SHAPE AND PERIOD OF LIMIT CYCLES BIFURCATING FROM A CLASS OF HAMILTONIAN PERIOD ANNULUS

R. PROHENS[1] AND J. TORREGROSA[2]

ABSTRACT. In this work we are concerned with the problem of shape and period of isolated periodic solutions of perturbed analytic radial Hamiltonian vector fields in the plane. Françoise develop a method to obtain the first non vanishing Poincaré-Pontryagin-Melnikov function. We generalize this technique and we apply it to know, up to any order, the shape of the limit cycles bifurcating from the period annulus of the class of radial Hamiltonians. We write any solution, in polar coordinates, as a power series expansion in terms of the small parameter. This expansion is also used to give the period of the bifurcated periodic solutions. We present the concrete expression of the solutions up to third order of perturbation of Hamiltonians of the form $H = H(r)$. Necessary and sufficient conditions that show if a solution is simple or double are also presented.

Keywords: Polynomial differential equation; bifurcation of limit cycles; shape, number, location and period of limit cycles

1. INTRODUCTION

Consider a Hamiltonian vector field having a continuous domain of closed trajectories (period annulus) in a neighbourhood of an equilibrium point. A classical problem in perturbation theory is the boundedness of the number of isolated periodic solutions bifurcating from the period annulus through analytic perturbations. In last years the shape and period of such periodic solutions have also been studied. Let us consider the equation

$$\begin{cases} \dot{x} = -\frac{\partial}{\partial y}H(x,y) + \varepsilon P(x,y,\varepsilon), \\ \dot{y} = \frac{\partial}{\partial x}H(x,y) + \varepsilon Q(x,y,\varepsilon), \end{cases} \tag{1}$$

where $H(x,y)$, $P(x,y,\varepsilon)$ and $Q(x,y,\varepsilon)$ are analytic functions and ε is a small parameter. Let us assume that, in (r,θ)-polar coordinates, H only depends on r, $H = H(r)$, and that the equilibrium point is at the origin. When $\varepsilon = 0$, we call this equation of Hamiltonian radial type. In this work we are involved with former problems concerning the perturbed Hamiltonian equation (1).

Equation (1) writes in polar coordinates as a 1-form

$$dH + \varepsilon\omega(r,\theta,\varepsilon) = 0, \tag{2}$$

see [4, §2.6] for instance. For equation (2) let $r(\theta; \rho, \varepsilon)$ be the solution such that $r(0; \rho, \varepsilon) = \rho$. Since for $\varepsilon = 0$ we have that $H(r(\theta; \rho, 0)) \equiv H(\rho)$ for any ρ and θ, then $r(\theta; \rho, \varepsilon)$ can be written as the explicit power series expansion in ε, with

coefficients depending on the angle θ and the initial condition ρ,

$$r(\theta; \rho, \varepsilon) = r_0(\theta) + \varepsilon r_1(\theta) + \varepsilon^2 r_2(\theta) + \dots, \qquad (3)$$

where $r_0(\theta) = \rho$ for all θ. Even thinking that $r_i(\theta)$ depends on ρ, to simplify the reading, we do not make it explicit. For sake of shortness we will refer this expansion as the *shape of the orbit*.

The goal of this paper is to give shape and period of the bifurcated limit cycles from radial Hamiltonians, up to any order in ε, under a unified treatment. We observe that the method described in this work also applies to get lower bounds of the number of limit cycles of polynomial vector fields, however this is not the aim of this paper. In [24] this problem, among others, is studied for equations like (2). Our work is based on a generalization of the method introduced by Françoise. This method, see [6], provides formulas for computing the first nonzero derivative of the return map associated to orbits of the perturbed Hamiltonian system with $H(x, y) = x^2 + y^2$.

The method is developed in Section 2. In the rest of sections, as an application, we study small perturbations of linear centers, isochronous centers with a polynomial linearization (linear-like) and non-linear radial Hamiltonians. Some of the presented examples have been studied previously but, to the best of our knowledge, the study of the period for the perturbed periodic orbits is new, except for the van der Pol equation for which we present some improvements. All the computations are made with the algebraic manipulator MAPLE.

The Poincaré-Pontryagin-Melnikov theory applied to equation (1) gives necessary and sufficient conditions so that, up to first order, limit cycles bifurcate from the period annulus. In this setting the first Poincaré-Pontryagin-Melnikov function takes the form

$$F_1(r) = \frac{1}{2\pi} \int_0^{2\pi} \beta(r, \theta, 0) d\theta$$

where $\omega(r, \theta, \varepsilon) = \alpha(r, \theta, \varepsilon) dr + \beta(r, \theta, \varepsilon) d\theta$. Then from each simple zero, ρ, of $F_1(r)$ emerges a hyperbolic limit cycle of (1) which tends to the curve $H(r) = H(\rho)$ when ε goes to zero, see [23]. This result gives the shape of the limit cycle up to order 0 in ε.

The general expressions of the functions $r_i(\theta)$ appearing in (3) are given in Theorem 1. Although in this result we only present the concrete expressions of the first three coefficients, higher order terms are computed in the applications. Moreover the method also provides an explicit way to obtain the series expansion of the period of periodic solutions, see Corollary 4.

