T}$, $\exp|_T$ is injective;

f) $\mathcal{T}$ is invariant under translation by $2\pi i$.

Proof. Recall that $\exp : \mathbb{C} \to \mathbb{C}^*$ is a holomorphic cover and, in particular, a local homeomorphism. Let us check each one of the above properties:

a) The boundary of D is a loop arround 0, hence its preimage under the exponential map must be a $2\pi i$-periodic continuous curve. Since exp is a local homeomorphism, this curve cannot have self-intersections. The exterior of D is mapped to the right of this curve. There is $R > 0$ such that $D \subseteq D(0, R)$, therefore the right half plane with $\operatorname{Re} z > \log R$ is contained in H.

b) Again, since exp is a holomorphic cover, every component T of $\mathcal{T}$ must be a Jordan domain. Since the point $f(0)$ is in the interior of D, the components of $\mathcal{V}$ must be bounded away from the origin. Taking a logarithm, this gives a lower bound for the real part of the points in $\mathcal{T}$. On the other hand, the logarithm of an unbounded set must have points with unbounded real part.

c) Tracts of F must have disjoint closures. Indeed, if the boundaries of two tracts had a common point z_0, take a small enough neighbourhood U of z_0 so that $V = f(\exp(U))$ does not intersect any singular value of f. Now since exp is a conformal map and the restriction of f to $f^{-1}(V)$ has no critical points, the map $f \circ \exp$ is a covering from U to V and hence a local homeomorphism. Since D is a Jordan domain, the intersection of the boundary of D with U is a piece of arc. Thus, the intersection of the boundary of $\mathcal{T}$ with U must also be a piece or arc, all the points in $\partial\mathcal{T} \cap U$ must be common to both tracts. If the tracts are different, this cannot happen for all the common points. There must be a point z_0' with a neighbourhood U' such that $\partial\mathcal{T} \cap U'$ has points belonging only to boundary of one of the tracts. Hence it would not be homeomorphic to an arc, raising a contradiction.
Suppose now that there was a sequence of points z_k each one belonging to a different tract and converging to a finite point z. Consider $w_k = f(\exp(z_k))$ which is a converging sequence in ∂D and let $w \in D$ be its limit. Since $f(\exp z) = w$, the point z must be in the boundary of some tract. Let U be a small neighbourhood of z such that $V = f(\exp U)$ does not intersect $S(f)$. U contains infinitely many points of the sequence $\{z_k\}_k$ and hence $U \cap \mathcal{T}$ is a disjoint union of infinitely many pieces of arc. We get a contradiction because $V \cap \partial D$ is a piece of arc and $f \circ \exp$ must be a local homeomorphism. Thus, $z = \infty$.

As a consequence of b) and c), there exists a curve $\delta \subseteq \mathbb{C} \setminus \mathcal{V}$ joining 0 and ∞ (it has been drawn in blue in Figure 4.4). Hence, we can define a continuous branch of the logarithm on $\mathcal{T}$.

d) Let T be a tract of F. By Proposition 1.4.2, since $f \circ \exp : T \to W$ and $\exp : H \to W$ are both universal covers, they must be equivalent. That is, there exist a homeomorphism $F : T \to H$ making the diagram commute. Since $f \circ \exp_{|T}$ and exp are conformal, F must be conformal too.

e) The preimage of $\mathcal{V}$ under exp is compactly contained in the open band defined by two preimages of the curve δ. Therefore, it cannot contain vertical segments of length 2π and hence $\exp_{|T}$ is injective.

f) This is a direct consequence of the fact that the exponential map is $2\pi i$-periodic.

■

Definition 4.1.4 (Class $\mathcal{B}_{log}$). We will denote by $\mathcal{B}_{log}$ the subclass of $\mathcal{B}$ consisting of any function $F \in \mathcal{B}$, $F : \mathcal{T} \to H$ satisfying the properties a) to f) in Proposition 4.1.2.

Note that maybe not every function in $\mathcal{B}_{log}$ comes as a logarithmic transform of a function in $\mathcal{B}$, but we will only be interested in those ones.

Remark 4.1.1. A function $F \in \mathcal{B}_{log}$ need not to be $2\pi i$-periodic. Some authors add this condition to functions in class $\mathcal{B}_{log}$.

Corollary 4.1.3. *Let $F : \mathcal{T} \to H$ in $\mathcal{B}_{log}$ be a logamic transform of a function $f \in \mathcal{B}$. Every compact set $K \subseteq \mathbb{C}$ can intersect only a finite number of tracts of F.*

Proof. This is a direct consequence of propety (c) in the definition of class $\mathcal{B}_{log}$. If a compact set K was able to intersect an infinite number of tracts then there would be an accumulation point inside K contradicting the fact that tracts only accumulate at infinity. ∎

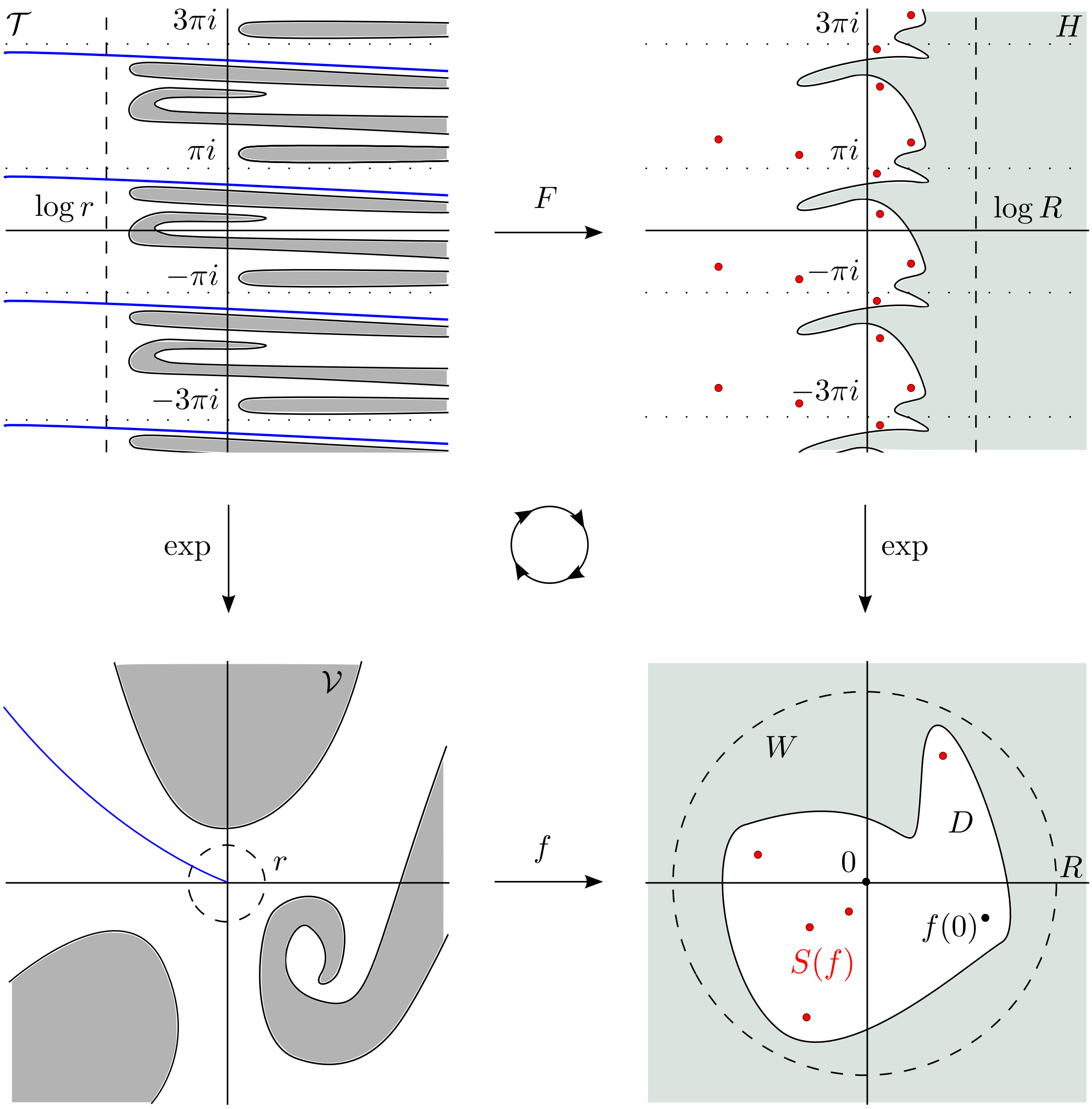

Figure 4.4: Logarithmic coordinates for a function $f \in \mathcal{B}$.

4.2 Expansivity and normalization

Using the Koebe-Bieberbach quarter theorem (Theorem 1.7.3), we can prove an expansivity property for the functions of the class $\mathcal{B}_{log}$.

Lemma 4.2.1 (Expansivity property). *Let $F : \mathcal{T} \to H$ be a function of class $\mathcal{B}_{log}$. There exists $R_0 > 0$ such that $|F'(z)| \geqslant 2$ when $Re\, F(z) \geqslant R_0$.*

Proof. Let T be a tract of F. Denote by F_T^{-1} the inverse of the conformal isomorphism $F : T \to H$. By property (a) we know that there exists $R > 0$ such that

$$\mathbb{H}_R := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > R\} \subseteq H.$$

It is clear that if $\operatorname{Re} F(z) \geqslant R$ the disk $D(F(z),\ \operatorname{Re} F(z) - R)$ is contained in $\mathbb{H}_R$, see Figure 4.5.

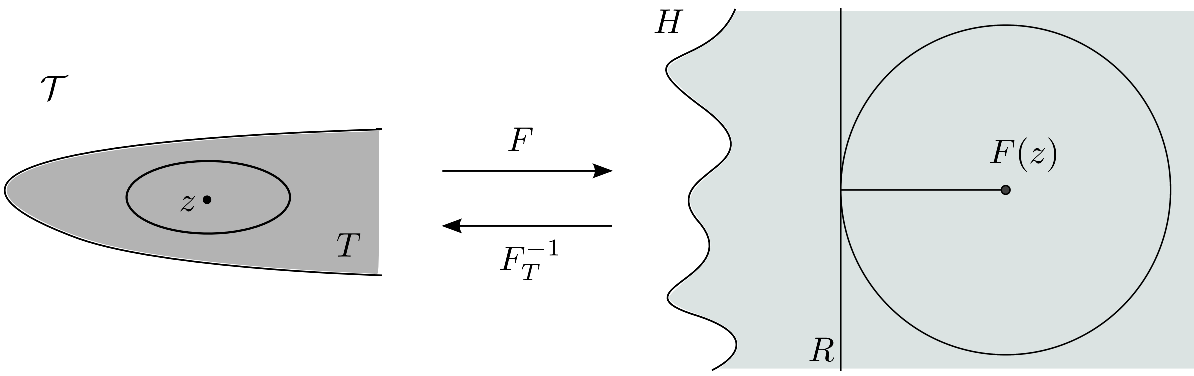

Figure 4.5: Scheme in the proof of Lemma 4.2.1.

Consider now for every value of z the map $\psi : \mathbb{D} \to D(F(z),\ \operatorname{Re} F(z) - R)$ consisting of a dilatation composed with a translation:

$$\psi(\zeta) = F(z) + \zeta|\operatorname{Re} F(z) - R|.$$

Then the map $F_T^{-1} \circ \psi : \mathbb{D} \to T$ is a conformal isomorphism. By the Koebe-Bieberbach quarter theorem (Theorem 1.7.3),

$$D((F_T^{-1} \circ \psi)(0),\ |(F_T^{-1} \circ \psi)'(0)|/4) \subseteq (F_T^{-1} \circ \psi)(\mathbb{D}) = F_T^{-1}(D(F(z),\ \operatorname{Re} F(z) - R)).$$

Using the chain rule,

$$(F_T^{-1} \circ \psi)'(0) = (F_T^{-1})'(\psi(0)) \cdot \psi'(0) = (F_T^{-1})'(F(z)) \cdot |\operatorname{Re} F(z) - R|$$

and since $F_T^{-1}(F(z)) = z$,

$$(F_T^{-1})'(F(z)) \cdot F'(z) = 1 \quad \Leftrightarrow \quad (F_T^{-1})'(F(z)) = \frac{1}{F'(z)}$$

we have

$$(F_T^{-1} \circ \psi)'(0) = \frac{|\operatorname{Re} F(z) - R|}{F'(z)}.$$

Hence,

$$D\left(z,\ \frac{|\operatorname{Re} F(z)-R|}{4|F'(z)|}\right) \subseteq F_T^{-1}(D(F(z),\ \operatorname{Re} F(z)-R)) \subseteq T.$$

Property (e) says that $\exp|_T$ is univalent, then since the exponential is a $2\pi i$-periodic function T cannot contain any vertical segment of length 2π. Then,

$$\frac{|\operatorname{Re} F(z)-R|}{4|F'(z)|} \leqslant \pi \quad \Leftrightarrow \quad \frac{1}{4\pi}|\operatorname{Re} F(z)-R| \leqslant |F'(z)|.$$

Observe that there exists always a value $R_0 > R$ such that if $\operatorname{Re} F(z) \geqslant R_0$ then

$$|F'(z)| \geqslant \frac{1}{4\pi}|\operatorname{Re} F(z)-R| \geqslant \frac{1}{4\pi}|R_0-R| \geqslant 2,$$

it is enough to pick $R_0 > 8\pi + R$. ∎

Definition 4.2.1 (Normalized function). We say that $F : \mathcal{T} \to H$ is *normalized* if H is the right half plane $\mathbb{H}$ and furthermore the expansivity property holds for all $z \in \mathcal{T}$.

Remark 4.2.1. Given any function $F \in \mathcal{B}_{log}$ we can always normalize it by restricting to $\mathbb{H}_{R_0}$ and using the change of variables $w = z - R_0$.

4.3 Symbolic dynamics and combinatorics

Definition 4.3.1 (Julia set for $\mathcal{B}_{log}$). If $F \in \mathcal{B}_{log}$, then

$$J(F) := \{z \in \overline{\mathcal{T}}\ :\ F^n(z) \text{ is defined and in } \overline{\mathcal{T}} \text{ for all } n \geqslant 0\}.$$

Note that $J(F)$ is the set of points which can be iterated infinitely many times. Recall that F is defined only in $\mathcal{T}$, hence if one iterate gets out of $\mathcal{T}$ then the orbit of the point is truncated. Here the nomenclature can be confusing; apparently there is no relation between $J(f)$ and a possible definition of Julia set for such function. Lemma 4.3.1 explains what the relation with $J(f)$ is, justifying the use of this name.

Definition 4.3.2 (Escaping set for $\mathcal{B}_{log}$). If $F \in \mathcal{B}_{log}$, then

$$I(F) := \{z \in J(F)\ :\ \lim_{n\to\infty} \operatorname{Re} F^n(z) = \infty\}.$$

Note that $I(F) \subseteq J(F)$ by definition. If we can understand the structure of $I(F)$ for functions $F \in \mathcal{B}_{log}$ then next lemma tells us what information we get from the original function $f \in \mathcal{B}$ that has F as logarithmic transform.

Lemma 4.3.1. *If $f \in \mathcal{B}$ and F is a logarithmic transform of f, then* $\exp(I(F)) \subseteq I(f)$. *Furthermore, if F is normalized then* $\exp(J(F)) \subseteq J(f)$.

Proposition 4.10.2 is an extension of this to a particular subclass of $\mathcal{B}_{log}$. Finally, let us define

$$J^K(F) := \{z \in J(F)\ :\ \forall n \geqslant 1,\ \operatorname{Re} F^n(z) \geqslant K\}$$

for $K > 0$. Observe that every $z \in I(F)$ eventually enters $J^K(F)$ for all K.
Let $F : \mathcal{T} \to H$ be a logarithmic transform. We denote by $\mathcal{A}$ the set of tracts of F and call it the symbolic alphabet associated to F. Thus, there is a one-to-one correspondence between the tracts of F and symbols in $\mathcal{A}$ which, by Corollary 4.1.3, must be countable.

Definition 4.3.3 (External address). Let $F \in \mathcal{B}_{log}$ and let $z \in J(F)$. For each $j \geqslant 0$, let $T_j \in \mathcal{A}$ be the (unique) tract of F with $F^j(z) \in \overline{T_j}$. Then the sequence

$$\underline{s} := \operatorname{addr}(z) := T_0 T_1 T_2 \ldots \in \mathcal{A}^{\mathbb{N}}$$

is called the *external address* (or *itinerary*) of z.

Given an external address $\underline{s} \in \mathcal{A}^{\mathbb{N}}$, we introduce

$$J_{\underline{s}} := \{z \in J(F) \ : \ \operatorname{addr}(z) = \underline{s}\}, \quad I_{\underline{s}} := \{z \in I(F) \ : \ \operatorname{addr}(z) = \underline{s}\}.$$

and also, for $K > 0$

$$J^K_{\underline{s}} := \{z \in J^K(F) \ : \ \operatorname{addr}(z) = \underline{s}\}.$$

There may be itineraries $\underline{s} \in \mathcal{A}^{\mathbb{N}}$ that are not realized by any point, thus these sets may be empty for some $\underline{s}$.

Definition 4.3.4 (Shift operator). The one-sided *shift operator* with alphabet $\mathcal{A}$ is a map

$$\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$$

defined by

$$\sigma(T_0 T_1 T_2 \ldots) = T_1 T_2 \ldots$$

This map makes the following diagram commutative:

$$\begin{tikzcd} J(F) \arrow[r, "F"] \arrow[d, "\operatorname{addr}"'] & J(F) \arrow[d, "\operatorname{addr}"] \\ \mathcal{A}^{\mathbb{N}} \arrow[r, "\sigma"'] & \mathcal{A}^{\mathbb{N}} \end{tikzcd}$$

i.e. for all $z \in J(F)$,

$$\sigma(\operatorname{addr}(z)) = \operatorname{addr}(F(z)).$$

4.4 General properties of class $\mathcal{B}_{log}$

Lemma 4.4.1 (Exponential separation of orbits). *Let $F \in \mathcal{B}^n_{log}$ and let T be a tract of F. If $\omega, \zeta \in T$ are such that $|\omega - \zeta| \geqslant 8\pi$, then*

$$|F(\omega) - F(\zeta)| \geqslant \exp(|\omega - \zeta|/8\pi) \cdot \min\{Re\, F(\omega), Re\, F(\zeta)\}.$$

Proof. We can assume without loss of generality that $\operatorname{Re} F(\omega) \geqslant \operatorname{Re} F(\zeta)$. By property (e) in the definition of class $\mathcal{B}_{log}$, T has height at most 2π, $\operatorname{dist}(z, \partial T) \leqslant \pi$ for all $z \in T$ and hence by the standard estimate and Pick's theorem we have

$$\frac{|\omega - \zeta|}{2\pi} \leqslant \operatorname{dist}_T(\omega, \zeta) = \operatorname{dist}_{\mathbb{H}}(F(\omega), F(\zeta).$$

Let $\xi \in \mathbb{H}$ be a point such that $\operatorname{Re} \xi = \operatorname{Re} F(\zeta)$ and $\operatorname{dist}_{\mathbb{H}}(F(\zeta, \xi) = \operatorname{dist}_{\mathbb{H}}(F(\zeta), F(\omega))$. This is possible because along the line $\{z \in \mathbb{H} \ : \ \operatorname{Re} z = \operatorname{Re} F(\zeta)\}$ the distance function

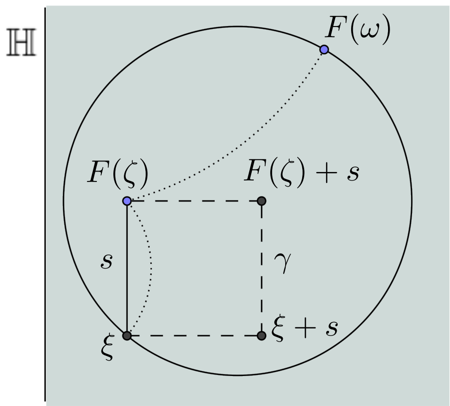

Figure 4.6: Scheme in the proof of Lemma 4.4.1.