Theorem 2 provides necessary conditions for which equation (2) has a periodic orbit written as (3) up to order 0, 1 or 2. For each simple zero of F_1, Theorem 3.(i) gives the explicit expression of $r_1(\theta)$ and the initial condition $r_1(0)$. In each step of our procedure we obtain not only the Poincaré-Pontryagin-Melnikov function of order i, F_i, but also the expression of $r_i(\theta)$. It is important to remark that to get the shape up to order i, in ε, it is necessary take into account terms that appear for orders higher than i. For example the expression of $r_1(0)$ that appears in Theorem 3.(i) depends on the first and second order terms. The shape of the limit cycles that bifurcates from a perturbed analytic system can also be studied,

in cartesian coordinates, in an implicit form. This is the approach of the works of Giacomini, Llibre and Viano, see [12, 13, 14, 15].

Section 3 is devoted to check the accuracy of our results. We start with a system which exhibits an explicit algebraic limit cycle and then the well known van der Pol equation is considered. For this classical example the shape, the maximum amplitude and the period of the limit cycle are provided up to higher order in ε. We improve some previous works given in the literature. Approximations of type (3) are presented in [18, §9.2], where a van der Pol oscillator is considered to give a procedure for carrying approximations, up to third order, of the total energy stored in a oscillation. In [26], the shape of closed trajectories of rotated vector fields is also studied and an implementation to the van der Pol equation is presented. The period and frequency of the periodic solution are studied in [1] and [3] and its Fourier expansion in [20]. The amplitude up to order 10 is done in [5] and up to order 25 in [3] where all the computations use exact rational arithmetic. Finally, in [19] the maximum amplitude for large values of ε is computed numerically. Our results give, for $\varepsilon \in (0, 1.34)$, values for the maximum amplitude which coincide up to six digits with the numerical approximations.

Another classical examples are the polynomial Liénard equations. In Section 4 next Liénard quintic equations are considered,

$$\begin{cases} \dot{x} & = & -y + \sum_{i=1}^{n} \varepsilon^i (a_{1,i} x + a_{3,i} x^3 + a_{5,i} x^5), \\ \dot{y} & = & x. \end{cases} \tag{4}$$

Perko, see [21, Th. 6, §3.8], proves that former equation up to first order analysis has exactly two limit cycles for some choice of the parameters. Proposition 10 provides the shape as well as the period for such perturbed limit cycles. Blows and Perko, in [2], give explicit sufficient conditions for which equation (4) has a double limit cycle. This problem is also studied in [14], where an algorithm for computing the saddle-node limit cycle bifurcation hypersurface in a neighbourhood of the origin of the parameter space is provided. As a generalization of these conditions for any system of type (2), Theorem 3.(ii) provides necessary and sufficient conditions to have two limit cycles bifurcating from each double zero of F_1. Moreover we get the initial conditions $r_1(0)$. An application of this result is done in Proposition 11 where the shape and period of both limit cycles are explicitly computed up to higher order. As a consequence of Theorem 2 it can be seen how this method also applies when F_1 vanish identically and a higher order study is necessary.

The method presented can be also used to study perturbations of some isochronous centers. In Section 5 we study some polynomial isochronous centers for which the linearization and its inverse are both polynomial. These transformations are called Jacobian changes, see [28] where a small class of them, the Tame transformations, are also considered. From a quadratic Tame transformation we can recover a cubic system studied in [27]. For these type of systems, as in the previous families, a higher order analysis can be done. A similar study is presented in [24] when the transformation is not polynomial but birational.

As an illustration, in the proof of Proposition 13, we prove that, for ε small enough, system

$$\begin{cases} \dot{u} & = & -v, \\ \dot{v} & = & u + \dfrac{\varepsilon}{2}\left(v - u^2 + 2v^2 - 4u^2v + 2u^4\right), \end{cases}$$

has a limit cycle cycle whose shape, up to second order approximation, is

$$r(\theta; 1, \varepsilon) = 1 + r_1(\theta)\varepsilon + r_2(\theta)\varepsilon^2 + \cdots,$$

where

$$\begin{aligned}
r_1(\theta) & = & -\frac{3}{4}\cos\theta - \frac{1}{8}\sin(2\theta) + \frac{1}{16}\cos(3\theta) + \frac{1}{16}\sin(4\theta) - \frac{1}{80}\cos(5\theta), \\
r_2(\theta) & = & -\frac{307}{1024} + \frac{39}{80}\sin\theta + \frac{213}{512}\cos(2\theta) - \frac{17}{960}\sin(3\theta) - \frac{1}{1280}\cos(4\theta) \\
& & -\frac{153}{1600}\sin(5\theta) + \frac{523}{15360}\cos(6\theta) + \frac{23}{2240}\sin(7\theta) - \frac{9}{1024}\cos(8\theta) \\
& & -\frac{1}{320}\sin(9\theta) + \frac{9}{25600}\cos(10\theta),
\end{aligned}$$

and the corresponding period is

$$\begin{aligned}
T(\varepsilon) \approx 2\pi\Bigg(& 1 + \frac{251}{320}\varepsilon^2 + \frac{23683041}{14336000}\varepsilon^4 + \frac{117449035916903}{20230963200000}\varepsilon^6 \\
& + \frac{37072650080368252263539}{16330562973204480000000}\varepsilon^8\Bigg).
\end{aligned}$$

This system, with the Tame transformation $(u, v) = (x, y + Cx^2)$, becomes a perturbed isochronous cubic system previously studied by Toni in [27].

In last section we exemplify, perturbing the Hamiltonian $H = (2r^2 + r^4)/4$, how the method presented in this paper also works for other radial Hamiltonian equations.

Finally we recall some related works, where the computation of the Poincaré first return map is considered, are Gasull and Torregrosa [10, 11], Iliev [16], Iliev and Perko [17], Poggiale [22] and Roussarie [25]. The results of these papers allow to compute the first non null term of the Poincaré map, $F_n(r) = r_n(2\pi)$, just under the assumption that $F_i(r) = r_i(2\pi) \equiv 0$, for $i = 1, \ldots, n - 1$, but when an arbitrary Hamiltonian H is considered, up to our knowledge, there are no general methods to compute explicitly such functions for any order.

We thank Armengol Gasull for his comments and suggestions during the realization of this work.

2. DESCRIBING THE METHOD AND MAIN RESULTS

We start by giving some definitions and notations. Let us assume that, in equation (1), we have that

$$P(x, y, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i P_i(x, y), \quad Q(x, y, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i Q_i(x, y),$$

where P_i, Q_i are analytic functions. In polar coordinates given by $(x, y) = (r \cos \theta, r \sin \theta)$ equation (1) writes as

$$
\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{r} H_\theta + \frac{1}{r} \sum_{i=1}^{\infty} \varepsilon^i R_i(r, \theta), \\
\frac{d\theta}{dt} &= \frac{1}{r} H_r + \frac{1}{r} \sum_{i=1}^{\infty} \varepsilon^i S_i(r, \theta),
\end{aligned}
\tag{5}
$$

or

$$
\frac{dr}{d\theta} = \frac{-H_\theta + \sum_{i=1}^{\infty} \varepsilon^i R_i(r, \theta)}{H_r + \sum_{i=1}^{\infty} \varepsilon^i S_i(r, \theta)},
\tag{6}
$$

where $H_\theta = \partial H/\partial \theta$, $H_r = \partial H/\partial r$ and

$$
\begin{aligned}
R_i(r, \theta) &= r \cos \theta \, P_i(r \cos \theta, r \sin \theta) + r \sin \theta \, Q_i(r \cos \theta, r \sin \theta), \\
S_i(r, \theta) &= \cos \theta \, Q_i(r \cos \theta, r \sin \theta) - \sin \theta \, P_i(r \cos \theta, r \sin \theta).
\end{aligned}
\tag{7}
$$

Since the Hamiltonian is of radial type we have that $H = H(r)$, hence $H_\theta \equiv 0$ and $dH = H_r \, dr = H'(r) \, dr$. Therefore equation (6) writes as the 1-form

$$
dH + \sum_{i=1}^{\infty} \varepsilon^i w_i(r, \theta) = 0
\tag{8}
$$

where

$$
\omega_i(r, \theta) = \alpha_i(r, \theta) \, dr + \beta_i(r, \theta) \, d\theta,
\tag{9}
$$

being $\alpha_i(r, \theta) = S_i(r, \theta)$ and $\beta_i(r, \theta) = -R_i(r, \theta)$. We point out that both, α_i and β_i, are 2π-periodic functions in θ.

The results of this work are based in a special decomposition, given in Lemma 5, of an arbitrary 1-form written in polar coordinates. It is reminiscent of the decompositions used by Françoise [6, 7, 8] and it is a generalization of Lemma 2.3 of Gasull and Torregrosa given in [11]. This generalization has been also described in [16] and [17]. Basically, the idea of the decomposition for any 1-form $\omega(r, \theta) = \alpha(r, \theta)dr + \beta(r, \theta)d\theta$, is described in the following. Since, in general, $\int_0^\theta \beta(r, \psi) \, d\psi$ is not a 2π-periodic function in θ, we decompose ω as $h(r, \theta) \, dH + dG(r, \theta) + F(r) \, d\theta$, where $G(r, \theta)$ is 2π-periodic in θ. This fact will make easier the calculation of the integrals of the 1-form $dG(r, \theta) + F(r) \, d\theta$ that appear in the proof of Theorem 1.

Next theorem gives the way to obtain the power series expansion (3) of any solution of equation (8). We note that it is not only an existential result but also an explicit method to compute all the functions that appear in its statement.