$\operatorname{dist}_{\mathbb{H}}(F(\zeta),\ \cdot\ )$ takes the value 0 at $F(\zeta)$ and increases to ∞ as z moves away of $F(\zeta)$ (geodesics between points with the same real part are arcs of circles orthogonal to $\partial\mathbb{H}$, their Euclidean length increases as they separate and hence the same does their hyperbolic length). See Figure 4.6. Let $s = |F(\zeta) - \xi| > 0$. The straight segments $\overline{F(\zeta)F(\omega)}$ and $\overline{F(\zeta)\xi}$ have the same hyperbolic length but since $\operatorname{Re} F(\omega) \geqslant \operatorname{Re} F(\zeta) = \operatorname{Re}\xi$, the points of the first one may be further away of $\partial\mathbb{H}$ than the points of the second one and hence $|F(\zeta) - F(\omega)| \geqslant s$ to compensate this. Let γ be the dashed curve in Figure 4.6 consisting of three straight segments connecting $F(\zeta)$ to ξ through $F(\zeta)+s$ and $\xi+s$, let

$$\gamma_1 := \overline{\xi(\xi+s)}, \quad \gamma_2 := \overline{(\xi+s)(F(\zeta)+s)}, \quad \gamma_3 := \overline{(F(\zeta)+s)F(\zeta)}, \quad \gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

On the one hand,

$$l_{\mathbb{H}}(\gamma_2) = \int_{\gamma_2} \frac{1}{\operatorname{Re} z}|dz| \leqslant \sup_{z\in\gamma} \left|\frac{1}{\operatorname{Re} z}\right| \cdot l(\gamma_2) = \frac{1}{\operatorname{Re}\xi + s} \cdot s \leqslant 1,$$

while on the other hand,

$$l_{\mathbb{H}}(\gamma_1) = \int_{\gamma_1} \frac{1}{\operatorname{Re} z}|dz| = \int_{\operatorname{Re}\xi}^{\operatorname{Re}\xi+s} \frac{dt}{t} = \big[\log t\big]_{\operatorname{Re}\xi}^{\operatorname{Re}\xi+s} = \log(\operatorname{Re}\xi + s) - \log(\xi)$$

and since γ_1 and γ_3 are indistinguishable in terms of the real parts of their points

$$l_{\mathbb{H}}(\gamma_1) = l_{\mathbb{H}}(\gamma_3) = \log(\operatorname{Re} F(\zeta) + s) - \log(\operatorname{Re} F(\zeta)) = \log\left(\frac{\operatorname{Re} F(\zeta) + s}{\operatorname{Re} F(\zeta)}\right).$$

Hence, we get

$$l_{\mathbb{H}}(\gamma) = l_{\mathbb{H}}(\gamma_1) + l_{\mathbb{H}}(\gamma_2) + l_{\mathbb{H}}(\gamma_3) = 2\log\left(\frac{\operatorname{Re} F(\zeta) + s}{\operatorname{Re} F(\zeta)}\right) + 1$$

and thus

$$\frac{|\omega - \zeta|}{2\pi} \leqslant \operatorname{dist}_{\mathbb{H}}(F(\omega), F(\zeta)) = \operatorname{dist}_{\mathbb{H}}(F(\zeta), \xi) < l_{\mathbb{H}}(\gamma) = 2\log\left(\frac{\operatorname{Re} F(\zeta) + s}{\operatorname{Re} F(\zeta)}\right) + 1.$$

Isolating s from the above expression,

$$|F(\omega) - F(\zeta)| \geqslant s > \operatorname{Re} F(\zeta) \left[\exp\left(\frac{|\omega - \zeta|}{4\pi} - \frac{1}{2} \right) - 1 \right].$$

It is easy to check that $e^{x-1/2} - 1 > e^{x/2}$ for $x \geqslant 2$, since $|\omega - \zeta| \geqslant 8\pi$ by assumption,

$$\frac{|\omega - \zeta|}{4\pi} \geqslant 2$$

and the result holds. ∎

Lemma 4.4.2 (Growth of real parts). *Let $F \in \mathcal{B}^n_{log}$. If $\zeta, \omega \in J(F)$ are distinct points with the same external address $\underline{s}$, then*

$$\lim_{k\to\infty} \max\{Re\, F^k(\zeta), Re\, F^k(\omega)\} = \infty.$$

Proof. Assume to the contrary that there exists $S > 0$ such that $\operatorname{Re} F^k(\zeta), \operatorname{Re} F^k(\omega) < S$ (and hence $\max\{\operatorname{Re} F^k(\zeta), \operatorname{Re} F^k(\omega)\} < S$) for infinitely many $k \in \mathbb{N}$. Observe that since ζ and ω have the same external address, $F^k(\zeta)$ and $F^k(\omega)$ will belong to the same tract for every k.
Let T be a tract of F. Property (b) in the definition of the $\mathcal{B}_{log}$ says that the real parts of points in T are bounded from below. Recall also that, by property (e), T cannot contain vertical segments of length greater than 2π. Thus, $\overline{T} \cap \{z \in \mathbb{C} \ : \ \operatorname{Re} z \leqslant S\}$ is a compact set and has bounded imaginary parts.
Every tract T that intersects this set must also intersect the vertical line $\{\operatorname{Re} z = S\}$ because they are Jordan domains with unbounded real part. Note that, up to translations by a multiple of $2\pi i$, only a finite finite number of tracts intersect this line. Otherwise we could find a sequence in $\overline{T}$ with points all belonging to different tracts accumulating at a finite point, contradicting property (c). Thus, there exists a constant $C > 0$ such that if $\operatorname{Re} F^k(\zeta), \operatorname{Re} F^k(\omega) < S$ then

$$|F^k(\zeta) - F^k(\omega)| < C.$$

Note that S and C are independent of the value of k. By the expansivity property (Lemma 4.2.1)

$$|F_T^{-1}(F(z))| = \frac{1}{F'(z)} \leqslant \frac{1}{2}.$$

Therefore,

$$|F^{(k-1)}(\zeta) - F^{(k-1)}(\omega)| = |F_T^{-1}(F^k(\zeta)) - F_T^{-1}(F^k(\omega))| \leqslant \frac{1}{2} \cdot |F^k(\zeta) - F^k(\omega)| \leqslant \frac{C}{2}$$

and by induction

$$|\zeta - \omega| = |(F_T^{-1})^k(F^k(\zeta)) - (F_T^{-1})^k(F^k(\omega))| \leqslant \frac{1}{2^k} \cdot |F^k(\zeta) - F^k(\omega)| \leqslant \frac{C}{2^k}.$$

Now since this happens for infinitely many $k \in \mathbb{N}$, we can take the limit

$$|\zeta - \omega| \leqslant \lim_{k\to\infty} \frac{C}{2^k} = 0$$

and hence $\zeta = \omega$ proving the result. ∎

Theorem 4.4.3 (**Existence of unbounded continua in** $J_{\underline{s}}$). *For every* $F \in \mathcal{B}_{log}$ *there exists* $K \geqslant 0$ *with the following property: if* $z_0 \in J^K(F)$ *and* $\underline{s}$ *is the external address of* z_0*, then there exists an unbounded closed connected set* $A \subseteq J_{\underline{s}}$ *with* $dist(z_0, A) \leqslant 2\pi$.

Proof. We may assume without loss of generality that F is normalized. Choose $K > 0$ large enough so that no bounded component of $\mathbb{H} \cap \overline{T}$ intersects the line $\{z \in \mathbb{C} : \operatorname{Re} z = K\}$ for any tract T of F. Let $z_0 \in J^K(F)$ and consider $z_k := F^k(z_0) \in \mathbb{H}_K$ for all $k \geqslant 1$. Let us introduce a bit of notation. Given an unbounded set $S \subseteq \mathbb{C}$ such that

$$S \setminus B_{2\pi}(z_k) = S \setminus \{z \in \mathbb{C} \; : \; |z - z_k| < 2\pi\}$$

has exactly one unbounded component, we will denote by $X_k(S)$ this component. Denote by T_k the tract of F containing x_k, then $\overline{T_k} \setminus B_{2\pi}(z_k)$ has only one unbounded component and hence $X_k(\overline{T_k})$ is well defined. Note that by property (e) of functions in class $\mathcal{B}_{log}$, the straight segment

$$z_k + i[-2\pi, 2\pi] \subseteq B_{2\pi}(z_k)$$

intersects $\overline{T_k}$ transversally. See Figure 4.7.

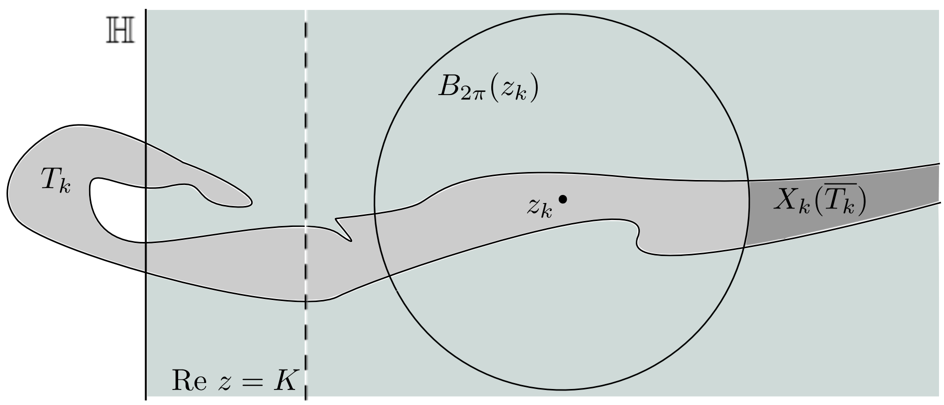

Figure 4.7: Scheme in the proof of Theorem 4.4.3.

Therefore $X_k(\overline{T_k})$ is non-empty and contained in $\mathbb{H}$ for all $k \geqslant 1$. Hence, we can consider $F^{-1}_{T_{k-1}}(X_k(\overline{T_k})) \subseteq T_{k-1}$ which is an unbounded Jordan domain and by the expansivity of F

$$\operatorname{dist}(F^{-1}_{T_{k-1}}(X_k(T_k)), z_{k-1}) \leqslant \frac{1}{2} \operatorname{dist}(X_k(T_k), z_k) = \frac{2\pi}{2} = \pi.$$

Thus, $F^{-1}_{T_{k-1}}(X_k(T_k)) \cap B_{2\pi}(z_{k-1}) \neq \emptyset$ and $\operatorname{dist}(X_{k-1}(F^{-1}_{T_{k-1}}(X_k(T_k))), z_{k-1}) = 2\pi$. Let $A_0 = X_0(\overline{T_0})$. Define inductively the following sequence of sets

$$A_k := X_0(F^{-1}_{T_0}(X_1(F^{-1}_{T_1}(\cdots(X_{k-1}(F^{-1}_{T_{k-1}}(X_k(\overline{T_k})))) \cdots))))$$

for $k \geqslant 1$. Each A_k is an unbounded Jordan domain and its closure in $\widehat{\mathbb{C}}$, $\widehat{A}_k$, is a continuum which has distance 2π to z_0. Since as we said before $F^{-1}_{T_{k-1}}(X_k(\overline{T_k})) \subseteq T_{k-1}$, $\{\widehat{A}_k\}_{k=0}^{\infty}$ is a sequence of nested continua, Proposition 2.1.1 applies and

$$A := \bigcap_{k \geqslant 0} \widehat{A}_k$$

is a continuum. We have

$$\operatorname{dist}(A, z_0) \leqslant \sup_{k \geqslant 0} \operatorname{dist}(\widehat{A}_k, z_0) = 2\pi$$

and by the Boundary bumping theorem (Theorem 2.2.3), A is unbounded. ■

4.5 Functions satisfying a head-start condition

Now we introduce a sufficient condition that guarantees that every escaping point is on a ray tail.

Definition 4.5.1 (Head-start condition). We will define it in three steps:

- Let T and T' be tracts and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a (not necessarily strictly) monotonically increasing continuous function with $\varphi(x) > x$ for all $x \in \mathbb{R}$. We say that the pair (T, T') satisfies the head-start condition for φ if, for all $z, w \in \overline{T}$ with $F(z), F(w) \in \overline{T'}$,

$$\operatorname{Re} w > \varphi(\operatorname{Re} z) \Rightarrow \operatorname{Re} F(w) > \varphi(\operatorname{Re} F(z)).$$

- An external address $\underline{s}$ satisfies the head-start condition for φ if all consecutive pairs of tracts (T_k, T_{k+1}) satisfy the head-start condition for φ, and if for all distinct $z, w \in J_{\underline{s}}$, there is $M \in \mathbb{N}$ such that $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w))$ or $\operatorname{Re} F^M(w) > \varphi(\operatorname{Re} F^M(z))$.
- We say that F satisfies a head-start condition if every external address of F satisfies the head-start condition for some φ. If the same function φ can be chosen for all external addresses, we say that F satisfies the uniform head-start condition for φ.

Notice that in the second part we require that the head-start condition cannot be a void condition for any itinerary. Furthermore, if $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w))$ and the head-start condition is satisfied for that pair of tracts then for all $n > M$, $\operatorname{Re} F^n(z) > \varphi(\operatorname{Re} F^M(w))$ and similarly if $\operatorname{Re} F^M(w) > \varphi(\operatorname{Re} F^M(z))$.

The head-start condition allows us to order the points in $J_{\underline{s}}$ by the growth of their real parts. At this point it would be good to recall the definitions of strict total order (Definition 2.4.3) and order topology (Definition 2.4.4) given in Section 2.4.

Definition 4.5.2 (Speed ordering). Let $\underline{s}$ be an external address satisfying the head-start condition for φ. For $z, w \in J_{\underline{s}}$, we say that $z \succ w$ if there exists $K \in \mathbb{N}$ such that $\operatorname{Re} F^K(z) > \varphi(\operatorname{Re} F^K(w))$. We extend this order to the closure $\widehat{J_{\underline{s}}}$ in $\widehat{\mathbb{C}}$ by the convention that $\infty \succ z$ for all $z \in J_{\underline{s}}$.

Note that this definition is consistent. If there existed K_1, K_2 such that $\operatorname{Re} F^{K_1}(z) > \varphi(\operatorname{Re} F^{K_1}(w))$ and $\operatorname{Re} F^{K_2}(w) > \varphi(\operatorname{Re} F^{K_2}(z))$ then we would raise a contradiction because as we have said before once we are in one of these situations and the head-start condition is satisfied then it is preserved by iteration.

Lemma 4.5.1. *Equivalently, $z \succ w$ if and only if there exists $n_0 \in \mathbb{N}$ such that $Re\, F^n(z) > Re\, F^n(w)$ for all $n > n_0$. Hence the speed ordering does not depend on the choice of the function φ of the head-start condition.*

Proof. If $z \succ w$, since $\varphi(x) > x$ for all $x \in \mathbb{R}$, then $\operatorname{Re} F^n(z) > \varphi(\operatorname{Re} F^n(w)) > \operatorname{Re} F^n(w)$ for all large enough values of n.
Conversely, since $z \neq w$ and $\underline{s}$ satisfies the head-start condition for some function φ, then by definition $\exists M \in \mathbb{N}$ such that either $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w)) > \operatorname{Re} F^M(w)$ or $\operatorname{Re} F^M(w) > \varphi(\operatorname{Re} F^M(z)) > \operatorname{Re} F^M(z)$ and this is preserved under iteration. But we are assuming that $\operatorname{Re} F^n(z) > \operatorname{Re} F^n(w)$ for all $n > n_0$. Thus, it must happen that $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w))$ and therefore $z \succ w$. ■

Lemma 4.5.2. *$\widehat{J_{\underline{s}}}$ together with the speed ordering $\succ$ is a totally ordered space.*

Proof. We will see that it is a transitive and total binary relation in $\widehat{J_{\underline{s}}}$:

- Let $a \prec b$ and $b \prec c$, then there are $k, l \in \mathbb{N}$ such that $\operatorname{Re} F^k(a) > \varphi(\operatorname{Re} F^k(b))$ and $\operatorname{Re} F^l(b) > \varphi(F^l(c))$. Take $m = \max\{k, l\}$. Then we have

$$\operatorname{Re} F^m(a) > \varphi(\operatorname{Re} F^m(b)) > \operatorname{Re} F^m(b) > \varphi(\operatorname{Re} F^m(c))$$

  where the first and last inequalities hold because of the head-start condition. Thus $a \prec c$.

- To be a total order, every pair of elements must be comparable under it. Given $z, w \in \widehat{J_{\underline{s}}}$, $z \neq w$, the head-start condition requests that either $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w))$ or $\operatorname{Re} F^M(w) > \varphi(\operatorname{Re} F^M(z))$ for some $M \in \mathbb{N}$. In the first case we would have $z \succ w$ while the second one implies $w \succ z$.

Finally, note that it is a non-reflexive relation and thus a strict order. ■

Corollary 4.5.3 (Growth of real parts). *Consider $F \in \mathcal{B}^n_{log}$. Let $\underline{s}$ be an external address that satisfies the head-start condition for φ and let $z, w \in J_{\underline{s}}$. If $z \succ w$, then $z \in I(F)$. In particular, $J_{\underline{s}} \setminus I_{\underline{s}}$ consists of at most one point.*

Proof. Recall that Lemma 4.4.2 stated that if $z, w \in J_{\underline{s}}$ then

$$\lim_{k\to\infty} \max\{\operatorname{Re} F^k(z), \operatorname{Re} F^k(w)\} = \infty.$$

By Lemma 4.5.1, if $z \succ w$ then $\operatorname{Re} F^n(z) > \operatorname{Re} F^n(w)$ for all sufficiently large n. Then, from a moment on $\max\{\operatorname{Re} F^n(z), \operatorname{Re} F^n(w)) = \operatorname{Re} F^n(z)$ so

$$\infty = \lim_{k\to\infty} \max\{\operatorname{Re} F^k(z), \operatorname{Re} F^k(w)\} = \lim_{k\to\infty} \operatorname{Re} F^k(z).$$

which means that $z \in I(F)$. ■

Proposition 4.5.4 (Speed order topology vs subspace topology). *Let $\underline{s}$ be an external address satisfying the head-start condition for φ. Then the topology of $\widehat{J_{\underline{s}}}$ as a subset of the Riemann sphere $\widehat{\mathbb{C}}$ agrees with the order topology induced by $\succ$.*

Proof. The map id : $\widehat{J}_{\underline{s}} \subseteq \widehat{\mathbb{C}} \to (\widehat{J}_{\underline{s}}, \prec)$ is continuous. Indeed, let us see that the preimages of $(a,\infty)_\prec$ and $(-\infty,b)_\prec$ are open sets in $\widehat{J}_{\underline{s}}$ with the subspace topology. Take $z \in (a,\infty)_\prec$ and consider $k \in \mathbb{N}$ minimal such that $\operatorname{Re} F^k(z) > \varphi(\operatorname{Re} F^k(a))$. It must exists because $z \succ a$. Then since φ, Re and F are continuous functions, there exists a small enough neighbourhood $V \subseteq \widehat{\mathbb{C}}$ of z in satisfying $\operatorname{Re} F^k(w) > \varphi(\operatorname{Re} F^k(a))$ for all $w \in V$ and $V \cap \widehat{J}_{\underline{s}} \subseteq (a,\infty)_\prec$. Hence $(a,\infty)_\prec$ is open in $\widehat{J}_{\underline{s}} \subseteq \widehat{\mathbb{C}}$. Analogously, $(-\infty,b)_\prec$ is open too. Now we shall see that id^{-1} is continuous too. But recall that a continuous one-to-one map from a compact space onto a Hausdorff space is a homeomorphism. Therefore, by 2.4.4 id is a homeomorphism between $\widehat{J}_{\underline{s}} \subseteq \widehat{\mathbb{C}}$ and $(\widehat{J}_{\underline{s}}, \prec)$. ■

Corollary 4.5.5 (**Arcs in** $J_{\underline{s}}$)**.** *Let $\underline{s}$ be an external address satisfying the head-start condition for φ. Every component of $\widehat{J}_{\underline{s}}$ is either a point or is homeomorphic to $[0,1]$ preserving the order.*

Proof. Connected components of $\widehat{J}_{\underline{s}}$ are compact metric spaces. It is a direct consequence of Proposition 4.5.4 and the Order characterization of the arc (Corollary 2.5.3) from Section 2.5. ■