Theorem 1. *Let us consider polar coordinates in the plane. Let $H = H(r)$ be an analytic function and let ρ be a real number such that $H(r) = H(\rho)$ gives a closed curve in the plane such that $H'(\rho)$ does not vanish. Consider $r(\theta; \rho, \varepsilon)$ the solution of equation (8) with $r(0; \rho, \varepsilon) = \rho$. For each positive integer, n, there exist functions $F_n(r)$, $G_n(r, \theta)$ and $h_n(r, \theta)$, 2π-periodic in θ, such that the 1-form*

$$
W_n = -\sum_{i=1}^{n} \omega_i h_{n-i},
$$

writes as $W_n = W_n^0 + W_n^1$, *where* $W_n^0 = h_n dH + dG_n$ *and* $W_n^1 = F_n d\theta$. *By way of notation we take* $h_0 \equiv 1$. *If we write the solutions of equation* (8) *in the form*

$$r(\theta; \rho, \varepsilon) = r_0(\theta) + \varepsilon r_1(\theta) + \varepsilon^2 r_2(\theta) + \varepsilon^3 r_3(\theta) + \cdots,$$

then we have that

$$r_0(\theta) = \rho,$$

$$r_1(\theta) = r_1(0) + \frac{1}{H'(\rho)}\Big(F_1(\rho)\theta + G_1(\rho, \theta) - G_1(\rho, 0)\Big),$$

$$r_2(\theta) = r_2(0) + \frac{1}{H'(\rho)}\Big(-\frac{1}{2}H''(\rho)\left(r_1^2(\theta) - r_1^2(0)\right) + F_1'(\rho)\int_0^\theta r_1(\psi)d\psi$$

$$+ \frac{\partial G_1}{\partial r}(\rho, \theta)r_1(\theta) - \frac{\partial G_1}{\partial r}(\rho, 0)r_1(0) + F_2(\rho)\theta + G_2(\rho, \theta) - G_2(\rho, 0)\Big),$$

$$r_3(\theta) = r_3(0) + \frac{1}{H'(\rho)}\Big(-H''(\rho)(r_1(\theta)r_2(\theta) - r_1(0)r_2(0))$$

$$- \frac{1}{6}H'''(\rho)(r_1^3(\theta) - r_1^3(0)) + F_1'(\rho)\int_0^\theta r_2(\psi)d\psi + \frac{1}{2}F_1''(\rho)\int_0^\theta r_1^2(\psi)d\psi$$

$$+ \frac{\partial G_1}{\partial r}(\rho, \theta)r_2(\theta) - \frac{\partial G_1}{\partial r}(\rho, 0)r_2(0) + \frac{1}{2}\frac{\partial^2 G_1}{\partial r^2}(\rho, \theta)r_1^2(\theta) - \frac{1}{2}\frac{\partial^2 G_1}{\partial r^2}(\rho, 0)r_1^2(0)$$

$$+ F_2'(\rho)\int_0^\theta r_1(\psi)d\psi + \frac{\partial G_2}{\partial r}(\rho, \theta)r_1(\theta) - \frac{\partial G_2}{\partial r}(\rho, 0)r_1(0)$$

$$+ F_3(\rho)\theta + G_3(\rho, \theta) - G_3(\rho, 0)\Big).$$

In the above result, and in the rest of the paper, we have written $r_i(\theta)$ instead of $r_i(\theta; \rho)$, in order to simplify notation, when it is not necessary to make explicit the dependence in ρ. Moreover, from the expressions of $r_i(\theta)$, we can obtain necessary conditions to know if a solution of equation (8) is periodic as it is proved in next theorem.

Theorem 2. *Let us assume hypotheses of Theorem 1 and let us define*

$$C_1 = F_1(\rho),$$

$$C_2 = F_1'(\rho)\int_0^{2\pi} r_1(\psi)d\psi + 2\pi F_2(\rho),$$

$$C_3 = F_1'(\rho)\int_0^{2\pi} r_2(\psi)d\psi + \frac{1}{2}F_1''(\rho)\int_0^{2\pi} r_1^2(\psi)d\psi + F_2'(\rho)\int_0^{2\pi} r_1(\psi)d\psi + 2\pi F_3(\rho).$$

Hence, conditions $C_1 = 0$, $C_1 = C_2 = 0$ *and* $C_1 = C_2 = C_3 = 0$, *are necessary so that* $r(\theta; \rho, \varepsilon) = \rho + O(\varepsilon)$, $r(\theta; \rho, \varepsilon) = \rho + r_1(\theta)\varepsilon + O(\varepsilon^2)$ *and* $r(\theta; \rho, \varepsilon) = \rho + r_1(\theta)\varepsilon + r_2(\theta)\varepsilon^2 + O(\varepsilon^3)$ *are periodic solutions up to order* 0, 1 *and* 2 *in* ε *of equation* (8), *respectively.*

We note that in previous result, nevertheless, nothing is said about sufficient conditions guaranteeing the existence of such limit cycle. In next theorem we give sufficient conditions so that, close to the solution $H(r) = H(\rho)$ of the unperturbed equation, the equation (8) has isolated periodic solutions for ε small enough.

Theorem 3. *Let us assume the hypotheses of Theorem 1.*

(i) *If $F_1(\rho) = 0$ and $F_1'(\rho) \neq 0$ then equation (8) has a hyperbolic limit cycle close to $r(\theta; \rho, \varepsilon) = \rho + r_1(\theta)\varepsilon + O(\varepsilon^2)$, where $r_1(\theta)$ is given in Theorem 1 and*

$$r_1(0) = -\frac{F_2(\rho)}{F_1'(\rho)} - \frac{1}{2\pi}\int_0^{2\pi} \frac{G_1(\rho, \theta) - G_1(\rho, 0)}{H'(\rho)}d\theta.$$

(ii) *If $F_1(\rho) = F_1'(\rho) = F_2(\rho) = 0$ and $F_1''(\rho) \neq 0$ then equation (8) gives rise for ε small enough two (no) limit cycles when $\Delta > 0$ ($\Delta < 0$). Here we take $\Delta = B^2 - 4AC$ where*

$$A = \pi F_1''(\rho),$$

$$B = 2\pi F_2'(\rho) + F_1''(\rho)\int_0^{2\pi} g_1(\theta, \rho)\,d\theta,$$

$$C = 2\pi F_3(\rho) + F_2'(\rho)\int_0^{2\pi} g_1(\theta, \rho)\,d\theta + \frac{1}{2}F_1''(\rho)\int_0^{2\pi} g_1^2(\theta, \rho)\,d\theta,$$

being $g_1(\theta, \rho) = (G_1(\theta, \rho) - G_1(0, \rho))/H'(\rho)$. When the two limit cycles exist, they can be written as $r(\theta; \rho, \varepsilon) = \rho + r_1(\theta)\varepsilon + O(\varepsilon^2)$ where $r_1(\theta)$ is given in Theorem 1. Moreover the corresponding $r_1(0)$ is each of the two solutions of the equation $A\lambda^2 + B\lambda + C = 0$.

(iii) *If $F_1(\rho) \equiv 0$, $F_2(\rho) = 0$ and $F_2'(\rho) \neq 0$ then equation (8) has a hyperbolic limit cycle close to $r(\theta; \rho, \varepsilon) = \rho + r_1(\theta)\varepsilon + O(\varepsilon^2)$, where $r_1(\theta)$ is given in Theorem 1 and*

$$r_1(0) = -\frac{F_3(\rho)}{F_2'(\rho)} - \frac{1}{2\pi}\int_0^{2\pi} \frac{G_1(\rho, \theta) - G_1(\rho, 0)}{H'(\rho)}d\theta.$$

The procedure used in the proof of the former theorem provides explicit expressions for $r_i(0)$, $i = 2, 3, \ldots$ We do not write down these expressions because of their complexity, but in next sections we compute them for the families described in the introduction.

We observe that, from the expressions that appear in Theorem 3, if F_1 has a simple zero then the shape up to first order depends on the functions F_1, G_1, corresponding to first order, and F_2 that comes from the second order study. When F_1 has a double zero or vanishes identically we also need the expression of F_3 that comes from the third order. This happens in general for any order.

Next result gives the expression of the period of the perturbed periodic orbits.

Corollary 4. *Let us assume the hypotheses of Theorem 1. Then the period of the periodic solution, $r(\theta; \rho, \varepsilon)$, of equation (8) given in Theorem 3 satisfies*

$$T(\varepsilon; \rho) = \int_0^{2\pi} \frac{r(\theta; \rho, \varepsilon)}{H'(r(\theta; \rho, \varepsilon)) + \sum_{i=1}^{\infty} \varepsilon^i S_i(r(\theta; \rho, \varepsilon), \theta)}d\theta,$$

where S_i is defined in (7) and developing it up to first order we obtain

$$T(\varepsilon; \rho) \approx 2\pi\left(\frac{\rho}{H'(\rho)} - \varepsilon\rho H''(\rho)r_1(0)\right).$$

For proving our main results we need to introduce some notation and technical lemmas.

Lemma 5. *Let* $\Omega = \alpha(r,\theta)dr + \beta(r,\theta)d\theta$ *be an arbitrary analytic 1-form, 2π-periodic in θ, let $H = H(r)$ be an analytic function, let ρ be a real number such that $H(r) = H(\rho)$ gives a closed curve in the plane and $H'(\rho)$ does not vanish. Then taking next functions, 2π-periodic in θ,*

$$F(r) = \frac{1}{2\pi} \int_0^{2\pi} \beta(r,\psi)d\psi,$$

$$G(r,\theta) = \int_0^\theta \beta(r,\psi)\,d\psi - F(r)\theta,$$

$$h(r,\theta) = \left(\alpha(r,\theta) - \frac{\partial G(r,\theta)}{\partial r}\right)/H'(r),$$

we can write

$$\Omega = \Omega^0 + \Omega^1, \tag{10}$$

where

$$\Omega^0 = h\,dH + dG \quad and \quad \Omega^1 = F\,d\theta. \tag{11}$$

Moreover

$$\int_{H=\rho} \Omega = \int_{H=\rho} \Omega^1. \tag{12}$$

Proof. To get decomposition (10) fulfilling equalities (11), it is necessary that conditions $h(r,\theta)H'(r) + \partial G(r,\theta)/\partial r = \alpha(r,\theta)$ and $\partial G(r,\theta)/\partial\theta + F(r) = \beta(r,\theta)$ are met. By imposing the periodicity of $G(r,\theta)$, i.e. that $G(r,0) = G(r,2\pi)$, the decomposition follows. From these conditions and by taking into account that $\int_{H=\rho} \Omega = \int_{H=\rho} \beta(r,\theta)\,d\theta$, equality (12) follows straightforward. $\qquad\square$