Corollary 4.5.6 (**Uniqueness of the unbounded component in** $J_{\underline{s}}$)**.** *Let $\underline{s}$ be an external address satisfying the head-start condition for φ and let $K \geqslant 0$ be such that $J_{\underline{s}}^K \neq \emptyset$. Suppose that if $z_0 \in J_{\underline{s}}^K$ there exists an unbounded closed connected set $A \subseteq J_{\underline{s}}$ with dist$(z_0, A) \leqslant 2\pi$. Then $J_{\underline{s}}$ has a unique unbounded component, which is a closed arc to infinity.*

Proof. We deduced the existence of this unbounded component in Theorem 4.4.3. By Corollary 4.5.5, connected components need to be arcs. Since infinity is the largest point in $(\widehat{J}_{\underline{s}}, \prec)$ it cannot be an interior point of the arc. Otherwise it would contradict Proposition 4.5.4. Thus, infinity must be the extreme point of a closed arc and all other components must be bounded. ■

Proposition 4.5.7 (**Points in the unbounded component of** $J_{\underline{s}}$)**.** *Let $\underline{s}$ be an external address that satisfies the head-start condition for φ. Then there exists $K' \geqslant 0$ such that $J_{\underline{s}}^{K'}$ is either empty or contained in the unbounded component of $J_{\underline{s}}$ (and this component is a closed arc). The value K' depends on F and φ, but not on $\underline{s}$.*

Proof. As usual, we can assume F to be normalized without loss of generality, $F \in \mathcal{B}^n_{log}$, which means that $H = \mathbb{H}$ and the expansivity property of F holds in all $\mathcal{T}$. Let $K \geqslant 0$ be like in Theorem 4.4.3 and Corollary 4.5.6 and set

$$K' := \max\{\varphi(0) + 1,\, K\}.$$

Let $z_0 \in J_{\underline{s}}^{K'}$ and consider $z_k := F^k(z_0)$ and

$$S_k := \{w \in J_{\sigma^k(\underline{s})} \ : \ w \succ z_k \text{ or } w = z_k\}$$

for all $k \geqslant 0$. By Corollary 4.5.6 there exists a unique unbounded component A_k of S_k and is an arc to infinity. Theorem 4.4.3 tells us that dist$(z_k, A_k) \leqslant 2\pi$.
Recall that by property (d) of the logarithmic transforms, the restriction to every tract $F_{|T} : T \to H$ is a conformal isomorphism. We want to apply $F_{|T}^{-1}$ to A_k and so before we

have to check that $A_k \subseteq \mathbb{H}$. Let $w \in J_{\sigma^k(\underline{s})}$ with $\operatorname{Re} w \leqslant 0$. Because of the monotonicity of φ, $\varphi(\operatorname{Re} w) \leqslant \varphi(0)$. On the other hand, since $z_0 \in J_{\underline{s}}^{K'}$ we have that for all $k \geqslant 1$

$$\operatorname{Re} z_k \geqslant K' \geqslant \varphi(0) + 1 > \varphi(0).$$

Putting these together we get $\operatorname{Re} z_k > \varphi(0) \geqslant \varphi(\operatorname{Re} w)$ and thus $w \prec z_k$. Therefore $w \notin S_k$ and, in particular, $w \notin A_k$. Hence $A_k \subseteq \mathbb{H}$ for all $k > 1$ and $F^{-1}_{|T_{k-1}}(A_k)$ is well defined, where T_i denotes the tract of F containing the point z_i. By construction,

$$F^{-1}_{|T_{k-1}}(A_k) \subseteq J_{\sigma^{k-1}(\underline{s})} \subseteq T_{k-1}.$$

Furthermore, $F^{-1}_{|T_{k-1}}(A_k) \subseteq A_{k-1}$. Indeed, if $w \in F^{-1}_{|T_{k-1}}(A_k)$ then either $F(w) \succ z_k$ (and $\operatorname{Re} F(w) > \varphi(\operatorname{Re} z_k)$) or $F(w) = z_k$. Since the speed ordering is a total order, w and z_{k-1} must be comparable. If $w \prec z_{k-1}$ then $\operatorname{Re} z_{k-1} > \varphi(\operatorname{Re} w)$ and, by the head-start condition,

$$\operatorname{Re} z_k = \operatorname{Re} F(z_{k-1}) > \varphi(\operatorname{Re} F(w)) > \operatorname{Re} F(w)$$

contradicting what we obtained before: $\operatorname{Re} F(w) > \varphi(\operatorname{Re} z_k)$ or $F(w) = z_k$. Therefore, the only possibilities are $w \succ z_{k-1}$ or $w = z_{k-1}$, which means that $w \in S_{k-1}$ and $F^{-1}_{|T_{k-1}}(A_k) \subseteq S_{k-1}$. Also, the preimage of an unbounded component cannot be bounded, therefore we have $F^{-1}_{|T_{k-1}}(A_k) \subseteq A_{k-1}$.
Finally, the expansivity property of F gives

$$\operatorname{dist}(z_0, A_0) \leqslant \frac{1}{|F'(w)|^k} \operatorname{dist}(z_k, A_k) \leqslant \frac{1}{2^k} \operatorname{dist}(z_k, A_k) \leqslant \frac{\pi}{2^{k-1}}$$

for all $k \geqslant 0$. Taking the limit we obtain $\operatorname{dist}(z_0, A_0) = 0$ and since A_0 is a closed set, $z_0 \in A_0$. ■

Theorem 4.5.8 (Ray tails). *Suppose that $F \in \mathcal{B}_{log}$ satisfies a head-start condition. Then for every escaping point z, there exists $k \in \mathbb{N}$ such that $F^k(z)$ is on a ray tail γ. This ray tail is the unique arc in $J(F)$ connecting $F^k(z)$ to ∞ (up to reparametrization).*

Proof. Let $z \in I(F) \subseteq J(F)$ and $\underline{s} = \sigma(z)$ its external address. Since F satisfies a head-start condition, this means that $\underline{s}$ satisfies the head-start condition for some function φ. Let K' be defined like in Proposition 4.5.7, then if $J_{\underline{s}}^{K'} \neq \emptyset$ it is contained in the unique unbounded component of $J_{\underline{s}}$ which is an arc to infinity and we will denote by A. The orbit of z tends to ∞, thus there exists $k \geqslant 0$ such that $F^k(z)$ enters $J_{\underline{s}}^{K'}$. Consider the set

$$\gamma_k := \{w \in I_{\sigma^k(\underline{s})} \ : \ w \succ F^k(z) \text{ or } w = F^k(z)\},$$

which, by Corollary 4.5.3, equals S_k (in the notation of Proposition 4.5.7) because $I_{\sigma^k(\underline{s})} = J_{\sigma^k(\underline{s})}$. Summarizing, we have

$$F^k(z) \in J_{\underline{s}}^{K'} \subseteq A \subseteq \gamma_k \subseteq I_{\underline{s}} = J_{\underline{s}}.$$

In fact, A is the only connected component of γ_k, $A = \gamma_k$. The extreme points of γ_k, $F^k(z)$ and ∞, both belong to A. By Corollary 4.5.5, all the intermediate points must belong to A too. Therefore, γ_k is an injective curve connecting $F^k(z)$ to ∞ and is unique. (!!!: falta) ■

4.6 Bounded slope and linear head-start condition

Since the head-start condition is quite technical and it is not easy to check it directly, we introduce some geometric facts that imply a linear head-start condition.

Definition 4.6.1 (**Linear head-start condition**). Let $K > 1$ and $M > 0$. We say that an external address $\underline{s}$ satisfies the *linear head-start condition with constants* K *and* M if it satisfies the head-start condition for

$$\varphi(t) := K \cdot t^+ + M$$

where $t^+ := \max\{t, 0\}$.

Observe that φ is an admissible function for the head-start condition. Either $\varphi'(t) = K > 1$ or $\varphi'(t) = 0$, in both cases $\varphi'(t) \geqslant 0$ and hence φ is a monotonically increasing function. On the other hand, if $t < 0$ we have $\varphi(t) = 0 > t$ and if $t \geqslant 0$,

$$\varphi(t) = Kt + M > Kt > t$$

because $M > 0$ and $K > 1$.

Definition 4.6.2 (**Bounded slope**). Let $F \in \mathcal{B}_{log}$. We say that the tracts of F have *bounded slope* (with constants $\alpha, \beta > 0$) if

$$|\operatorname{Im} z - \operatorname{Im} w| \leqslant \alpha \max\{\operatorname{Re} z, \operatorname{Re} w, 0\} + \beta$$

whenever z and w belong to a common tract of F. We denote the class of all functions with this property by $\mathcal{B}_{log}(\alpha, \beta)$ and use $\mathcal{B}^n_{log}(\alpha, \beta)$ to denote those that are also normalized.

Lemma 4.6.1 (**Characterization of bounded slope**). *A function F belongs to $\mathcal{B}_{log}(\alpha, \beta)$ if and only if exists a curve $\gamma : [0, \infty) \to \mathcal{T}$ with $|F(\gamma(t))| \to \infty$ and*

$$\limsup_{t\to\infty} \frac{|Im\,\gamma(t)|}{Re\,\gamma(t)} < \infty.$$

Hence, if one tract of F has bounded slope, then all tracts do.

Proof. Assume that $F \in \mathcal{B}_{log}(\alpha, \beta)$. Condition (b) in the definition of class $\mathcal{B}_{log}$ guarantees that every point in T can be connected to ∞ by a curve γ with unbounded real part. Since the restriction of F to each tract is a conformal isomorphism, it must happen that $|F(\gamma(t))|$ tends to ∞ as t approaches ∞. Let w be any point in T, eventually

$$\max\{\operatorname{Re}\gamma(t), \operatorname{Re} w, 0\} = \operatorname{Re}\gamma(t)$$

as $t \to \infty$. Suppose that

$$\limsup_{t\to\infty} \frac{|\operatorname{Im}\gamma(t)|}{\operatorname{Re}\gamma(t)} = \infty$$

then since $F \in \mathcal{B}_{log}(\alpha, \beta)$

$$\limsup_{t\to\infty} \frac{|\operatorname{Im}\gamma(t)|}{\operatorname{Re}\gamma(t)} = \limsup_{t\to\infty} \frac{|\operatorname{Im}\gamma(t)|}{\operatorname{Re}\gamma(t)} - \frac{|\operatorname{Im} w| + \beta}{\operatorname{Re}\gamma(t)} \leqslant \limsup_{t\to\infty} \frac{|\operatorname{Im}\gamma(t) - \operatorname{Im} w| - \beta}{\max\{\operatorname{Re}\gamma(t), \operatorname{Re} w, 0\}} \leqslant \alpha$$

raising a contradiction. Conversely, let $\gamma : [0, \infty) \to T$ satisfying

$$\lim_{t\to\infty} |F(\gamma(t))| = \infty, \quad \limsup_{t\to\infty} \frac{|\operatorname{Im}\gamma(t)|}{\operatorname{Re}\gamma(t)} = K < \infty.$$

Assume to the contrary that for every $\alpha, \beta > 0$ there exist $z, w \in T$ such that

$$|\operatorname{Im} z - \operatorname{Im} w| > \alpha \max\{\operatorname{Re} z, \operatorname{Re} w, 0\} + \beta.$$

If we fix a point $w \in T$ and the value of β this tells us that we can find points $z \in T$ such that

$$\frac{|\operatorname{Im} z - \operatorname{Im} w| - \beta}{\operatorname{Re} z}$$

is arbitrary large. But, by property (e), the whole tract T is contained in the band

$$B(\gamma) := \{z \in \mathbb{C} \ : \ \exists t \geqslant 1, \ |\operatorname{Im} z - \operatorname{Im}\gamma(t)| < 2\pi\}$$

around γ. Therefore, we could find points in γ where

$$\frac{|\operatorname{Im}\gamma(t) - \operatorname{Im} w| - \beta}{\operatorname{Re}\gamma(t)}$$

is arbitrary large, contradicting our assumption. ∎

Lemma 4.6.2 (Linear separation of orbits). *Let $F \in \mathcal{B}^n_{log}$ and let $\alpha, \beta > 0$. Let T be a tract of F, and suppose that $z, w \in \overline{T}$ satisfy $Re\, F(w) \geqslant Re\, F(z)$ and $|Im\, F(w) - Im\, F(z)| \leqslant \alpha Re\, F(w) + \beta$.*

(a) There exists a constant $\delta = \delta(\alpha, \beta)$ with the following property: if $|z - w| \geqslant \delta$, then

$$Re\, F(w) > e^{|z-w|/16\pi} Re\, F(z).$$

(b) Let $K \geqslant 1$ and $Q \geqslant 0$. Then there is a constant $\delta = \delta(\alpha, \beta, K, Q)$ with the following property: if $|z - w| \geqslant \delta$, then

$$Re\, F(w) > K\, Re\, F(z) + |z - w| + Q.$$

Proof. Let us prove (a) first, (b) is an extension of it.

(a) Set $\delta := \alpha + \beta + 2$ and $\delta := \max\{\delta', 16\pi \log \delta'\}$. We have

$$\begin{aligned} |F(w) - F(z)| &\leqslant |\operatorname{Re} F(w) - \operatorname{Re} F(z)| + |\operatorname{Im} F(w) - \operatorname{Im} F(z)| \\ &\leqslant \operatorname{Re} F(w) + \alpha \operatorname{Re} F(w) + \beta = (\alpha + 1)\operatorname{Re} F(w) + \beta \end{aligned}$$

because F is normalized ($\operatorname{Re} F(w), \operatorname{Re} F(z) > 0$) and hence $0 \leqslant \operatorname{Re} F(w) - \operatorname{Re} F(z) \leqslant \operatorname{Re} F(w)$. The expansivity of F gives

$$|F(z) - F(w)| \geqslant 2|z - w| \geqslant 2\delta \geqslant 2\delta' = 2\alpha + 2\beta + 4 > \alpha + \beta + 1.$$

Therefore, putting both inequalities together we get

$$\alpha + 1 + \beta < |F(z) - F(w)| \leqslant (\alpha+1)\mathrm{Re}\, F(w) + \beta$$

and conclude that $\mathrm{Re}\, F(w) > 1$. Thus,

$$|F(z) - F(w)| \leqslant (\alpha+1)\mathrm{Re}\, F(w) + \beta < (\alpha+1+\beta)\mathrm{Re}\, F(w) < \delta'\mathrm{Re}\, F(w).$$

We are assuming that $|z-w| \geqslant \delta \geqslant \delta' > 2$, then Lemma 4.4.1 gives

$$|F(z) - F(w)| \geqslant \exp\left(\frac{|z-w|}{8\pi}\right)\min\{\mathrm{Re}\, F(z), \mathrm{Re}\, F(w)\} = \exp\left(\frac{|z-w|}{8\pi}\right)\mathrm{Re}\, F(z)$$

and hence

$$\mathrm{Re}\, F(w) > \frac{|F(z)-F(w)|}{\delta'} \geqslant \frac{\exp(|z-w|/8\pi)\mathrm{Re}\, F(z)}{\delta'}.$$

We claim that

$$\frac{1}{\delta'}\exp\left(\frac{x}{8\pi}\right) \geqslant \exp\left(\frac{x}{16\pi}\right)$$

for all $x \geqslant 16\pi \log \delta'$ and in particular for $|z-w| \geqslant \delta \geqslant 16\pi\log\delta'$. Indeed,

$$\exp\left(\frac{x}{8\pi} - \frac{x}{16\pi}\right) = \exp\left(\frac{x}{16\pi}\right) \geqslant \delta' \quad \Leftrightarrow \quad \frac{x}{16\pi} \geqslant \log\delta' \quad \Leftrightarrow \quad x \geqslant 16\pi\log\delta'.$$

Thus,

$$\mathrm{Re}\, F(w) > \frac{\exp(|z-w|/8\pi)\mathrm{Re}\, F(z)}{\delta'} \geqslant e^{|z-w|/16\pi}\mathrm{Re}\, F(z).$$

(b) Let $\delta_0 = \delta(\alpha,\beta)$ from (a) and pick $\delta \geqslant \delta_0 + 1/2$ large enough so that all $x \geqslant \delta - 1/2 \geqslant \delta_0$ satisfy

$$e^{x/16\pi} > x + K + Q + 1/2.$$

Let $z' \in T$ the point with $\mathrm{Re}\, F(z') = \max\{1, \mathrm{Re}\, F(z)\}$ and $\mathrm{Im}\, F(z') = \mathrm{Im}\, F(z)$. It is well defined because F is normalized and $F_T : T \to \mathbb{H}$ is a conformal isomorphism. Then, by the expansivity of F

$$|z - z'| \leqslant \frac{1}{2}|F(z) - F(z')| = \frac{1}{2}|\mathrm{Re}\, F(z) - \mathrm{Re}\, F(z')| \leqslant \frac{1}{2}$$

because either $\mathrm{Re}\, F(z) = \mathrm{Re}\, F(z')$ or $\mathrm{Re}\, F(z) \leqslant 1$ and $\mathrm{Re}\, F(z') = 1$. We have

$$|w - z'| \geqslant |w-z| - |z-z'| \geqslant |w-z| - 1/2 \geqslant \delta - \frac{1}{2} \geqslant \delta_0$$

and also $|\mathrm{Im}\, F(w) - \mathrm{Im}\, F(z')| = |\mathrm{Im}\, F(w) - \mathrm{Im}\, F(z')| \leqslant \alpha\mathrm{Re}\, F(w) + \beta$. In the proof of (a) we have seen that the hypothesis on z and w imply that $\mathrm{Re}\, w > 1$. Hence,

$$\mathrm{Re}\, F(w) \geqslant \max\{1, \mathrm{Re}\, F(z)\} = \mathrm{Re}\, F(z').$$

Therefore we can apply (a) to z', w,

$$\begin{aligned}\mathrm{Re}\, F(w) &> e^{\frac{|w-z'|}{16\pi}}\mathrm{Re}\, F(z') > (|w-z'| + K + Q + 1/2)\cdot \mathrm{Re}\, F(z') \geqslant \\ &\geqslant K\mathrm{Re}\, F(z') + |w - z'| + Q + 1/2 \geqslant K\mathrm{Re}\, F(z) + |w-z| + Q.\end{aligned}$$

■

Note that the constant δ only depends on α, β, K, Q but not on the function nor the tract. The same constant is valid for all the tracts of F at the same time.

Corollary 4.6.3 (**Linear separation of orbits**). *Let $F \in \mathcal{B}^n_{log}(\alpha, \beta)$ and let $\underline{s}$ be an external address. If $z, w \in J_{\underline{s}}$ with $|z-w| \geqslant \delta(\alpha, \beta, K, 0)$, then*

$$\mathit{Re}\, F^k(z) > K \mathit{Re}\, F^k(w) + |z-w| \quad \mathit{or} \quad \mathit{Re}\, F^k(w) > K \mathit{Re}\, F^k(z) + |z-w|$$

for all $k \geqslant 1$.