Remark 6.   *(i) A noteworthy fact is that previous lemma does not depends on the interval of integration on which we define the functions $F(r)$ and $G(r,\theta)$. The integrals depend only on the length of the interval, i.e. one could consider not only the interval $[0,2\pi]$ but also any interval of length 2π obtaining an equivalent decomposition. This approach is done in [24].*

(ii) From the proof of previous lemma one can see that there are different ways to define the function $G(r,\theta)$ so that $\Omega^0 = h\,dH + dG$. However, since $G(r,\theta)$ is forced to be a 2π-periodic function, there is a unique way in which $G(r,\theta)$ can be chosen, except for the constant of integration.

Definition 7.   *(i) Let $f(\theta;\rho,\varepsilon) = \sum_{i\geq 0} f_i\,\varepsilon^i$ be given. We define*

$$\mathcal{D}_i(f) = f_i = \frac{1}{i!}\frac{d^i}{d\varepsilon^i}f(\theta;\rho,\varepsilon)\Big|_{\varepsilon=0}.$$

(ii) Let us consider the 1-form $\omega = \alpha(r,\psi,\varepsilon)dr + \beta(r,\psi,\varepsilon)d\psi$ and, for each $\theta \in [0,2\pi]$, the curve $\gamma_\varepsilon(\theta)$ given by $r(\psi;\rho,\varepsilon) = \rho + \varepsilon r_1(\psi) + \varepsilon^2 r_2(\psi) + \cdots$, whenever $0 \leq \psi \leq \theta$. For each i we define

$$\mathcal{Q}_i(\omega,\gamma_\varepsilon(\theta)) = \frac{1}{i!}\left(\frac{d^i}{d\varepsilon^i}\int_{\gamma_\varepsilon(\theta)}\omega\right)\Big|_{\varepsilon=0}.$$

By way of notation we write $\mathcal{Q}_i(\omega) = \mathcal{Q}_i\left(\omega,\gamma_\varepsilon(\theta)\right)$.

Next result follows straightforward from the power series expansion, in a neighbourhood of a point, of a one variable function.

Lemma 8. *Let $f(r)$ be an analytic function. If $r = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \cdots$, then $f(r) = f_0(r_0) + \varepsilon f_1(r_0) + \varepsilon^2 f_2(r_0) + \cdots$ where*

$$f_0(r_0) = \mathcal{D}_0(f) = f(r_0),$$

$$f_1(r_0) = \mathcal{D}_1(f) = f'(r_0)r_1,$$

$$f_2(r_0) = \mathcal{D}_2(f) = f'(r_0)r_2 + \frac{1}{2}f''(r_0)r_1^2,$$

$$f_3(r_0) = \mathcal{D}_3(f) = f'(r_0)r_3 + f''(r_0)r_1 r_2 + \frac{1}{6}f'''(r_0)r_1^3,$$

$$f_4(r_0) = \mathcal{D}_4(f) = f'(r_0)r_4 + \frac{1}{2}f''(r_0)(2r_1 r_3 + r_2^2) + \frac{1}{2}f'''(r_0)r_1^2 r_2 + \frac{1}{24}f^{(4)}(r_0)r_1^4,$$

for ε small enough.

Lemma 9. *In polar coordinates, let us consider the 1-form $\omega = \alpha(r,\theta)dr + \beta(r,\theta)d\theta$, where α and β are analytic functions, 2π-periodic in θ, and let γ_ε be a curve written as $r = r(\theta;\rho,\varepsilon) = \rho + \varepsilon r_1(\theta) + \varepsilon^2 r_2(\theta) + \cdots$. Then, for all n*

$$\mathcal{Q}_n(\omega) = \int_0^\theta \left(\mathcal{D}_n\left(\beta(\rho,\psi)\right) + \sum_{i=0}^{n-1} \mathcal{D}_i\left(\frac{\partial r_{n-i}(\psi)}{\partial \psi} \alpha(\rho,\psi) \right) \right) d\psi, \qquad (13)$$

and $\mathcal{Q}_0(\omega) = \int_0^\theta \beta(\rho,\psi)d\psi$. In particular we have that

(i) if $\alpha \equiv 0$ then

$$\mathcal{Q}_n(\omega) = \int_0^\theta \mathcal{D}_n(\beta(\rho,\psi))\, d\psi,$$

(ii) if there exists a function W such that $dW = \omega$ then

$$\mathcal{Q}_n(\omega) = \mathcal{D}_n(W)(\theta) - \mathcal{D}_n(W)(0).$$

Proof. For simplicity we will write $r(\theta,\varepsilon)$ instead of $r(\theta;\rho,\varepsilon)$. From Definition 7.(i) and using the parametrization of γ_ε, we have that

$$\frac{\partial^n}{\partial\varepsilon^n}\int_{\gamma_\varepsilon(\theta)}\omega\bigg|_{\varepsilon=0} = \frac{\partial^n}{\partial\varepsilon^n}\int_0^\theta \left(\alpha(r(\psi,\varepsilon),\psi)\frac{\partial}{\partial\psi}r(\psi,\varepsilon) + \beta(r(\psi,\varepsilon),\psi) \right) d\psi \bigg|_{\varepsilon=0}$$

$$= \int_0^\theta \sum_{i=0}^n \binom{n}{i} \frac{\partial^i}{\partial\varepsilon^i}\alpha(r(\psi,\varepsilon),\psi) \frac{\partial^{n-i}}{\partial\varepsilon^{n-i}}\left(\frac{\partial}{\partial\psi}r(\psi,\varepsilon) \right) d\psi \bigg|_{\varepsilon=0} + n!\int_0^\theta \mathcal{D}_n(\beta(\rho,\psi))d\psi.$$

$$(14)$$

Using again the parametrization of γ_ε, we have that

$$\frac{\partial^{n-i}}{\partial\varepsilon^{n-i}}\left(\frac{\partial}{\partial\psi}r(\psi,\varepsilon) \right) = \frac{\partial^{n-i}}{\partial\varepsilon^{n-i}}\left(\sum_{j=1}^\infty \varepsilon^j \frac{\partial}{\partial\psi}r_j(\psi) \right) = \sum_{j=0}^\infty c_{n,i,j}\varepsilon^j \frac{\partial}{\partial\psi}r_{n-i+j}(\psi),$$

for some real numbers $c_{n,i,j}$ such that $c_{n,i,0} = (n-i)!$. Hence,

$$\int_0^\theta \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} \frac{\partial^i}{\partial \varepsilon^i} \alpha(r(\psi,\varepsilon),\psi) \frac{\partial^{n-i}}{\partial \varepsilon^{n-i}} \left(\frac{\partial}{\partial \psi} r(\psi,\varepsilon) \right) d\psi \bigg|_{\varepsilon=0}$$

$$= \int_0^\theta \frac{1}{n!} \sum_{i=0}^{n-1} \binom{n}{i} \frac{\partial^i}{\partial \varepsilon^i} \alpha(r(\psi,\varepsilon),\psi) \, c_{n,i,0} \frac{\partial}{\partial \psi} r_{n-i}(\psi) \, d\psi$$

$$= \int_0^\theta \sum_{i=0}^{n-1} \mathcal{D}_i \left(\frac{\partial r_{n-i}(\psi)}{\partial \psi} \alpha(\rho,\psi) \right) d\psi,$$

which finishes the proof of expression (13). $\qquad\qquad\qquad\qquad\square$

We end this section with the proofs of the main results.

Proof of Theorem 1. First, let us obtain the decomposition of the 1-form $W_n = -\sum_{i=1}^n \omega_i \, h_{n-i}$ as $W_n = W_n^0 + W_n^1$, by induction on n. In case $n = 1$ we have, assuming $h_0 \equiv 1$, that $W_1 = -\omega_1$. Since ω_1 is under hypotheses of Lemma 5, we have that there exist functions $F_1(r)$ and $G_1(r,\theta)$, $h_1(r,\theta)$, 2π-periodic in θ, such that the 1-form W_1 decomposes as $W_1 = W_1^0 + W_1^1$, where $W_1^0 = h_1 \, dH + dG_1$ and $W_1^1 = F_1 \, d\theta$. Thus, applying Lemma 5 to W_n for each $n \geq 2$, it turns out that W_n is well defined and decomposes as in the statement.

To obtain the expressions of $r_i(\theta)$, $i = 0,\ldots,3$, we consider $r(\theta;\rho,\varepsilon)$ as the solution of the initial value problem given by equation (8) with $r(0;\rho,\varepsilon) = \rho$. For purposes of notation we briefly write it as γ_ε. By integrating equation (8) over γ_ε, we have that

$$\int_{\gamma_\varepsilon} dH + \sum_{i=1}^\infty \varepsilon^i \int_{\gamma_\varepsilon} \omega_i(r,\theta) = 0.$$

If we collect the constant term in ε, then we get

$$H(r_0(\theta)) - H(r_0(0)) = 0.$$

Hence, as H is an analytic function, we get that $r_0(\theta) = r_0(0) = \rho$, for all $\theta \in [0, 2\pi]$. The rest of the expressions are proved as follows. From equation (8) we consider next equality

$$\left(\sum_{i=0}^n h_i \, \varepsilon^i \right) \left(dH + \sum_{i=1}^\infty w_i \, \varepsilon^i \right) = 0,$$

which is equivalent to

$$dH + \sum_{n=1}^\infty \left(\sum_{i=1}^n w_i h_{n-i} + h_n \, dH \right) \varepsilon^n = 0.$$

Hence, from the definition of W_n, we may write equation (8) as

$$dH - \sum_{n=1}^\infty (F_n \, d\theta + dG_n) \, \varepsilon^n = 0,$$

for some functions F_n and G_n the existence of which has been shown just before. By integrating last equation over γ_ε we get,

$$\int_{\gamma_\varepsilon} dH = \sum_{i=1}^{n} \int_{\gamma_\varepsilon} (F_n \, d\theta + dG_n) \, \varepsilon^i + O\left(\varepsilon^{n+1}\right). \tag{15}$$

Collecting terms in ε^i, $i \geq 1$, both in the left and right hand sides of previous expression, we get

$$\mathcal{Q}_i(dH) = \sum_{j=0}^{i-1} \mathcal{Q}_j(F_{i-j} d\theta + dG_{i-j}). \tag{16}$$

Let us assume that last condition is imposed for i from 1 to $n-1$. This forces that equality (15) is satisfied up to powers of order $n-1$ in ε. Let us recall that γ_ε is parametrized by the angle θ, as (3).