Proof. We will proceed by induction. The case $k = 1$ is a particular case of (b) in Lemma 4.6.2. Let δ be the constant provided by the lemma in this initial case. Since z and w have the same external address, $F^k(z)$ and $F^k(w)$ will belong to the same tract of F for any value of k. Assume that the statement is true for some value $k = n$, suppose that $|F^{n-1}(z) - F^{n-1}(w)| \geqslant \delta$ and for instance

$$\operatorname{Re} F^n(w) > K \operatorname{Re} F^n(z) + |F^{n-1}(z) - F^{n-1}(w)|$$

(if $\operatorname{Re} F^n(z) > K \operatorname{Re} F^n(w) + |F^{n-1}(z) - F^{n-1}(w)|$ the procedure is analogous). Then we have

$$\begin{aligned} |F^n(z) - F^n(w)| \geqslant |\operatorname{Re} F^n(z) - \operatorname{Re} F^n(w)| &\geqslant \operatorname{Re} F^n(w) - \operatorname{Re} F^n(z) > \\ &> K \operatorname{Re} F^n(z) + |F^{n-1}(z) - F^{n-1}(w)| - \operatorname{Re} F^n(z) = \\ &= (K-1) \operatorname{Re} F^n(z) + |F^{n-1}(z) - F^{n-1}(w)| \geqslant \delta \end{aligned}$$

because $K \geqslant 1$, $\operatorname{Re} F(z) \geqslant 0$ and $|F^{n-1}(z) - F^{n-1}(w)| \geqslant \delta$ by assumption. Now we have to distinguish two cases: (!!!: falta algo?) $\operatorname{Re} F^{n+1}(w) \geqslant \operatorname{Re} F^{n+1}(z)$ (resp. $\operatorname{Re} F^{n+1}(z) \geqslant \operatorname{Re} F^{n+1}(w)$), the bounded slope of the tracts of F gives

$$|\operatorname{Im} F^{n+1}(w) - \operatorname{Im} F^{n+1}(z)| \leqslant \alpha \max\{\operatorname{Re} F^{n+1}(z), \operatorname{Re} F^{n+1}(w), 0\} + \beta = \alpha \operatorname{Re} F^{n+1}(w) + \beta$$

(resp. $|\operatorname{Im} F^{n+1}(w) - \operatorname{Im} F^{n+1}(z)| \leqslant \alpha \operatorname{Re} F^{n+1}(z) + \beta$). Recall that δ does not depend on the tract. ■

4.7 Wiggling of the tracts

Definition 4.7.1 (**Bounded wiggling**). Let $F \in \mathcal{B}_{log}$ and let T be a tract of F. We say that T has *bounded wiggling* if there exist $K > 1$ and $\mu > 0$ such that for every $z_0 \in \overline{T}$, every point z on the hyperbolic geodesic of T that connects z_0 to ∞ satisfies

$$(\operatorname{Re} z)^+ > \frac{1}{K} \operatorname{Re} z_0 - \mu.$$

We say that $F \in \mathcal{B}_{log}$ has *uniformly bounded wiggling* if the wiggling of all tracts of F is bounded by the same constants K, μ.

Lemma 4.7.1 (Domains with bounded wiggling). *Let V be an unbounded Jordan domain such that $\exp_{|V}$ is injective. Suppose that there are $K, M > 0$ such that every $z_0 \in V$ can be connected to ∞ by a curve $\gamma \subseteq V$ satisfying*

$$Re\, z \geqslant \frac{Re\, z_0}{K} - M$$

for all $z \in \gamma$. Then there is $M' > 0$ that depends only on M such that, for every $z_0 \in V$,

$$Re\, z \geqslant \frac{Re\, z_0}{K} - M'$$

for all z on the geodesic connecting z_0 to ∞.

Proposition 4.7.2 (Head-start and wiggling for bounded slope). *Let $F \in \mathcal{B}^n_{log}(\alpha, \beta)$ and let $K > 1$. Then the following are equivalent:*

(a) For some $M > 0$, F satisfies the uniform linear head-start condition with constants K and M.

(b) For some $\mu > 0$, the tracts of F have uniformly bounded wiggling with constants K and μ.

(c) For some $M' > 0$, the following holds: if T is a tract of F and $z, w \in \overline{T}$ with $Re\, w > K(Re\, z)^+ + M'$ and $|Im\, F(z) - Im\, F(w)| \leqslant \alpha \max\{Re\, F(z), Re\, F(w)\} + \beta$, then $Re\, F(w) > K\, Re\, F(z) + M'$.

Proof. We will see that (c)⇒(a)⇒(b)⇒(c).

(c)⇒(a) This implication is direct. Let $z, w \in J_{\underline{s}}$ for some external address $\underline{s}$. In particular, $F(z), F(w)$ belong to the same tract of F. The bounded slope of F gives

$$|\mathrm{Im}\, F(z) - \mathrm{Im}\, F(w)| \leqslant \alpha \max\{\mathrm{Re}\, F(z), \mathrm{Re}\, F(w)\} + \beta.$$

Suppose that $\mathrm{Re}\, w > \varphi(\mathrm{Re}\, z) = K(\mathrm{Re}\, z)^+ + M$, then by (c)

$$\mathrm{Re}\, F(w) > \varphi(\mathrm{Re}\, F(z)) = K(\mathrm{Re}\, F(z))^+ + M = K \mathrm{Re}\, F(z) + M.$$

We also need to check that $\exists k \in \mathbb{N}$ such that either $\mathrm{Re}\, F^k(z) > \varphi(\mathrm{Re}\, F^k(w))$ or $\mathrm{Re}\, F^k(w) > \varphi(\mathrm{Re}\, F^k(z))$. But this is a consequence of Corollary 4.6.3, if $|z - w| \geqslant \delta(\alpha, \beta, K, 0)$ then this is true for all $k \geqslant 1$. Recall that by the expansivity of F, there exists $j \in \mathbb{N}$ such that $|F^j(w) - F^j(z)| > 2$ and then Lemma 4.4.1 applies and thus $|F^k(z) - F^k(w)|$ eventually must be greater than any constant $\delta(\alpha, \beta, K, 0)$. Hence F satisfies the uniform linear head-start condition for φ.

(b)⇒(c) Let $\delta = \delta(\alpha, \beta, K, 0)$ from Lemma 4.6.2 and set

$$\widetilde{M} := K \cdot (\mu + 2\pi(\alpha + \beta)), \quad M := \max\{\delta, \widetilde{M}, 1\}.$$

Let T be a tract of F and let $z, w \in \overline{T}$ be such that

$$\mathrm{Re}\, w > K(\mathrm{Re}\, z)^+ + M, \quad |\mathrm{Im}\, F(z) - \mathrm{Im}\, F(w)| \leqslant \alpha \max\{\mathrm{Re}\, F(z), \mathrm{Re}\, F(w)\} + \beta.$$

Note that, in particular, $\operatorname{Re} w \geqslant (\operatorname{Re} z)^+ \geqslant \operatorname{Re} z$ and

$$\begin{aligned}|z-w| &\geqslant |\operatorname{Re} z - \operatorname{Re} w| = \operatorname{Re} w - \operatorname{Re} z > K(\operatorname{Re} z)^+ + M - \operatorname{Re} z \geqslant \\ &\geqslant K(\operatorname{Re} z)^+ + M - (\operatorname{Re} z)^+ = (K-1)(\operatorname{Re} z)^+ + M \geqslant M \geqslant \delta.\end{aligned}$$

By Lemma 4.6.2 (b), we only need to show that $\operatorname{Re} F(w) > \operatorname{Re} F(z)$. Suppose to the contrary that $\operatorname{Re} F(w) \leqslant F(z)$. Then by Lemma 4.6.2

$$\operatorname{Re} F(z) > K \operatorname{Re} F(w) + |z-w| > |z-w| > M \geqslant 1$$

Recall from the preliminaries, that the geodesic of $\mathbb{H}$ connecting $F(w)$ to ∞ is the horizontal ray

$$\Gamma(t) := F(w) + t, \quad t \geqslant 0.$$

and hence $\gamma(t) := F_T^{-1}(F(w)+t)$ is the geodesic of T connecting w to ∞. Let $y \in \Gamma$ such that $\operatorname{Re} y = \operatorname{Re} F(z)$ and let σ be the straight segment from $F(z)$ to y. We can estimate the hyperbolic length of σ as follows

$$\begin{aligned}\operatorname{dist}_{\mathbb{H}}(F(z),\Gamma) &\leqslant \operatorname{dist}_{\mathbb{H}}(F(z),y) \leqslant l_{\mathbb{H}}(\sigma) = \int_\sigma \frac{1}{\operatorname{Re} z}|dz| \leqslant \sup_{z\in\sigma}\left|\frac{1}{\operatorname{Re} z}\right| \cdot l(\sigma) = \\ &= \frac{1}{\operatorname{Re} F(z)} \cdot |\operatorname{Im} F(z) - \operatorname{Im} F(w)| \leqslant \frac{\alpha \max\{\operatorname{Re} F(z), \operatorname{Re} F(w)\} + \beta}{\operatorname{Re} F(z)} = \\ &= \frac{\alpha \operatorname{Re} F(z) + \beta}{\operatorname{Re} F(z)} = \alpha + \frac{\beta}{\operatorname{Re} F(z)} \leqslant \alpha + \beta.\end{aligned}$$

Now by the standard estimate and Pick's theorem we have

$$\operatorname{dist}(z,\gamma) \leqslant 2\pi \operatorname{dist}_{\mathcal{T}}(z,\gamma) = 2\pi \operatorname{dist}_{\mathbb{H}}(F(z),\Gamma) \leqslant 2\pi(\alpha+\beta)$$

i.e. γ intersects the Euclidean ball $B(z, 2\pi(\alpha+\beta))$ and hence

$$\min_{\zeta\in\gamma} \operatorname{Re}\zeta \leqslant \operatorname{Re} z + 2\pi(\alpha+\beta).$$

By the bounded wiggling condition, $(\operatorname{Re}\zeta)^+ > 1/K \cdot \operatorname{Re} w - \mu$ for all $\zeta \in \gamma$,

$$\begin{aligned}\operatorname{Re} w &< K((\operatorname{Re}\zeta)^+ + \mu) \leqslant K(\operatorname{Re} z + 2\pi(\alpha+\beta) + \mu) \leqslant \\ &\leqslant K(\operatorname{Re} z)^+ + \widetilde{M} \leqslant K(\operatorname{Re} z)^+ + M.\end{aligned}$$

This contradicts our assumptions, thus we have proved the claim. Applying Lemma 4.6.2 (b) we get $\operatorname{Re} F(w) > K\operatorname{Re} F(z) + M$.

(a)⇒(b) Assume that F satisfies the uniform linear head-start condition with constants K and M for some $M > 0$. Let T be a tract of F and $z \in \overline{T}$. By Theorem 4.5.8, there is $k \in \mathbb{N}$ such that $F^k(z)$ is on a ray tail. Therefore, there exists a set $\Gamma \subseteq I(F) \cap \mathbb{H}$ such that $\Gamma \cup \{\infty\}$ is an arc. Since $I(F) \subseteq \mathcal{T}$ is $2\pi i$-periodic and the real part of the points in $\mathbb{H}$ is bounded from below, we can choose a copy of Γ such that $\operatorname{dist}(F(z),\Gamma) \leqslant \kappa$ where $\kappa > 0$ is a constant independent of z and T. Since $F_{|T}$ is a conformal isomorphism, let $\zeta \in T$ be such that $\operatorname{dist}(F(z), \Gamma) = \operatorname{dist}(F(z), F(\zeta))$. Then

$$|z - \zeta| \leqslant \frac{1}{2}|F(z) - F(\zeta)| < \frac{\kappa}{2}$$

and ζ can be connected to ∞ along the curve $\gamma := F_T^{-1}(\Gamma) \subseteq I(F)$. At this point we claim that

$$\operatorname{Re}\zeta \leqslant K(\operatorname{Re} w)^+ + M = \varphi(\operatorname{Re} w)$$

for all $w \in \gamma$. Suppose by way of contradiction that there exists $w_0 \in \gamma$ such that

$$\operatorname{Re}\zeta > K(\operatorname{Re} w_0)^+ + M = \varphi(\operatorname{Re} w_0).$$

Since F satisfies the uniform head-start condition for φ,

$$\operatorname{Re} F^n(\zeta) > \varphi(\operatorname{Re} F^n(w_0)) = K(\operatorname{Re} F^n(w_0))^+ + M = K\operatorname{Re} F^n(w_0) + M$$

for all $n > 1$. Since $\zeta \prec w_0$ in γ, where $\prec$ denotes the speed ordering, there must exist $n_0 \in \mathbb{N}$ such that

$$\operatorname{Re} F^{n_0}(w) > \varphi(\operatorname{Re} F^{n_0}(\zeta)) = K(\operatorname{Re} F^{n_0}(\zeta))^+ + M = K\operatorname{Re} F^{n_0}(\zeta)) + M$$

but then using the head-start condition

$$\operatorname{Re} F^{n_0}(w) > K(K\operatorname{Re} F^{n_0}(w) + M) + M = K^2\operatorname{Re} F^{n_0}(w) + (KM + M),$$
$$0 > (K^2 - 1)\operatorname{Re} F^{n_0}(w) + (KM + M)$$

which is not possible because $K^2 - 1, \operatorname{Re} F^{n_0}(w), KM + M \geqslant 0$. Therefore we have proved our claim. Then,

$$(\operatorname{Re} w)^+ \geqslant \frac{\operatorname{Re}\zeta}{K} - \frac{M}{K}$$

for all $w \in \gamma$ and there is a curve $\gamma' \subseteq T$ extending γ to z such that

$$(\operatorname{Re} w)^+ \geqslant \frac{\operatorname{Re}\zeta}{K} - \frac{M}{K} - \frac{\kappa}{2} \geqslant \frac{\operatorname{Re} z}{K} - \frac{\kappa}{K} - \frac{M}{K} - \frac{\kappa}{2}$$

for all $w \in \gamma'$. Recall that by property (b) of the logarithmic transform, the real part of the points in the tracts of F is bounded from below. Call

$$L := \inf_{z \in \mathcal{T}} \operatorname{Re} z$$

then we have

$$\operatorname{Re} w \geqslant (\operatorname{Re} w)^+ - L \geqslant \frac{\operatorname{Re} z}{K} - \frac{\kappa}{K} - \frac{M}{K} - \frac{\kappa}{2} - L =: \frac{\operatorname{Re} z}{K} - \mu_1, \quad \mu_1 > 0$$

for all $w \in \gamma'$. However, γ' may not be the geodesic connecting z to ∞. Let us call $\tilde{\gamma}$ this geodesic and let $w \in \tilde{\gamma}$. By Lemma 4.7.1, there exists $\mu_2 > 0$ depending only on μ_1 such that

$$\operatorname{Re} w \geqslant \frac{\operatorname{Re} z}{K} - \mu_2$$

for all $w \in \tilde{\gamma}$. Finally,

$$(\operatorname{Re} w)^+ \geqslant \operatorname{Re} w \geqslant \frac{\operatorname{Re} z}{K} - \mu_2 > \frac{\operatorname{Re} z}{K} - \frac{\mu_2}{2} =: \frac{\operatorname{Re} z}{K} - \mu, \quad \mu > 0$$

and therefore the tracts of F have uniformly bounded wiggling with constants K and μ.

■

4.8 Geometry of functions of finite order

Definition 4.8.1 (Order of an entire function). The order (at infinity) of an entire function is defined as

$$\rho(f) := \lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|}.$$

Equivalently,

$$\rho(f) = \inf\{m \in \mathbb{R}\cup\{\infty\} \ : \ f(z) = O(\exp(|z|^n)) \text{ as } z \to \infty\}.$$

We say that f has finite order if $\rho(f) < \infty$, that is $\log\log|f(z)| = O(\log|z|)$ as $|z| \to \infty$.

Example 4.8.1. The functions $f_m(z) = \exp(|z|^m)$ have finite order m, while $g(z) = \exp(\exp(z))$ has not finite order. Indeed,

$$\rho(g) = \lim_{r\to\infty} \sup_{|z|=r} \frac{|z|}{\log|z|} = \lim_{r\to\infty} \frac{r}{\log r} = +\infty.$$

We want to characterize the functions $F \in \mathcal{B}_{log}$ that come out as logarithmic coordinates of functions $f \in \mathcal{B}$ of finite order.

Definition 4.8.2 (Finite order for $\mathcal{B}_{log}$). We say that $F \in \mathcal{B}_{log}$ has finite order if

$$\log \operatorname{Re} F(w) = O(|\operatorname{Re} w|)$$

as $\operatorname{Re} w \to \infty$ in $\mathcal{T}$.

Lemma 4.8.1. *A function $f \in \mathcal{B}$ has finite order if and only if any logarithmic transform $F \in \mathcal{B}_{log}$ of f has finite order in the sense of Definition 4.8.2.*

Proof. If F is a logarithmic transform of f we have

$$f(z) = \exp F(w), \quad z = \exp(w)$$

and hence

$$\lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|} = \lim_{r\to\infty} \sup_{|e^w|=r} \frac{\log\log|\exp(F(w))|}{\log|\exp(w)|} =$$

$$= \lim_{r\to\infty} \sup_{|e^{\operatorname{Re} w}|=r} \frac{\log\log|\exp(\operatorname{Re} F(w))|}{\log|\exp(\operatorname{Re} w)|} = \lim_{\operatorname{Re} w\to\infty} \frac{\log \operatorname{Re} F(w)}{\operatorname{Re} w}$$

showing the equivalence. ∎

Theorem 4.8.2 (Spiral theorem). *Suppose that $F \in \mathcal{B}^n_{log}$ has finite order. Then the tracts of F have bounded slope.*

Proof. Let T be a tract of F and set

$$\varrho := \sup\left\{\frac{\log \operatorname{Re} F(z)}{\operatorname{Re} z} \ :\ z \in \mathcal{T},\ \operatorname{Re} z, \operatorname{Re} F(z) \geqslant 1\right\} < \infty$$

which is well defined because F has finite order. Consider the central geodesic $\gamma : [1,\infty) \to T$ defined by $\gamma(t) = F_T^{-1}(t)$. The length of the geodesic is given by

$$l_{\mathbb{H}}(\gamma([1,t])) = \int_1^t \frac{dt}{t} = \big(\log t\big]_1^t = \log t$$

Then for every $t \geqslant 1$, using the standard estimate, Pick's theorem and the finite order of F

$$|\gamma(t)| - |\gamma(1)| \leqslant |\gamma(t) - \gamma(1)| \leqslant 2\pi l_T(\gamma([1,t])) = 2\pi l_{\mathbb{H}}(\gamma([1,t])) = 2\pi \log t \leqslant 2\pi\varrho \operatorname{Re}\gamma(t).$$

Thus, we have found an asymptotic curve satisfying

$$|\operatorname{Im}\gamma(t)| \leqslant |\gamma(t)| \leqslant 2\pi\varrho\operatorname{Re}\gamma(t) + |\gamma(1)|$$

which by Lemma 4.6.1 is equivalent to the bounded slope condition. ∎

This is a version of the Ahlfors non-spiralling theorem.

Theorem 4.8.3 (Finite order functions have good geometry). *Suppose that $F \in \mathcal{B}^n_{log}$ has finite order. Then the tracts of F have bounded slope and (uniformly) bounded wiggling.*

Proof. By Theorem 4.8.2, $F \in \mathcal{B}^n_{log}(\alpha,\beta)$ for some α,β. Since F has finite order, there exist $\varrho, M \geqslant 0$ such that

$$\log \operatorname{Re} F(z) \leqslant \varrho\operatorname{Re} z + M$$

for all $z \in \mathcal{T}$. Let T be a tract of F and consider $z \in \overline{T}$. Assume by now that $\operatorname{Re} F(z) \geqslant 1$. As we saw in the preliminaries, the geodesic connecting z to ∞ in $\mathbb{H}$ is a horizontal ray, consider its preimage

$$\gamma(t) := F_T^{-1}(F(z)+t), \quad t \geqslant 0.$$

Using Pick's theorem, we can compute the distance between z and the points of γ:

$$\operatorname{dist}_{\mathcal{T}}(z,\gamma(t)) = \operatorname{dist}_{\mathbb{H}}(F(z),\gamma(t)) = \int_1^{1+t}\frac{dt}{t} = \big(\log(t)\big]_1^{t+1} = \log(1+t).$$

By the standard estimate,

$$\operatorname{Re} z - \operatorname{Re}\gamma(t) \leqslant |z-\gamma(t)| \leqslant 2\pi\operatorname{dist}_{\mathcal{T}}(z,\gamma(t)) = 2\pi\log(1+t),$$

since we are assuming $\operatorname{Re} F(z) \geqslant 1$,

$$2\pi\log(1+t) \leqslant 2\pi\log(\operatorname{Re} F(z)+t) = 2\pi\log(\operatorname{Re} F(\gamma(t)),$$

and using the finite order condition,

$$2\pi\log(\operatorname{Re} F(\gamma(t)) \leqslant 2\pi\log(\operatorname{Re} F(z)) \leqslant 2\pi(\varrho \operatorname{Re}\gamma(t) + M).$$

Thus,

$$\operatorname{Re} z - \operatorname{Re}\gamma(t) \leqslant 2\pi(\varrho \operatorname{Re}\gamma(t) + M)$$

or equivalently,

$$\operatorname{Re} z \leqslant (2\pi\varrho + 1) \operatorname{Re}\gamma(t) + 2\pi M,$$

$$(\operatorname{Re}\gamma(t))^+ = \operatorname{Re}\gamma(t) \geqslant \frac{1}{2\pi\varrho+1}\operatorname{Re} z - \frac{2\pi M}{2\pi\varrho+1}.$$

Since z was arbitrary, this tells us that F has uniformly bounded wiggling with constants

$$K = 2\pi\varrho + 1 > 1, \quad \mu = \frac{2\pi M}{2\pi\varrho+1} > 0.$$

It remains to cover the case where $\operatorname{Re} F(z) < 1$. Let w be a point in the geodesic connecting z to ∞ such that $\operatorname{Re} F(w) = 1$. If z_1, z_2 are any two points in the geodesic connecting z to w, the expansivity of F gives

$$|z_1 - z_2| \leqslant \frac{1}{2}|F(z) - F(w)| \leqslant \frac{1}{2}$$

and hence $\gamma'(t)$ has bounded Euclidean diameter. Thus, this can be absorbed in the constant μ and does not matter for the bounded wiggling. ■

4.9 Proof of the main theorem

Now we have all the tools to prove the theorem for functions of class $\mathcal{B}$ of finite order. Note that class $\mathcal{B}$ is closed under finite composition, but the composition of two functions of finite order need not to have finite order in general. The following lemma completes the argument.

Lemma 4.9.1 (Linear head-start is preserved by composition). *Let $F_i : \mathcal{T}_{F_i} \to \mathbb{H}$ be in $\mathcal{B}^n_{log}$, for $i = 1, 2, \ldots, n$. Then there is an $a \geqslant 0$ so that if $\tau_a(z) = z - a$ then $G_a := \tau_a \circ F_n \circ \cdots \circ F_1 \in \mathcal{B}^n_{log}$ on appropriate tracts $\mathcal{T}_a \subseteq \mathcal{T}_{F_1}$, so that G_a is a conformal isomorphism from each component of $\mathcal{T}_a$ onto $\mathbb{H}$. If all F_i have bounded slope and satisfy uniform linear head-start conditions, then G_a also has bounded slope and satisfies a uniform linear head-start condition.*

Proof. Since each F_i is normalized, $F_i : \mathcal{T}_{F_i} \to \mathbb{H}$. For every $i \geqslant 2$ we can define

$$a_i := \sup_{z \in \mathcal{T}_{F_i} \setminus \mathbb{H}} \operatorname{Re} F_i(z) \geqslant 0$$

if $\mathcal{T}_{F_i} \setminus \mathbb{H} \neq \emptyset$ and $a_i := 0$ otherwise. Then $F_i^{-1}(\mathbb{H}_{a_i}) \subseteq \mathbb{H}$ for $i = 2, \ldots, n$, where

$$\mathbb{H}_R := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > R\}.$$

Set $a = a_n \geqslant 0$ and consider

$$\mathcal{T}_a := (F_n \circ \cdots \circ F_1)^{-1}(\mathbb{H}_a) = F_1^{-1} \circ \cdots \circ F_n^{-1}(\mathbb{H}_a) \subseteq \mathcal{T}_{F_1}.$$

Note that for every i we have chosen a branch of F_i^{-1} to a specified tract T_{i_j} of F_i so that $F_{i|T_{i_j}}$ is a conformal isomorphism. Thus, the restriction of $F := (F_n \circ \cdots \circ F_1) : \mathcal{T}_a \to \mathbb{H}_a$ to every connected component of $\mathcal{T}_a$ is a composition of conformal isomorphisms and hence, a conformal isomorphism. The geometric properties of the tracts in the definition of class $\mathcal{B}_{log}$ are inherited from F_1, $\mathcal{T}_a \subseteq \mathcal{T}_{F_1}$. Thus, the function

$$G_a := \tau_a \circ F : \mathcal{T}_a \to \mathbb{H}$$

belongs to $\mathcal{B}_{log}^n$. Moreover, since the characteristic of having bounded slope is intrinsic to the geometry of the tracts, as $\mathcal{T}_a \subseteq \mathcal{T}_{F_1}$ and F_1 has bounded slope, G_a has bounded slope too.

It remains to show that G_a satisfies a uniform linear head-start condition. Let α, β be such that $F_i \in \mathcal{B}_{log}^n(\alpha, \beta)$ for $i = 1, \ldots, n$. Note that if $F \in \mathcal{B}_{log}^n(\alpha, \beta)$, then $F \in \mathcal{B}_{log}^n(\alpha', \beta')$ for all $\alpha' \geqslant \alpha$ and $\beta' \geqslant \beta$. For instance, if $F_i \in \mathcal{B}_{log}^n(\alpha_i, \beta_i)$, take as α and β the maximums of α_i and β_i respectively. Assume that F_i satisfies the uniform linear head-start condition with constants $K_i > 1$ and $M_i > 0$ and let M_i' be the constant provided by Proposition 4.7.2 (c). Now set

$$K = \max_i K_i, \quad M = \max\{\delta,\ \max_i M_i'\}$$

where $\delta = \delta(\alpha, \beta, K, 0)$ from Lemma 4.6.2 (b). Fix i and let T be a tract of F_i. Let $w, z \in T$ such that

$$\operatorname{Re} w > K(\operatorname{Re} z)^+ + M$$

and such that $F_i(w)$ and $F_i(z)$ belong to the same tract of F_{i+1} (if $i = n$, we use the convention $F_{i+1} = F_1$). Note that in particular $\operatorname{Re} w > \operatorname{Re} z$. Then

$$|w - z| \geqslant |\operatorname{Re} w - \operatorname{Re} z| = \operatorname{Re} w - \operatorname{Re} z > K(\operatorname{Re} z)^+ + M - \operatorname{Re} z \geqslant M \geqslant \delta.$$

The inequality $K(\operatorname{Re} z)^+ + M - \operatorname{Re} z \geqslant M$ holds because if $\operatorname{Re} z > 0$,

$$K(\operatorname{Re} z)^+ + M - \operatorname{Re} z = K\operatorname{Re} z + M - \operatorname{Re} z = (K-1)\operatorname{Re} z + M > M$$

while if $\operatorname{Re} z \leqslant 0$, $K(\operatorname{Re} z)^+ + M - \operatorname{Re} z \geqslant M = M - \operatorname{Re} z \geqslant M$. In this situation, Corollary 4.6.3 gives that

$$\operatorname{Re} F_i(z) > K\operatorname{Re} F_i(w) + |w - z| \quad \text{ or } \quad \operatorname{Re} F_i(w) > K\operatorname{Re} F_i(z) + |w - z|$$

and in particular, since $|w - z| \geqslant M$, we have

$$\operatorname{Re} F_i(z) > K\operatorname{Re} F_i(w) + M \quad \text{ or } \quad \operatorname{Re} F_i(w) > K\operatorname{Re} F_i(z) + M.$$

Finally, by Proposition 4.7.2 (c), there is some $M' > 0$ such that

$$\operatorname{Re} F(w) > K\operatorname{Re} F(z) + M'$$

which contradicts the inequality in the left hand side:

$$\frac{\operatorname{Re} F(z) - M}{K} > \operatorname{Re} F(w) > K\operatorname{Re} F(z) + M',$$

$$\operatorname{Re} F(z) - M > K^2\operatorname{Re} F(z) + K^2 M',$$

$$0 > (K^2 - 1)\operatorname{Re} F(z) + K^2 M' + M$$

but $K^2-1, \operatorname{Re} F(z), k^2M'+M \geqslant 0$. Therefore,

$$\operatorname{Re} F_i(w) > K \operatorname{Re} F_i(z) + M$$

and hence all F_i satisfy the uniform linear head-start condition with constants K, M. Let $\varphi(t) = Kt^+ + M$, if $w, z \in \mathcal{T}_a$ we have

$$\operatorname{Re} w > \varphi(\operatorname{Re} z) \Rightarrow \operatorname{Re} F_1(w) > \varphi(\operatorname{Re} F_1(z)) \Rightarrow \cdots \Rightarrow \operatorname{Re} F(w) > \varphi(\operatorname{Re} F(z))$$

and then

$$\begin{aligned}\operatorname{Re} G_a(w) &= \operatorname{Re} F(w) - a > \varphi(\operatorname{Re} F(z)) - a = K(\operatorname{Re} F(z))^+ + M - a = \\ &= K\operatorname{Re} F(z) + M - a > K\operatorname{Re} F(z) + M - Ka = K(\operatorname{Re} F(z) - a) + M = \\ &= K(\operatorname{Re} G_a(z)) + M = K(\operatorname{Re} G_a(z))^+ + M = \varphi(\operatorname{Re} G_a(z)).\end{aligned}$$

Hence, G_a satisfies the uniform linear head-start condition with constants K and M. ■

Now we proceed to prove the main theorem in [RRRS11] giving a partial positive result on strong Eremenko's conjecture.

Theorem 4.9.2 (Entire functions with dynamic rays). *Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in I(f)$ can be connected to ∞ by a curve γ such that ${f^n}_{|\gamma} \to \infty$ uniformly.*

Proof. Let $f_1, \dots f_n \in \mathcal{B}$ be functions of finite order. By applying a suitable affine change of variable to each f_i, we may assume without loss of generality that F_i is a normalized logarithmic transform of f_i. By Theorem 4.8.3, the tracts of F_i have bounded slope and (uniformly) bounded wiggling for all i. Then, by Proposition 4.7.2 each F_i satisfies a linear head-start condition. Applying Lemma 4.9.1, there is an $a \geqslant 0$ such that $G_a(z) := \tau_a \circ F_n \circ \cdots \circ F_1 \in \mathcal{B}^n_{log}$ satisfies a uniform linear head-start condition, where $\tau_a(z) = z - a$. On a sufficiently restricted domain $\mathcal{T} \subseteq \mathbb{H}$,

$$F := G_a \circ \tau_a^{-1} = \tau_a \circ F_n \circ \cdots \circ F_1 \circ \tau_a^{-1} : \mathcal{T} \to \mathbb{H}$$

is a logarithmic transform of $f = f_n \circ \cdots \circ f_1$ and satisfies a linear head-start condition. Now Theorem 4.5.8 tells us that every escaping point z is eventually mapped to a ray tail γ in $J(F)$ connecting $F^k(z)$ to ∞. Finally, by Proposition 3.7.3 z must be on a ray tail. ■

4.10 Disjoint-type functions

Among all the functions in class $\mathcal{B}$, in some sense the disjoint-type functions behave in a nicer way and enjoy better properties.

Definition 4.10.1 (Disjoint-type function). An entire function $f \in \mathcal{B}$ is said to be of disjoint-type if there is a bounded Jordan domain $D \subseteq \mathbb{C}$ such that $S(f) \subseteq D$ and the tracts of f, $\mathcal{V} = f^{-1}(\mathbb{C} \setminus \overline{D})$, satisfy that $\overline{\mathcal{V}} \cap \overline{D} = \emptyset$. Or, in other words, the tracts of f are compactly contained in $\mathbb{C} \setminus \overline{D}$.

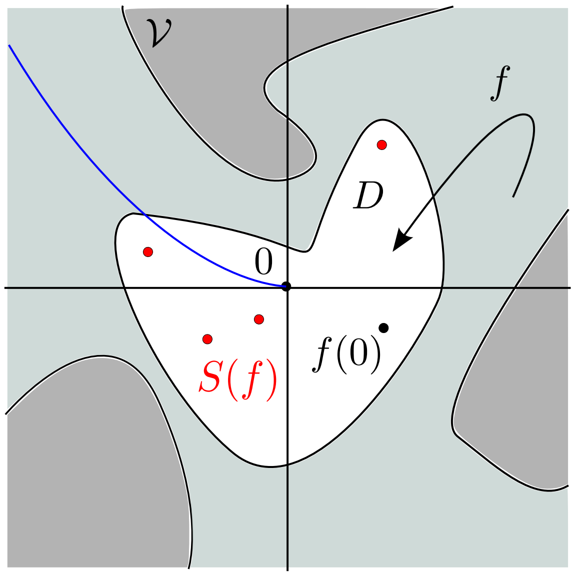

Figure 4.8: A disjoint-type function $f \in \mathcal{B}$.

Example 4.10.1. The function $f(z) =$ that we considered in Section 3.6 is of disjoint type.

The same notion can be defined in $\mathcal{B}_{log}$ easily. Then, the following lemma explains what is the relation between these classes of functions.

Definition 4.10.2 (Disjoint-type logarithmic transform). Let $F \in \mathcal{B}_{log}$, we say that $F : \mathcal{T} \to H$ is a disjoint-type function if $\overline{\mathcal{T}} \subseteq H$.

Lemma 4.10.1. *A function $f \in \mathcal{B}$ is of disjoint type if and only if f has a logarithmic transform of disjoint type.*

Remark 4.10.1. We may not be able to normalize a disjoint-type logarithmic transform preserving the disjoint-type property.

Proposition 4.10.2. *Let $f \in \mathcal{B}$ and let F be a logarithmic transform of F. If F is of disjoint type, then* $\exp(J(F)) = J(f)$.

Maybe this is the best benefit of the disjoint-type case. This tells us that the logarithmic transform F encodes all the information from the Julia set $J(f)$.

Lemma 4.10.3 (Characterization of disjoint-type). *A function $f \in \mathcal{B}$ is of disjoint-type if and only if $S(f)$ is contained in the immediate basin of an attracting fixed point of f.*

Proof. Let $f \in \mathcal{B}$ be of disjoint type, this means that $\mathcal{V} \cap W = \emptyset$ where $S(f) \subseteq D$, $W = \mathbb{C} \setminus \overline{D}$ and $\mathcal{V} = f^{-1}(W)$. The definition of $\mathcal{V}$ gives

$$f(\mathbb{C} \setminus \overline{\mathcal{V}}) \subseteq \mathbb{C} \setminus \overline{W} = D.$$

In particular, since $D \subseteq \mathbb{C} \setminus \overline{\mathcal{V}}$ by the disjoint-type property, we have $f(D) \subseteq D$. Recall that D is a bounded Jordan domain hence homeomorphic to a disk. Brouwer fixed point

theorem asserts that there exists an attracting fixed point $z_0 \in D$ and D is contained in its immediate basin of attraction. Thus, $S(f) \subseteq D \subseteq A^*_f(z_0)$.
Conversely, if $S(f) \subseteq A^*_f(z_0)$ we can take $D \subseteq A^*_f(z_0)$ a bounded Jordan domain containing $S(f)$ such that $f(D) \subseteq D$ (in particular $z_0 \in D$). Note that the whole $A^*_f(z_0)$ may not be a bounded Jordan domain, for instance it can be unbounded. This choice of the domain D leads to a disjoint-type logarithmic transform for f. Indeed, $f^{-1}(\mathbb{C} \setminus \overline{D}) \subseteq \mathbb{C} \setminus \overline{D}$ and

$$\mathcal{V} \cap D = f^{-1}(\mathbb{C} \setminus \overline{D}) \cap D = \emptyset.$$

■

Lemma 4.10.4 (Uniform expansion for disjoint-type maps). *Suppose $F : \mathcal{T} \to H$ is of disjoint type. Then there exists a constant $\Lambda > 1$ such that the derivative of F with respect to the hyperbolic metric on H satisfies $\|DF(z)\|_H = \lambda_{\mathcal{T}}(z)/\lambda_H(z) \geqslant \Lambda$ for all $z \in \mathcal{T}$. In particular, $dist_H(F(z), F(w)) \geqslant \Lambda dist_H(z, w)$ whenever z and w belong to the same tract of F.*

Proof. The derivative of $F : \mathcal{T} \to H$ with respect to the hyperbolic metric on H is defined as

$$\|DF(z)\|_H := |F'(z)| \cdot \frac{\rho_H(F(z))}{\rho_H(z)}.$$

For every tract T of $\mathcal{T}$, $F : T \to H$ is a conformal isomorphism by property (d). Therefore, F is a local isometry, this is

$$|F'(z)| \cdot \rho_H(F(z)) = \rho_{\mathcal{T}}(z)$$

and so

$$\|DF(z)\|_H = \frac{\rho_{\mathcal{T}}(z)}{\rho_H(z)}.$$

Since F is of disjoint-type, $\overline{T} \subsetneq H$, Pick's theorem (Theorem 1.6.3) tells us that

$$\rho_T(z) > \rho_H(z)$$

which gives $\|DF(z)\|_H > 1$. Now we have to see that, in fact, it is away from 1. By continuity, $\|DF(z)\|_H$ could only take values close to 1 in the boundary of T. If z tends to the boundary of T in $\mathbb{C}$ we know

$$\lim_{z \to \partial T} \frac{\rho_T(z)}{\rho_H(z)} \to \infty,$$

note that points in T are away from ∂H and hence $\rho_H(z)$ is bounded. Since T is an unbounded Jordan domain, we also have to check the directions with unbounded points. In this case, the real part of the points is unbounded. We would like that

$$\liminf_{\mathrm{Re}\, z \to +\infty} \frac{\rho_{\mathcal{T}}(z)}{\rho_H(z)} > 1.$$

By property (e), $\mathcal{T}$ cannot contain vertical segments of length 2π. Therefore, if $z \in \mathcal{T}$

$$\mathrm{dist}\,(z, \partial \mathcal{T}) \leqslant \pi$$

and hence, using the standard estimate,

$$\rho_{\mathcal{T}}(z) \geqslant \frac{1}{2\mathrm{dist}\,(z, \partial\mathcal{T})} \geqslant \frac{1}{2\pi}.$$

From that we have

$$\liminf_{\mathrm{Re}\, z \to +\infty} \frac{\rho_{\mathcal{T}}(z)}{\rho_H(z)} \geqslant \liminf_{\mathrm{Re}\, z \to +\infty} \frac{1}{2\pi \cdot \rho_H(z)} > 1$$

because $\rho_H(z) \to 0$ as $\mathrm{Re}\, z \to +\infty$, it goes away from the boundary. We conclude that there exists a constant $\Lambda > 1$ such that $\|DF(z)\|_H \geqslant \Lambda$. Furthermore, we have seen that

$$\rho_{\mathcal{T}}(z) \geqslant \Lambda \cdot \rho_H(z).$$

In particular, combining this with Pick's theorem applied to the conformal isomorphism F_T^{-1},

$$\mathrm{dist}_H(F(z), F(w)) = \mathrm{dist}_T(z, w) \geqslant \Lambda \cdot \mathrm{dist}_H(z, w).$$

■

Proposition 4.10.5 (Disjoint-type maps and linear head-start). *Let $F : \mathcal{T} \to H$ be a disjoint-type map in $\mathcal{B}_{log}(\alpha, \beta)$, and let $R > 0$ be such that $\mathbb{H}_R \subseteq H$. Then F satisfies a uniform linear head-start condition if and only if the map $\tilde{F} := F_{|F^{-1}(\mathbb{H}_R)}$ satisfies a head-start condition.*

You can find the corresponding proof in [RRRS11, Proposition 5.9]. Since the argument in the proof of the main theorem is based on normalized functions but we may not be able to normalize a function preserving the disjoint-type property, this proposition allows us to prove an analogue result for disjoint-type logarithmic transforms without the need of normalizing them.

Theorem 4.10.6 (Disjoint-type maps). *Let $f = f_1 \circ f_2 \circ \cdots \circ f_n$, where $f_i \in \mathcal{B}$ for all i, and all f_i have finite order. Suppose that $S(f) \subseteq F(f)$ and that $F(f)$ consists only of the immediate basin of an attracting fixed point of f. Then every component of $J(f)$ is a dynamic ray together with a single landing point; in particular, every point of $I(f)$ is on a ray tail of f.*

Proof. Since $S(f) \subseteq F(f) = A_f^*(z_0)$ where z_0 denotes an attracting fixed point of f (the only one), by the characterization in Lemma 4.10.3, f has a disjoint-type logarithmic transform F. Hence, by Proposition 4.10.2, $\exp(J(F)) = J(f)$. Following the reasoning in the proof of Theorem 4.9.2, there is $R > 0$ such that $F_{|F^{-1}(\mathbb{H}_R}$ satisfies a uniform linear head-start condition. By Proposition 4.10.5, F satisfies a uniform linear head-start condition. Thus, Theorem 4.5.8 and Proposition 3.7.3 apply and show that every component of $J(f)$ is a ray tail. ■

4.11 Existence of Cantor bouquets

Recall that we have introduced Cantor bouquets in Section 3.5. We will see that the Julia sets of the functions considered in this chapter contain Cantor bouquets. More precisely we will prove that if a logarithmic transform $F \in \mathcal{B}_{log}$ of some function $f \in \mathcal{B}$ satisfies a head-start condition, then $J(F)$ (and hence $J(f)$ too) contains a set with a brush structure.

Theorem 4.11.1 (Existence of absorbing brush). *Suppose that $F \in \mathcal{B}_{log}$ satisfies a head-start condition. Then there exists a closed $2\pi i$-periodic subset $X \subseteq J(F)$ with the following properties:*

(a) $F(X) \subseteq X$;

(b) each connected component C of X is a closed arc to infinity all of whose points except possibly the finite endpoint escape;

(c) every escaping point of F enters X after finitely many iterations. If F satisfies the uniform head-start condition for some function, then there exists $K' > 0$ such that $J^{K'}(F) \subseteq X$.

If, additionally, F is of disjoint type, then we may choose $X = J(F)$.

Proof. Let X denote the union of all unbounded components of $J(F)$. The set $J(F) \cup \{\infty\} \subseteq \widehat{\mathbb{C}}$ may have some connected components, let $\widehat{X}$ be the one containing ∞. Consider $E = \widehat{X} \cap \mathbb{C} \subsetneq \widehat{X}$. By the Boundary bumping theorem stated in the preliminaries, or more precisely by its Corollary 2.2.4, every connected component of E is unbounded, hence $E = X$. By Corollary 4.5.5, $J(F) \cup \{\infty\} \subseteq \widehat{\mathbb{C}}$ is a union of arcs (sets homeomorphics to I) and therefore is a compact set in $\widehat{\mathbb{C}}$. Recall that closed subsets of compact spaces are compact. Thus, $X = \widehat{X} \cap \mathbb{C}$ is a closed set in $\mathbb{C}$. Let us check the properties (a) to (c):

(a) By definition, $J(F)$ is an invariant set under F. An unbounded component of $J(F)$ cannot be mapped to one of its bounded components because this would lead to the existence of an asymptotic value in $J(F)$ which is not possible.

(b) By Corollary 4.5.5, each connected component of X is a closed arc to infinity and, by Corollary 4.5.3, all their points must escape with the only possible exception of the finite endpoint.

(c) Escaping points end up entering J^K for all $K > 0$. By Proposition 4.5.7, there is $K' \geqslant 0$ such that either $J_{\underline{s}}^{K'}$ is empty or it is contained in the unbounded component of $J_{\underline{s}}$, and hence in X.

The final claim follows from the fact that $J(F) \cup \{\infty\}$ is connected. Indeed,

$$J(F) \cup \{\infty\} = \bigcap_{n \geqslant 0} F^{-n}(\overline{H}) \cup \{\infty\}$$

and since $\{F^{-n}(\overline{H}) \cup \{\infty\}\}_{n=0}^{\infty}$ is a nested sequence of continua, Theorem 2.1.2 tells us that $J(F) \cup \{\infty\}$ is a continuum and, in particular, connected. ■

The fact that X is absorbing is very remarkable. If we understand the dynamics in X, since we can recover $J(F)$ taking preimages of X, then we understand the dynamics of the whole $J(F)$. The following two theorems are an extension of this result. The first one is about disjoint-type functions, compare it with Theorem 4.10.6. The second one deals with the general case. Both of them are proved in [BJR].

Theorem 4.11.2. *Let $f \in \mathcal{B}$ be a disjoint-type function such that $f = f_1 \circ \cdots \circ f_n$ for some $n \geqslant 1$ and such that $f_i \in \mathcal{B}$ have finite order for all i. Then the Julia set $J(f)$ is a Cantor bouquet.*

Theorem 4.11.3 (**Absorbing Cantor bouquets**)**.** *Let $f \in \mathcal{B}$ be a finite order function or, more generally, a finite composition of such functions. Then for every $R > 0$, there exists a Cantor bouquet $X \subseteq J(f)$ with $f(X) \subseteq X$ such that*

(1) $|f^j(z)| \geqslant R$ for all $z \in X$ and $j \geqslant 0$;

(2) there is $R' \geqslant R$ such that, if $z \in \mathbb{C}$ with $|f^j(z)| \geqslant R'$ for all j, then $z \in X$.

Therefore, the set X from Theorem 4.11.1 can be chosen such that X and $\exp(X)$ are Cantor bouquets.

Chapter 5

Dynamic rays for holomorphic self-maps of the punctured plane

The purpose of this chapter is to review what has been prooved about the analytic self-maps of $\mathbb{C}^*$ and discuss if we can apply the results of the previous chapter.

5.1 Analytic self-maps of the punctured plane

The study of the iteration of rational functions in the Riemann sphere $\widehat{\mathbb{C}}$ goes back to the times of Pierre Fatou and Gaston Julia at the beginning of the 20th century. A thing that we can do is to remove one point from $\widehat{\mathbb{C}}$ and study the analytical self-maps of this space. We can assume without loss of generality that the removed point was ∞ and hence we have the complex plane $\mathbb{C}$. There are two kinds of analytic self-maps of $\mathbb{C}$: polynomials and transcendental entire functions. The first ones are analytic analytic in the whole $\widehat{\mathbb{C}}$ but ∞ is a fixed point with no preimages in $\mathbb{C}$. For transcendental entire maps ∞ is an essential singularity, therefore they are analytic in $\mathbb{C}$ but cannot even be extended continuously to $\widehat{\mathbb{C}}$.

Beyond this, we can continue our construction and take away two points from $\widehat{\mathbb{C}}$ which we will assume that are ∞ and 0. The resulting space is called the punctured plane and denoted by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We want to describe what are the analytic self-maps of $\mathbb{C}^*$ depending on the behaviour of 0 and ∞. Let us begin with the more regular ones, assume that f is analytic in the whole Riemann sphere and hence rational. The points 0 and ∞ cannot have any image nor preimage in $\mathbb{C}^*$, therefore they are either fixed or form a two periodic orbit. If ∞ is fixed and has no finite preimages (poles), f must be a polynomial. Since 0 must be the only possible root of f, then it must be of the form

$$f(z) = kz^n, \quad k \neq 0,\ n \geqslant 2. \tag{i}$$

On the other hand, if f exchanges 0 and ∞, then the polynomial in the numerator cannot have any roots and 0 must be the only root of the denominator, hence

$$f(z) = k\frac{1}{z^n}, \quad k \neq 0,\ n \geqslant 2. \tag{ii}$$

Note that we have excluded the cases where f is constant or is a Möbius transformation. Their Julia set is empty and hence they must be treated separately. Assume now that one of the excluded points is an essential singularity, for instance ∞. Then f is an entire transcendental map. We have to distinguish two cases depending on the behaviour of 0. If 0 is a fixed point, then

$$f(z) = z^n \exp(g(z)), \quad n \geqslant 0 \tag{iii}$$

where $g(z)$ denotes a non-constant entire function, while if 0 is mapped to ∞ by f then

$$f(z) = \frac{1}{z^n} \exp(g(z)), \quad n \geqslant 1 \tag{iv}$$

with $g(z)$ as before. Finally, let us consider the maps for which 0 and ∞ are essential singularities and thus, are properly analytic in $\mathbb{C}^*$ and cannot be extended even continuously to a larger space. They are of the form

$$f(z) = z^m \exp(g(z) + h(1/z)), \quad m \in \mathbb{Z} \tag{v}$$

where $g(z)$ and $h(z)$ are non-constant entire functions. The cases (i) to (v) describe all the possible kinds of analytic self-maps of $\mathbb{C}^*$.

Note that this is the last interesting case. If we remove three points of $\widehat{\mathbb{C}}$, Montel's theorem (Theorem 3.2.1) tells us that the Julia set of the analytic self-maps there is trivial. This was first showed by Hans Rådström in [Råd53] without using this theorem.

5.2 Properties of the Julia set

Recall the definition of the Julia set from Definition 3.2.2. In analogy with the rational or entire transcendental cases, H. Rådström proved the following properties of the Julia set of an analytic self-map of $\mathbb{C}^*$. Compare with Proposition 3.2.4.

Proposition 5.2.1. *If f is a holomorphic self-map of $\mathbb{C}^*$ which is not a Möbius transformation, then*

(i) $J(f)$ is a non-empty perfect subset of $\mathbb{C}^*$;

(ii) $J(f)$ is completely invariant;

(iii) for every $p \in \mathbb{N}$, $J(f^p) = J(f)$.

Pierre Fatou and Gaston Julia proved that for a rational function, the Julia set is the closure of the set of repelling periodic points. The same was proved by Irvine N. Baker for transcendental entire functions. The following proposition is the analog for $\mathbb{C}^*$.

Proposition 5.2.2. *Let f be an analyitc self-map of $\mathbb{C}^*$. Then $J(f)$ is the closure of the set of repelling periodic points of f.*

This was originally proved by Prodipeswar Bhattacharyya in his PhD thesis *Iteration of Analytic Functions* (University of London, 1969) having as advisor Irvine N. Baker. Finally, this last property is also very standard.

Proposition 5.2.3. *Let f be an analytic self-map of $\mathbb{C}^*$. If $U \cap J(f) \neq \emptyset$, then for every compact set $K \subseteq \mathbb{C}^*$ there is $n_0 \in \mathbb{N}$ such that*

$$f^n(U) \supseteq K$$

for all $n > n_0$.

Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is a universal covering, we would like to relate the properties of the Julia set of an analytical self-map of $\mathbb{C}^*$ to the ones of the Julia set of the lift to $\mathbb{C}$.

Lemma 5.2.4. *Given g an analytic self-map $\mathbb{C}^*$ let f be an entire function sattisfying $\exp f(z) = g(\exp z)$. Then we have*

$$\exp^{-1} F(g) \subseteq F(f).$$

Proof. Let $z_0 \in \mathbb{C}$ be such that $\exp z_0 \in F(g)$. Let U be a neighbourhood of z_0 in $\mathbb{C}$ and consider $V = \exp U \subseteq \mathbb{C}^*$. ■

In particular, since $\mathbb{C}^* = F(g) \sqcup J(g)$ and $\mathbb{C} = F(f) \sqcup J(f)$ we have

$$\exp^{-1} J(g) = \exp^{-1}(\mathbb{C}^* \setminus F(g)) = \mathbb{C} \setminus \exp^{-1} F(g) \supseteq \mathbb{C} \setminus F(f) = J(f).$$

Walter Bergweiler proved in [Ber95] that $\exp^{-1} J(g) \subseteq J(f)$ and hence we have the following theorem.

Theorem 5.2.5. *Given g an analytic self-map $\mathbb{C}^*$ let f be an entire function sattisfying $\exp f(z) = g(\exp z)$. If f is not linear nor constant, then*

$$\exp^{-1} J(g) = J(f).$$

The connectivity of the Julia set was studied in [BD98] proving the following two surprising results.

Theorem 5.2.6. *If $f(z)$ is a transcendental self-map of $\mathbb{C}^*$, then $J(f)$ has no compact component. In particular, $J(f)$ has no singleton component.*

Note that in the polynomial case it can happen that $J(f)$ consists of Cantor dust and hence every point in $J(f)$ is a singleton component. For entire transcendental functions it is also possible that singleton components are dense in the Julia set.

Theorem 5.2.7. *If $f(z)$ is an analytic self-map of $\mathbb{C}^*$, then $J(f)$ has either one or infinitely many components.*

5.3 Classification of the Fatou components

Definition 5.3.1 (Simply/doubly connected). A connected set $X \subseteq \widehat{\mathbb{C}}$ is said to be *simply connected* if $\widehat{\mathbb{C}} \setminus X$ is connected or, equivalently, if every loop is contractible, $\pi_1(X) = 0$. X is called *doubly connected* if $\widehat{\mathbb{C}} \setminus X$ has exactly two connected components or, in terms of the fundamental group, every non-trivial loop is homotopic.

The following theorem is due to Irvine N. Baker. It was first proved in [Bak87] and there is another proof in [BD98, Theorem 1]. Check also [Kee88].

Theorem 5.3.1. *If $f(z)$ is an analytic self-map of $\mathbb{C}^*$ which is not a Möbius transformation, then the components of $F(f)$ are simply or doubly-connected. There is at most one doubly connected component, which must separate 0 from ∞, except in the case when $f(z) = kz^n$ for some $k \neq 0$ and $n \in \mathbb{Z} \setminus \{0, \pm 1\}$.*

Observe that in the omitted case in the above theorem, when $f(z) = kz^n$, the Julia set is the unit circle and the Fatou set consists of two punctured discs: the attracting basins of 0 and ∞ with the fixed points removed. Therefore, it has two doubly connected components and is an exception for this rule.

Theorem 5.3.2. *If A is a doubly-connected component of $F(f)$ for a self-map $f(z)$ of $\mathbb{C}^*$ and A is relatively compact in $\mathbb{C}^*$, then either*

(i) A is a Herman ring,

(ii) A is pre-periodic but not periodic, or

(iii) A is a wandering component.

Furthermore, in cases (ii) and (iii) for all $n \in \mathbb{N}$, $f^n(A)$ are relatively compact simply-connected components.

You can find the proof of this theorem in [BD98, Theorem 4]. Additionally, they show that there are examples for each of these cases.

5.4 The complex standard family

One reason that makes interesting to understand the dynamics of the analytic self-maps of $\mathbb{C}^*$ is that they arise as complexification of analytic self-maps of the circle. The standard family is an example that illustrates this phenomena.

Definition 5.4.1 (Standard family). The *standard family of circle maps* is given by

$$F_{\alpha\beta}(\theta) := \theta + \alpha + \beta \sin(\theta) \pmod{2\pi}$$

where $\alpha, \beta \in \mathbb{R}$ are two parameters and $\theta \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

They are perturbations of the rigid rotation of angle α,

$$R_\alpha(\theta) = \theta + \alpha \pmod{2\pi}.$$

Usually the parameters are restricted to $\alpha \in [0, 2\pi)$ and $\beta \geqslant 0$. Consider the lift of $F_{\alpha\beta}$ to $\mathbb{C}$

$$\widetilde{F}_{\alpha\beta}(z) = z + \alpha + \beta \sin(z)$$

which restricted to $\mathbb{R}$ and modulo 2π gives $F_{\alpha\beta}$. Note that

$$\widetilde{F}_{\alpha\beta}(z + 2\pi) = z + 2\pi + \alpha + \beta \sin(z + 2\pi) = z + \alpha + \beta \sin(z) + 2\pi = \widetilde{F}_{\alpha\beta}(z) + 2\pi.$$

We want to map the real line to the unit circle with a 2π-periodic map. This can be done semiconjugating $\widetilde{F}_{\alpha\beta}$ by $h(z) = \exp(iz)$,

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\widetilde{F}_{\alpha\beta}"] \arrow[d, "\exp(iz)"'] & \mathbb{C} \arrow[d, "\exp(iz)"] \\ \mathbb{C}^* \arrow[r, "\widehat{F}_{\alpha\beta}"'] & \mathbb{C}^* \end{tikzcd}$$

if $w = e^{iz}$,

$$\widehat{F}_{\alpha\beta}(w) = \widehat{F}_{\alpha\beta}(\exp(iz)) = \exp(i\widetilde{F}_{\alpha\beta}(z)) = e^{iz}e^{i\alpha}e^{i\beta\sin(z)} = we^{i\alpha}e^{\beta(w-1/w)/2}$$

because recall that $\sin(z) = (e^{iz} - e^{-iz})/(2i) = (w - 1/w)/(2i)$. Thus, we have

$$\widehat{F}_{\alpha\beta\,|\mathbb{S}^1} \equiv F_{\alpha\beta}$$

and hence we say that $\widehat{F}_{\alpha\beta}$ is a complexification of $F_{\alpha\beta}$. The resulting map $\widehat{F}_{\alpha\beta}$ is an anylitic self-map of $\mathbb{C}^*$ for which 0 and ∞ are essential singularities, type (iv) above. In Figure 5.1 you can see the phase space of two maps of this family exhibiting a very different character. For the one on the left hand side, there is an attracting cycle of period two in $\mathbb{S}^1$. The black regions correspond to the attracting basin. On the right hand side it seems that the Julia set fills up the whole plane.

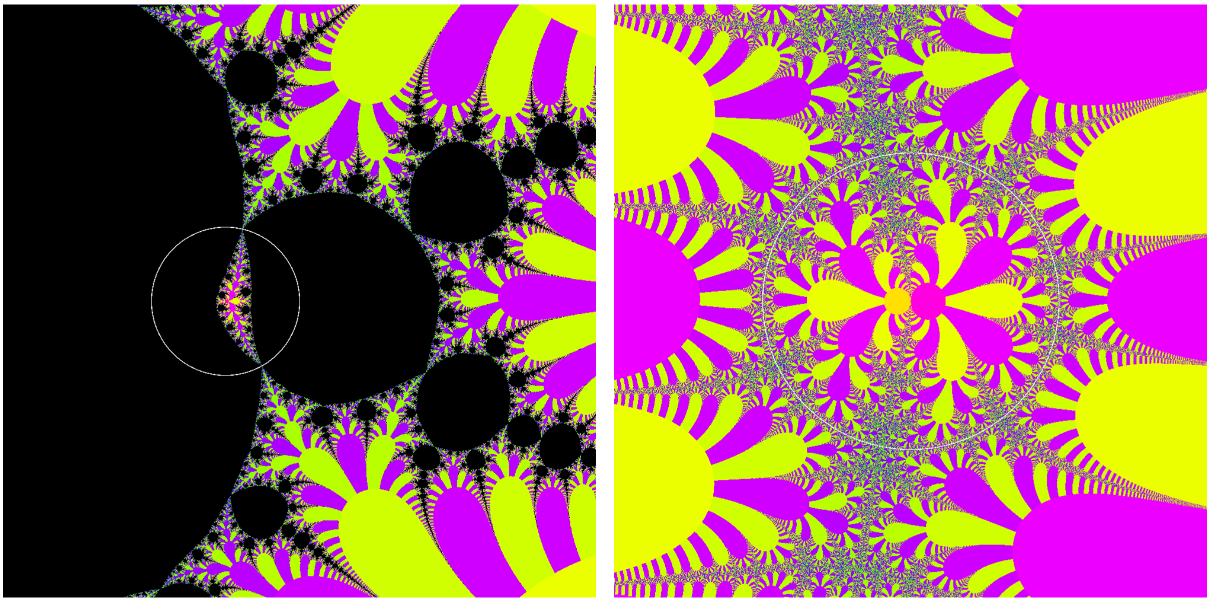

Figure 5.1: Phase space of the complex standard family $f_{\alpha,\beta}$. (L): $\alpha = 3.1$, $\beta = 0.8$ and $z \in [-3, 5] + i[-4, 4]$. (R): $\alpha = 3.1$, $\beta = 5$ and $z \in [-2, 2] + i[-2, 2]$. We have drawn in white the unit circle $\mathbb{S}^1$.

In Figure 5.2 we have drawn the phase spaces of the lifts of these maps, $\widetilde{F}_{\alpha\beta}$, for the same values of the parameters. Observe that now the unit circle $\mathbb{S}^1$ in the above images

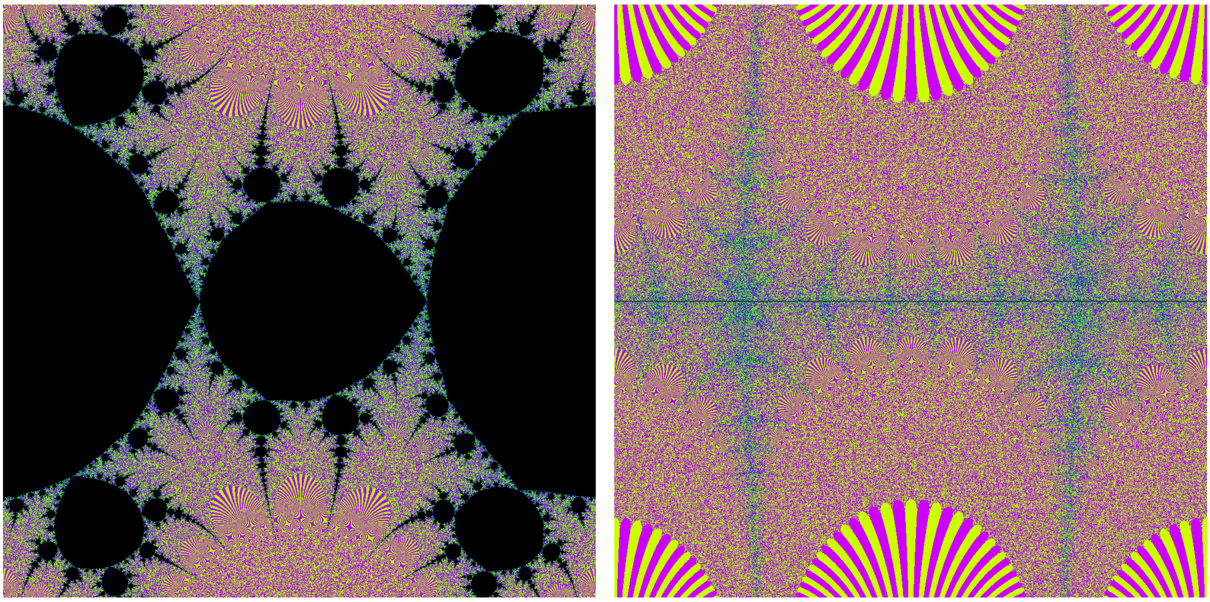

Figure 5.2: Phase space of the lift of the complex standard family $F_{\alpha,\beta}$. (L): $\alpha = 3.1$, $\beta = 0.8$. (R): $\alpha = 3.1$, $\beta = 5$. $z \in [-\pi, \pi] + i[-\pi, \pi]$.

corresponds to the real line.

Let us compute the singular values of this map. Critical points must satisfy

$$\widehat{F}'_{\alpha\beta}(w) = e^{\beta(w-1/w)/2}\left[e^{i\alpha} + \frac{\beta}{2}\left(w + \frac{1}{w}\right)\right] = 0$$

or, equivalently,

$$w^2 + \frac{2e^{i\alpha}}{\beta}w + 1 = 0$$

which leads to two critical points and the two corresponding critical values. Although $S(f)$ is finite, the results of [DT86] and [RRRS11] cannot be applied because the map $\widehat{F}$ is not entire. On the other hand, the critical points of the lift $\widetilde{F}_{\alpha\beta}$ satisfy

$$\widetilde{F}'_{\alpha\beta}(z) = 1 + \beta\cos z = 0 \quad \Leftrightarrow \quad \cos z = -\frac{1}{\beta}$$

and if $0 < \beta \leqslant 1$ this has infinitely many solutions on the real line and form an unbounded set. The distance of every critical point z_c to its image (a critical value) is

$$|z_c - \widetilde{F}_{\alpha\beta}(z_c)| = |\alpha + \beta\sin(z_c)| = \alpha + \beta\sqrt{1 + \frac{1}{\beta^2}}$$

thus, the critical values of $\widetilde{F}_{\alpha\beta}$ are unbounded. Hence, $S(\widetilde{F}_{\alpha\beta})$ is an unbounded set and we cannot use the results of [DT86] nor [RRRS11].

In [Fag99], Núria Fagella showed that the Julia set of $\widehat{F}_{\alpha\beta}$ contains an invariant set of curves such that the points on them tend exponentially fast to the essential singularities under iteration. The landing properties of these tails are also discussed there.

5.5 Logarithmic coordinates

Definition 5.5.1 (Class $\mathcal{B}^*$). We denote by $\mathcal{B}^*$ the class of all analytic self-maps of $\mathbb{C}^*$ for which 0 and ∞ are essential singularities and $S(f)$ is bounded away from 0 and ∞.

Definition 5.5.2 (Tract in $\mathcal{B}^*$). Let $f \in \mathcal{B}^*$ and let $A \subseteq \mathbb{C}$ be a topological anulus bounded away from 0 and ∞ containing $S(f)$. Denote by W_0 and W_∞ the components of $\mathbb{C} \setminus A$ containing respectively 0 and ∞. We call *tract* to each component V of $\mathcal{V} = f^{-1}(W_0) \cup f^{-1}(W_\infty)$.

The following theorem is a version of Lemma 4.1.1 for $\mathcal{B}^*$.

Theorem 5.5.1. *A tract $V \subseteq \mathbb{C}^*$ of a function $f \in \mathcal{B}^*$ is a Jordan domain containing either 0 or ∞ (but not both) in its closure and $f : V \to W$ is a universal covering where either $W = W_0$ or $W = W_\infty$.*

Proof. We will begin showing that $f : V \to W$ is a covering map. The map f is analytic on V and, by construction, it is surjective. Let $z \in W$ and consider $U \subsetneq W$ to be a neighbourhood of z. Let X be a component of $f^{-1}(U)$, we have to prove that X is mapped homeomorphically onto U. If $\overline{X}$ is compact, since $\overline{U}$ contains no critical values, $\overline{X}$ contains no critical points and hence $f : X \to U$ is a diffeomorphism. Suppose now that U was not compact and let $q \in U$. Then, we can choose an exhaustion of $\overline{U}$ by a continuous family of simple, closed curves γ_t, $0 \leqslant t \leqslant 1$ such that $\gamma_0 = \{q\}$ and $\gamma_1 = \partial\overline{U}$, see Figure 5.3. For small values of t, $f^{-1}(\gamma_t)$ is a simple closed curve around $f^{-1}(q)$, but since X is unbounded there must be a value $t_0 \leqslant 1$ such that γ_t is no longer a closed curve. Since $\overline{X}$ contains no critical points, this curve can be extended to ∞ and then this implies the existence of an asymptotic value on γ_{t_0} contradicting the assumption that $\gamma_{t_0} \subseteq W$ and all the asymptotic values are in $D = \mathbb{C} \setminus W$. Hence, $f : V \to W$ must be a covering map.

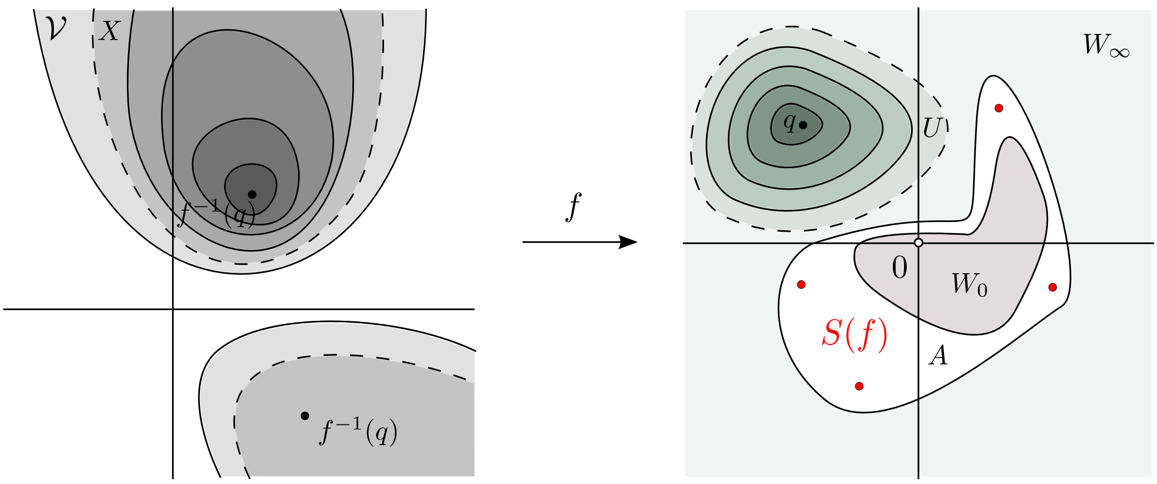

Figure 5.3: Sketch in the proof of Theorem 5.5.1.

The preimage of W under a covering map can be either a disc or a punctured disc. Let us discard the second case. Suppose to the contrary that V was a punctured disc with puncture point $a \in \widehat{\mathbb{C}}$. Great Picard theorem tells us that the puncture a cannot be one of the essential singularities because its image omits too many points. Suppose now that $a \neq 0, \infty$. If $W = W_\infty$, then a needs to be a pole but this is impossible because f is analytic on $\mathbb{C}^*$. On the other hand, if $W = W_0$ let f' be the function conjugated to f by $h(z) = 1/z$. The map f' is again a self-map of $\mathbb{C}^*$, we just have interchanged the singularities. Then we would have a punctured disc with puncture $a' = 1/a$ such that its image is a neighbourhood of ∞. Again, since f' is analytic on $\mathbb{C}^*$ this is impossible. We conclude that V has to be a disc. Since W_∞ has points as close to ∞ as we want and f is holomorphic on $\mathbb{C}^*$, the preimage of W_∞ must have an essential singularity in its boundary. Similarly, the preimage of W_0 also needs to contain an essential singularity in its boundary, otherwise we would obtain a contradiction by conjugating by $h(z) = 1/z$ as before.

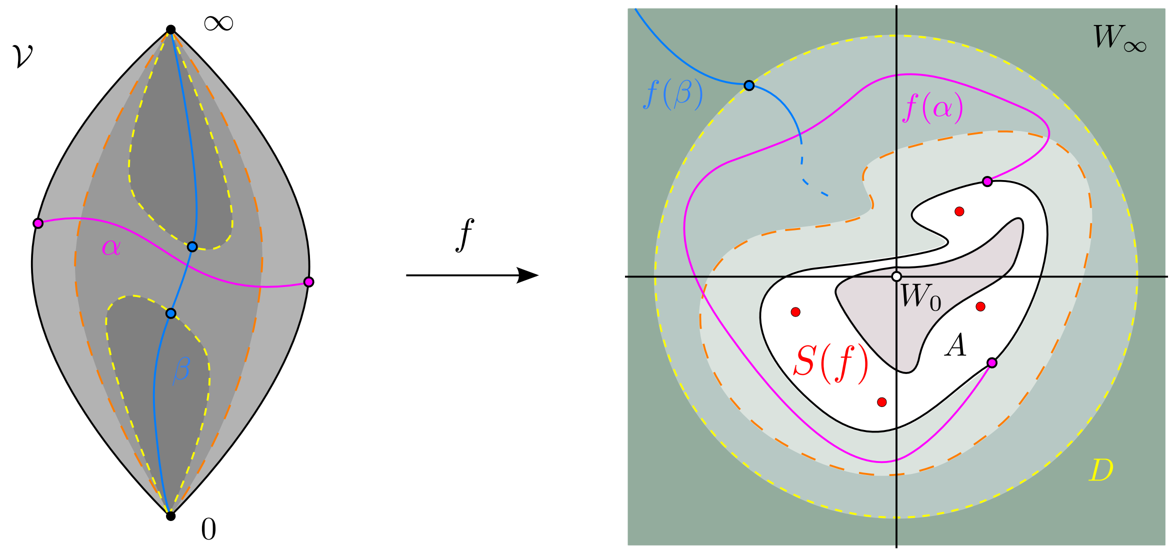

Figure 5.4: Sketch in the proof of Theorem 5.5.1.

It only remains to discard the case where V is a disc containing both 0 and ∞ in its boundary. Assume by now that $W = W_\infty$. See Figure 5.4. Suppose this was possible, then the boundary of V would consist of two disjoint curves mapped ∞-to-1 onto ∂W. Foliate W with a family of simple closed curves δ_t, $0 \leqslant t \leqslant 1$ such that $\delta_0 = \partial W$ and $\delta_1 = \infty$. Let B_t and U_t be respectively the bounded and unbounded components defined by the curve δ_t. We have that $S(f) \subseteq A \subseteq B_t$ for every value of t, therefore the preimages of U_t must be topological discs containing either 0, ∞ or both in their boundary. Observe that, since we are assuming ∂V contains 0 and ∞, for small values of t the set $\partial f^{-1}(U_t)$ will also contain both (look at the orange curve in the above figure). We claim there is some $0 < t_* < 1$ such that $\partial f^{-1}(U_{t_*})$ only contains one of these two points. Indeed, if this was not true, consider a curve $\alpha \subseteq V$ joining two points one from each components of ∂V and such that they have a different image under f (in pink in the above figure). Since f is continuous, $f(\alpha)$ must be a continuous curve in W_∞. Since the endpoints of this curve lie in ∂W_∞, the curve $f(\alpha)$ is bounded, let D be a disc centred at 0 containing $f(\alpha)$ (in yellow in the above figure). The preimage of $\mathbb{C} \setminus D$ cannot intersect α, therefore

each of its connected components can only have one of the essential singularities in its boundary. Hence, to this point we have proved that we can always enlarge the anulus A so that tracts only contain one of the essential singularities on the boundary.
We want to prove that indeed any of the tracts can contain both of the essential singularities independently on how you choose A. Consider now a curve $\beta \subseteq V$ joining 0 and ∞. Then both endpoint of $f(\beta)$ must be ∞ because otherwise we would find an asymptotic value in W_∞, which is not possible. Thus, both the final part and the begining part of β have points in $f^{-1}(\mathbb{C} \setminus D)$. Hence, if the component of $f^{-1}(\mathbb{C} \setminus D)$ does not have 0 and ∞ in its boundary necessarily there must be two components, one having 0 and the other one ∞ in their boundary. Using again a foliation of W_∞ by simple closed curves there must be a t_1 such that the preimage of $f^{-1}(U_t)$ passes from being connected to disconnected. The preimage at $t = t_1$ can be of any of the both types. Let t_2 be a parameter such that the preimage is of the other type. Between these two, there must be a foliation of curves of this second type filling all the space. This leads to the existence of a point such that both components of this family of curves accumulate. We raise a contradiction with the fact that f restricted to every tract needs to be a covering.
The proof for W_0 is analogous, you only have to conjugate by $h(z) = 1/z$. ■

Therefore, we can classify the tracts in $\mathcal{V}$ in four types

$$\mathcal{V} = \mathcal{V}_0^0 \sqcup \mathcal{V}_0^\infty \sqcup \mathcal{V}_\infty^0 \sqcup \mathcal{V}_\infty^\infty$$

where the lower index indicates if the tracts have 0 or ∞ in their closure and the upper index indicates if they are a covering of W_0 or W_∞.

Definition 5.5.3 (Logarithmic transform in $\mathcal{B}^*$). Let $f \in \mathcal{B}^*$ and consider $\mathcal{T} := \exp^{-1}(\mathcal{V})$ and $H := \exp^{-1}(W)$. We call *logarithmic transform* of f to the continuous functions $F : \mathcal{T} \to H$ making the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "\exp"'] & H \arrow[d, "\exp"] \\ \mathcal{V} \arrow[r, "f"'] & W \end{tikzcd}$$

commutative. The set H can be decomposed in $H = H_0 \sqcup H_\infty$ where

$$H_0 := \exp^{-1}(W_0), \quad H_\infty := \exp^{-1}(W_\infty).$$

The connected components of $\mathcal{T}$ are called *tracts* of F and can be classified in four subsets as well

$$\mathcal{T} = \mathcal{T}_0^0 \sqcup \mathcal{T}_0^\infty \sqcup \mathcal{T}_\infty^0 \sqcup \mathcal{T}_\infty^\infty.$$

We denote by $\mathcal{T}_i^j$ the set of tracts T of F such that $\exp T \in \mathcal{V}_i^j$, $i, j \in \{0, \infty\}$.

Theorem 5.5.2. *If $f \in \mathcal{B}^*$, then its logarithmic transform $F : \mathcal{T} \to H$ satisfies the following properties:*

a) H is the disjoint union of two $2\pi i$-periodic Jordan domains H_0 and H_∞, H_0 containing a left halft plane and H_∞ containing a right half plane;

b) *every component of $\mathcal{T}$ is an unbounded Jordan domain with real parts either bounded below and unbounded from above or unbounded below and bounded from above;*

d) *for every component T of $\mathcal{T}$, $F : T \to H$ is a conformal isomorphism;*

e) *for every component T of $\mathcal{T}$, $\exp_{|T}$ is injective;*

f) *$\mathcal{T}$ is invariant under translation by $2\pi i$.*

Proof. Recall that $\exp : \mathbb{C} \to \mathbb{C}^*$ is a holomorphic cover and, in particular, a local homeomorphism. Let us check each one of the above properties:

a) The boundary of A are two disjoint loops arround 0, hence its preimage under the exponential map must be a $2\pi i$-periodic continuous band. Since exp is a local homeomorphism, the boundaries of this band cannot have self-intersections. The domains W_∞ and W_0 are mapped to the right and to the left of this band respectively. There are $R_+, R_- > 0$ such that $A \subseteq D(0, R_+)$ and $W_0 \subseteq D(0, R_-)$, therefore the right half plane with $\operatorname{Re} z > \log R_+$ is contained in H_∞ and the left half plane with $\operatorname{Re} z < \log R_-$ is contained in H_0.

b) Again, since exp is a holomorphic cover, every component T of $\mathcal{T}$ must be a Jordan domain. Let r_- and r_+ be respectively a lower and an upper bound for the tracts of f containing respectively ∞ and 0 in their boundary. Taking a logarithm, this gives a lower and an upper bounds for the real part of the points in $\mathcal{T}$ containing respectively $+\infty$ and $-\infty$ in their boundary. On the other hand, the logarithm of an unbounded set must have points with unbounded real part.

c) This is the same argument as for entire functions. If the boundaries of two tracts had a common point z_0, take a small enough neighbourhood U of z_0 so that $V = f(\exp(U))$ does not intersect any singular value of f. Now since exp is a conformal map and the restriction of f to $f^{-1}(V)$ has no critical points, the map $f \circ \exp$ is a covering from U to V and hence a local homeomorphism. Since D is a Jordan domain, the intersection of the boundary of D with U is a piece of arc. Thus, the intersection of the boundary of $\mathcal{T}$ with U must also be a piece or arc, all the points in $\partial\mathcal{T} \cap U$ must be common to both tracts. If the tracts are different, this cannot happen for all the common points. There must be a point z_0' with a neighbourhood U' such that $\partial\mathcal{T} \cap U'$ has points belonging only to boundary of one of the tracts. Hence it would not be homeomorphic to an arc, raising a contradiction.
Suppose now that there was a sequence of points z_k each one belonging to a different tract and converging to a finite point z. Consider $w_k = f(\exp(z_k))$ which is a converging sequence in ∂D and let $w \in D$ be its limit. Since $f(\exp z) = w$, the point z must be in the boundary of some tract. Let U be a small neighbourhood of z such that $V = f(\exp U)$ does not intersect $S(f)$. U contains infinitely many points of the sequence $\{z_k\}_k$ and hence $U \cap \mathcal{T}$ is a disjoint union of infinitely many pieces of arc. We get a contradiction because $V \cap \partial D$ is a piece of arc and $f \circ \exp$ must be a local homeomorphism. Thus, $z = \infty$.

Like in the case of entire functions, as a consequence of b) and c), there exists a curve $\delta \subseteq \mathbb{C}^* \setminus \mathcal{V}$ joining 0 and ∞ (it has been drawn in blue in Figure 5.5). Hence, we can define a continuous branch of the logarithm on $\mathcal{T}$.

d) Let T be a tract of F. By Proposition 1.4.2, since $f \circ \exp : T \to W$ and $\exp : H \to W$ are both universal covers, they must be equivalent. That is, there exist a homeomorphism $F : T \to H$ making the diagram commute. Since $f \circ \exp_{|T}$ and exp are conformal, F must be conformal too.

e) The preimage of $\mathcal{V}$ under exp is compactly contained in the open band defined by two preimages of the curve δ. Therefore, it cannot contain vertical segments of length 2π and hence $\exp_{|T}$ is injective.

f) This is a direct consequence of the fact that the exponential map is $2\pi i$-periodic.

■

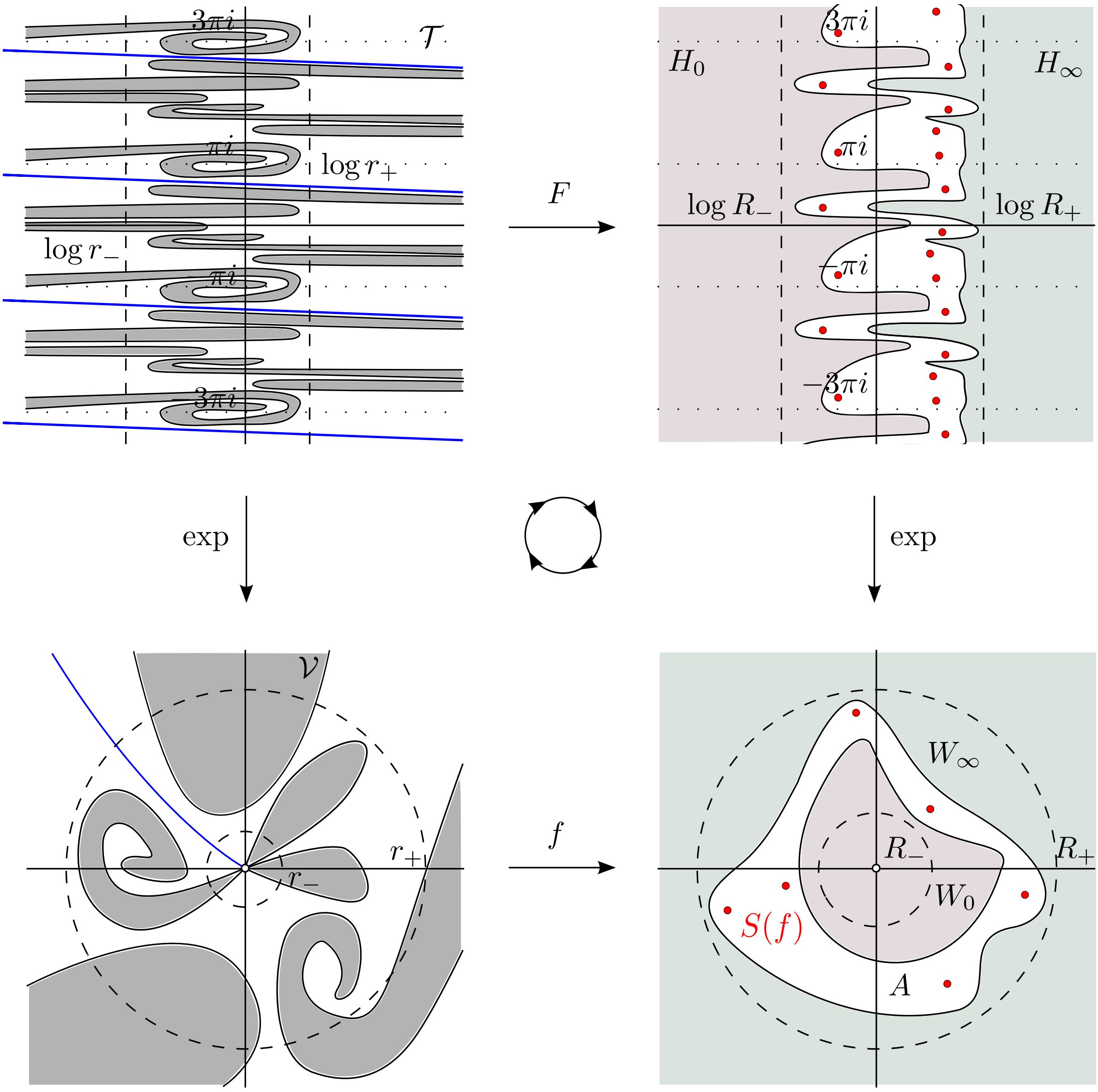

Figure 5.5: Logarithmic coordinates for a function $f \in \mathcal{B}^*$.

Definition 5.5.4 (**Class $\mathcal{B}^*_{log}$**). In analogy with class $\mathcal{B}_{log}$, we denote by $\mathcal{B}^*_{log}$ the class of functions $F : \mathcal{T} \to H$ satisfying the above properties.

Theorem 5.5.3 (**Reduction to $\mathcal{B}_{log}$**). *Let $F \in \mathcal{B}^*_{log}$ be the logarithmic transform of a function $f \in \mathcal{B}^*$. Then the map*

$$F_\infty := F_{|\mathcal{T}^\infty_\infty} : \mathcal{T}^\infty_\infty \to H_\infty$$

belongs to $\mathcal{B}_{log}$. Similarly, the map

$$F_0 := F_{|\mathcal{T}^0_0} : \mathcal{T}^0_0 \to H_0$$

is conjugated to a map $\widetilde{F}_0 \in \mathcal{B}_{log}$ by $h(z) = -\overline{z}$.

Proof. Conditions a) to f) in the definition of class $\mathcal{B}^*_{log}$ imply directly the conditions a) to f) in the definition of class $\mathcal{B}_{log}$ for the functions F_∞ and $\widetilde{F}_0$. ■

5.6 Escaping set and dynamic rays

Definition 5.6.1 (**Julia set for $\mathcal{B}^*_{log}$**). If $F \in \mathcal{B}^*_{log}$, then

$$J(F) = \{z \in \overline{\mathcal{T}} \ : \ F^n(z) \text{ is defined and in } \overline{\mathcal{T}} \text{ for all } n \geqslant 0\}.$$

Definition 5.6.2 (**Escaping set for analytic self-maps of $\mathbb{C}^*$**). Let f be a holomorphic self-map of $\mathbb{C}^*$. Then we define its escaping set as

$$I(f) := \{z \in \mathbb{C}^* \ : \ \lim_{n\to\infty} f^n(z) \notin \mathbb{C}^*\}$$

which contains two disjoint sets

$$I_0(f) := \{z \in \mathbb{C}^* \ : \ \lim_{n\to\infty} f^n(z) = 0\}$$
$$I_\infty(f) := \{z \in \mathbb{C}^* \ : \ \lim_{n\to\infty} f^n(z) = \infty\}.$$

Observe that the set

$$I_{0,\infty}(f) := I(f) \setminus \big(I_0(f) \sqcup I_\infty(f)\big)$$

consists of the points which have $\{0, \infty\}$ as limit set.

Lemma 5.6.1. *Let f and g are two holomorphic self-maps of $\mathbb{C}^*$ conjugated via $h(z) = 1/z$. Then*

$$I_0(g) = I_\infty(f), \quad I_\infty(g) = I_0(f).$$

Proof. Recall that conjugations map orbits to orbits. If a point z_0 converges to ∞ under f necessarily $h(z_0) = 1/z_0$ converges to $h(\infty) = 0$. ■

Definition 5.6.3 (**Escaping set for $\mathcal{B}^*_{log}$**). If $F \in \mathcal{B}^*_{log}$, then we define

$$I(F) := \{z \in J(F) \ : \ \lim_{n\to\infty} \operatorname{Re} F^n(z) = \pm\infty\}$$

which contains two disjoint sets

$$I_0(F) := \{z \in J(F) \ : \ \lim_{n\to\infty} \operatorname{Re} F^n(z) = -\infty\}$$
$$I_\infty(F) := \{z \in J(F) \ : \ \lim_{n\to\infty} \operatorname{Re} F^n(z) = +\infty\}.$$

Note that again there might be a set of points which have $\{0, \infty\}$ as limit set and jump infinitely many times from one side to the other.

Lemma 5.6.2. *If $f \in \mathcal{B}^*$ and F is a logarithmic transform of f, then* $\exp(I(F)) \subseteq I(f)$.

Proof. Points in $I(F)$ are such that $F^n(z) \in \mathcal{T}$ for all $n \in \mathbb{N}$ and tend to $\pm\infty$. Therefore, the forward orbit of $\exp(z)$ is contained in W and its orbit must tend to ∞ and 0 respectively. ∎

Observe that here ray tails are curves connecting points to 0 or ∞ such that all the points there converge uniformly to 0 or ∞.

5.7 Functions of finite order

Definition 5.7.1 (**Order of a holomorphic function on $\mathbb{C}^*$**). Let f be a holomorphic self-map of $\mathbb{C}^*$. The order at infinity of f is defined as

$$\rho_\infty(f) := \lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|}$$

while, if $h(z) = 1/z$, the order at 0 of f is given by

$$\rho_0(f) := \rho_\infty(h \circ f \circ h) = \lim_{r\to 0} \sup_{|z|=r} \frac{\log\log|1/f(z)|}{\log 1/|z|}.$$

Equivalently,

$$\rho_\infty(f) = \inf\{m \in \mathbb{R} \cup \{\infty\} \ : \ f(z) = O(\exp(|z|^n)) \text{ as } z \to \infty\},$$

and

$$\rho_0(f) = \inf\{m \in \mathbb{R} \cup \{\infty\} \ : \ f(z) = O(\exp(-|z|^n)) \text{ as } z \to 0\}.$$

We say that f has finite order if $\max\{\rho_\infty(f), \rho_0(f)\} < \infty$.

Example 5.7.1. The functions of the form $f(z) = \exp(|z|^m + |z|^{-n})$ have finite order, while $g_1(z) = \exp(\exp(z))$ or $g_2(z) = \exp(1/\exp(z))$ have not finite order. We will proof a more general result in the next section.

Lemma 5.7.1. *A function $f \in \mathcal{B}^*$ has finite order if and only if any logarithmic transform $F \in \mathcal{B}^*_{log}$ of f satisfies that the functions*

$$F_\infty := F_{|\mathcal{T}^\infty_\infty} : \mathcal{T}^\infty_\infty \to H_\infty$$

and $\widetilde{F}_0 \in \mathcal{B}_{log}$ conugated to

$$F_0 := F_{|\mathcal{T}^0_0} : \mathcal{T}^0_0 \to H_0$$

by $h(z) = -\overline{z}$ have finite order in the sense of Definition 4.8.2.

Proof. It is clear because the notion of order is local, it only matters a neighbourhood of 0 or ∞. Therefore, the analog proof for entire transcendental maps is valid here. ∎

5.8 Results and future work

Theorem 5.8.1 (**Holomorphic self-maps of the punctured plane with dynamic rays**). *Let $f \in \mathcal{B}^*$ be a function of finite order or a finite composition of such maps. Then every point $z \in I_\infty(f)$ can be connected to ∞ by a curve γ such that $f^n_{|\gamma} \to \infty$ uniformly. Similarly, every point $z \in I_0(f)$ can be connected to 0 by a curve γ such that $f^n_{|\gamma} \to 0$ uniformly.*

Proof. Let $F_i \in \mathcal{B}^*_{log}$ be the logarithmic transform of $f_i \in \mathcal{B}^*$. By Theorem 5.5.3, $F_{i,\infty}$ and $\widetilde{F}_{i,0}$ are in class $\mathcal{B}_{log}$. By Lemma 5.7.1, they have finite order. Now we can follow the same argument in the proof of Theorem 4.9.2. The logarithmic transforms can be normalized and by Theorem 4.8.3, the tracts of F_i have bounded slope and (uniformly) bounded wiggling for all i. Then, by Proposition 4.7.2 each F_i satisfies a linear head-start condition. Applying Lemma 4.9.1, there is an $a \geqslant 0$ such that $G_a(z) := \tau_a \circ F_n \circ \cdots \circ F_1 \in \mathcal{B}^n_{log}$ satisfies a uniform linear head-start condition, where $\tau_a(z) = z - a$. On a sufficiently restricted domain $\mathcal{T} \subseteq \mathbb{H}$,

$$F := G_a \circ \tau_a^{-1} = \tau_a \circ F_n \circ \cdots \circ F_1 \circ \tau_a^{-1} : \mathcal{T} \to \mathbb{H}$$

is a logarithmic transform of $f = f_n \circ \cdots \circ f_1$ and satisfies a linear head-start condition. Now Theorem 4.5.8 tells us that every escaping point z is eventually mapped to a ray tail γ in $J(F)$ connecting $F^k(z)$ to ∞. Finally, by Proposition 3.7.3 z must be on a ray tail.
On the other hand, the escaping points in the tracts of type $\mathcal{T}_0^\infty$ and $\mathcal{T}_\infty^0$ must eventually be mapped to a tract of the form $\mathcal{T}_0^0$ or $\mathcal{T}_\infty^\infty$. Then, by Proposition 3.7.3 they must also be on a ray tail. ■

Furthermore, by Theorem 4.11.1 the Julia set of such function $f \in \mathcal{B}^*$ must contain two Cantor bouquets, one attached to 0 and the other one to ∞.
In the future we plan to study what happens with the escaping points that jump between a neighbourhood of 0 and a neighbourhood of ∞. This set was called $I_{0,\infty}(f)$ before. We also have in mind to prove other structural theorems for this class of self-maps during the Ph.D. of the author.

5.9 A family of self-maps of $\mathbb{C}^*$

Observe that there is a huge family to which we can apply our results. Let P and Q be two polynomials of degree $\deg P = p$ and $\deg Q = q$. Consider the analytic self-map of the punctured plane

$$f(z) = \exp\big(P(z) + Q(1/z)\big).$$

The parameters of this family are the coefficients of P and Q.

Lemma 5.9.1. *Let P and Q be polynomials. The function*

$$f(z) = \exp\big(P(z) + Q(1/z)\big)$$

is critically finite and hence belongs to $\mathcal{B}^*$.

Proof. The critical points are solutions of

$$f'(z) = \exp\big(P(z) + Q(1/z)\big)\big(P'(z) - Q'(1/z)/z^2\big) = 0$$

or, equivalently since $z \neq 0$,

$$z^{q-1}\left(z^2 P'(z) - Q'(1/z)\right) = 0$$

which is a polynomial equation on z of degree $(q-1) + 2 + (p-q) = p+q$. Thus, $S(f)$ contains at most $p+q$ critical values. Let us study the asymptotic values:

$$\lim_{z\to\infty} \exp(P(z) + Q(1/z) = \lim_{z\to\infty} e^{z^p} = +\infty \text{ or } 0;$$

$$\lim_{z\to 0} \exp(P(z) + Q(1/z) = \lim_{z\to 0} e^{z^q} = +\infty \text{ or } 0.$$

Thus, there are no asymptotic values in $\mathbb{C}^*$ and $S(f)$ is composed of a finite number of critical values. ■

Lemma 5.9.2. *Let P and Q be polynomials. The function*

$$f(z) = \exp\big(P(z) + Q(1/z)\big)$$

has finite order.

Proof. Let us compute the orders at ∞ and 0 of this function:

$$\rho_\infty(f) = \lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|} = \lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|e^{z^p}|}{\log|z|} = \lim_{r\to\infty} \sup_{|z|=r} \frac{\log|z|^p}{\log|z|} = p$$

$$\rho_0(f) = \lim_{r\to 0} \sup_{|z|=r} \frac{\log\log 1/|f(z)|}{\log 1/|z|} = \lim_{r\to 0} \sup_{|z|=r} \frac{\log\log 1/|e^{z^q}|}{\log 1/|z|} = \lim_{r\to 0} \sup_{|z|=r} \frac{\log|z|^{-q}}{\log|z|^{-1}} = q.$$

■

Therefore, we can apply our theorem to this class of functions. Finally we would like to state a nice result about a related family of maps proved by Linda Keen.

Theorem 5.9.3. *Consider the following family of functions*

$$\mathcal{F} := \{f(z) = z^n \exp(P(z) + Q(1/z)) \;:\; n \in \mathbb{Z},\; P, Q \in \mathbb{C}[Z]\}.$$

Every function of the form

$$g(z) = z^n \exp(E(z) + H(1/z))$$

with E and H entire functions that is topologically conjugate to a function $f \in \mathcal{F}$, then g is holomorphically conjugate to some $h \in \mathcal{F}$.

Look at [Kee89] for more properties of this type of functions.

Bibliography

[Ale94] Daniel S. Alexander, *A history of complex dynamics: From Schroder to Fatou and Julia*, Aspects of Mathematics, vol. 24, F. Vieweg and Sohn. Braunschweig, 1994, ISBN: 3-528-06520-6.

[AO93] Jan M. Aarts and Lex G. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 897–918.

[Bak87] Irvine N. Baker, *Wandering domains for maps of the punctured plane*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), no. 2, 191–198.

[BD98] Irvine N. Baker and Patricia Domínguez, *Analytic self-maps of the punctured plane*, Complex Variables Theory Appl. **37** (1998), no. 1-4, 67–91.

[Ber91] Walter Bergweiler, *Periodic points of entire functions: proof of a conjecture of Baker*, Complex Variables Theory Appl. **17** (1991), no. 1-2, 57–72.

[Ber95] ———, *On the Julia set of analytic self-maps of the punctured plane*, Analysis **15** (1995), no. 3, 251–256.

[BF01] Krzysztof Barański and Núria Fagella, *Univalent Baker domains*, Nonlinearity **14** (2001), no. 3, 411–429.

[BJR] Krzysztof Barański, Xavier Jarque, and Lasse Rempe, *Brushing the hairs of transcendental entire functions*, preprint, arXiv:1101.4209v1 [math.DS], January 2011.

[Dev94] Robert L. Devaney, *Complex dynamics and entire functions*, Complex dynamical systems (Providence) (R. L. Devaney, ed.), Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc., 1994, pp. 181–206.

[Dev99] ———, *Cantor bouquets, explosions, and Knaster continua: dynamics of complex exponentials*, Publ. Mat. **43** (1999), no. 1, 27–54.

[DT86] Robert L. Devaney and Folkert Tangerman, *Dynamics of entire functions near the essential singularity*, Ergodic Theory Dynam. Systems **6** (1986), no. 4, 489–503.

[EL92] Alexandre E. Eremenko and Mikhail Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

[Ere89] Alexandre E. Eremenko, *On the iteration of entire functions*, Dynamical systems and ergodic theory (Warsaw) (Karol Krzyżewski, ed.), Banach Center Publ., vol. 23, PWN, 1989, p. 339–345.

[Fag99] Núria Fagella, *Dynamics of the complex standard family*, J. Math. Anal. Appl. **229** (1999), no. 1, 1–31.

[Fat26] Pierre Fatou, *Sur l'itération des fonctions transcendantes entières* (french), Acta Math. **47** (1926), no. 4, 337–370.

[FK92] Hershel M. Farkas and Irwin Kra, *Riemann surfaces*, second ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992, ISBN: 0-387-97703-1.

[GH81] Marvin J. Greenberg and John R. Harper, *Algebraic topology. A first course*, Mathematics Lecture Note Series, vol. 58, Westview Press, 1981, ISBN: 978-0805335576.

[GS01] Chaim Goodman-Strauss, *Compass and straightedge in the Poincaré disk*, Amer. Math. Monthly **108** (2001), no. 1, 38–49.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002, ISBN: 0-521-79540-0.

[Kee88] Linda Keen, *Dynamics of holomorphic self-maps of* $\mathbb{C}^*$, Holomorphic functions and moduli, Vol. I (Berkeley, CA) (D. Drasin, C. J. Earle, F. W. Gehring, I. Kra, and A. Marden, eds.), Math. Sci. Res. Inst. Publ., vol. 10, Springer, 1988, pp. 9–30.

[Kee89] ———, *Topology and growth of a special class of holomorphic self-maps of* $\mathbb{C}^*$, Ergodic Theory Dynam. Systems **9** (1989), no. 2, 321–328.

[KL07] Linda Keen and Nikola Lakic, *Hyperbolic geometry from a local viewpoint*, London Mathematical Society Student Texts, vol. 68, Cambridge University Press, Cambridge, 2007, ISBN: 0-521-86360-0.

[Kot87] Janina Kotus, *Iterated holomorphic maps on the punctured plane*, Dynamical systems (Berlin) (A. B. Kurzhanski and K. Sigmund, eds.), Lecture Notes in Econom. and Math. Systems, vol. 287, Springer, 1987, p. 10–28.

[Kot90] ———, *The domains of normality of holomorphic self-maps of* $\mathbb{C}^*$, Ann. Acad. Sci. Fenn. Ser. A I Math. **15** (1990), no. 2, 329–340.

[Mil06] John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, 2006, ISBN: 978-0-691-12488-9.

[Nad92] Sam B. Nadler, *Continuum theory. an introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York, 1992, ISBN: 0-824-78659-9.

[Råd53] Hans Rådström, *On the iteration of analytic functions.*, Math. Scand. **1** (1953), 85–92.

[RRRS11] Günter Rottenfußer, Johannes Rückert, Lasse Rempe, and Dierk Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) **173** (2011), no. 1, 77–125.

[Sch10] Dierk Schleicher, *Dynamics of entire functions*, Holomorphic dynamical systems (Berlin) (G. Gentili, J. Guenot, and G. Patrizio, eds.), Lecture Notes in Math., vol. 1998, Springer, 2010, p. 295–339.

Index