The condition to have equality (15) satisfied also for terms in ε^n follows from expression (16), using relations given in Lemma 9 and by the fact that $\mathcal{Q}_i$ is a linear operator. This condition writes as

$$\mathcal{D}_n(H)(\theta) - \mathcal{D}_n(H)(0) = \sum_{i=0}^{n-1} \left(\int_0^\theta \mathcal{D}_i(F_{n-i})(\psi) \, d\psi \right.$$
$$\left. + \mathcal{D}_i(G_{n-i})(\theta) - \mathcal{D}_i(G_{n-i})(0) \right). \tag{17}$$

In particular, when $n = 1$ we have that $\mathcal{D}_1(H) = H'(\rho)$ and, hence,

$$H'(\rho)(r_1(\theta) - r_1(0)) = F_1(\rho)\theta + G_1(\rho, \theta) - G_1(\rho, 0),$$

which gives the expression of $r_1(\theta)$ of the statement of this theorem. The expressions corresponding to $r_2(\theta)$ and $r_3(\theta)$ are also obtained from (17). □

Proof of Theorem 2. Let $r(\theta; \rho, \varepsilon)$ be the solution of the initial value problem given by equation (8) with $r(0; \rho, \varepsilon) = \rho$. From Theorem 1 we know that the first necessary condition so that $r(\theta; \rho, \varepsilon)$ is a solution up to order 0 in ε is that $r_0(\theta) \equiv \rho$. Even more, this solution will be a periodic solution if $r(2\pi; \rho, \varepsilon) = r(0; \rho, \varepsilon)$. Hence, conditions $r_1(2\pi) = r_1(0)$, $r_2(2\pi) = r_2(0)$ and $r_3(2\pi) = r_3(0)$, need to be satisfied to have a periodic solution. Hence, again from Theorem 1 and since $G_n(r, \theta)$, $n = 1, 2, \ldots$, is a 2π-periodic function on θ, each one of previous conditions is, respectively, equivalent to $C_1 = 0$, $C_2 = 0$ and $C_3 = 0$, as we would prove. □

Proof of Theorem 3. Since $F_1(\rho) = 0$, as a consequence of the Implicit Function Theorem, the existence of limit cycles near $r = \rho$, for ε small enough, is guaranteed. To see details we refer to [2]. Even more, from Theorem 1 we have that

$$r_1(\theta) = r_1(0) + \frac{1}{H'(\rho)} \left(G_1(\rho, \theta) - G_1(\rho, 0)\right).$$

In the case (i), the condition $C_2 = 0$ given in Theorem 2 allow us to compute $r_1(0)$ from

$$2\pi F_1'(\rho) r_1(0) + F_1'(\rho) \int_0^{2\pi} \frac{G_1(\rho, \theta) - G_1(\rho, 0)}{H'(\rho)} d\theta + 2\pi F_2(\rho) = 0.$$

The equivalent equation of degree one for $r_2(0)$ can be easily obtained substituting the explicit expression of $r_1(\theta)$ in the condition $C_3 = 0$ given in Theorem 2. The procedure described in Lemma 9 provides the sequence of equations of degree one that satisfies $r_i(0)$ for all $i \geq 2$.

In the case (ii), from condition $C_3 = 0$ of Theorem 2, $r_1(0)$ satisfies the quadratic equation of the statement. This equation may have two or no solutions according to the sign of its discriminant, Δ. The values of $r_i(0)$ for all $i \geq 2$ can be obtained in a similar way as in the previous case.

The case (iii) follows similarly as case (i) and $r_1(0)$ comes from condition $C_3 = 0$. □

Proof of Corollary 4. The expression for the period is an immediate consequence of the explicit computation of (3) given in Theorem 1. Substituting it in equation (5), corresponding to $d\theta/dt$, and computing its power expansion in ε, the proof ends by integrating the variable θ over the interval $[0, 2\pi]$. □

3. TEST EXAMPLES

In this section we apply the results of Section 2 to describe the periodic solution of some families of differential equations used as test examples. The first one exhibits an explicit algebraic limit cycle of degree two and hence all the expressions can be checked explicitly. The second one is the well-known van der Pol equation. For these differential equations the results have been tested doing a comparison with some previous works that use different techniques.

3.1. **An example with an explicit algebraic limit cycle.** Next system

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x + \varepsilon x - \varepsilon y(x^2 + y^2 + \varepsilon x^2 - 1), \end{cases} \tag{18}$$

has the algebraic curve $x^2 + y^2 + \varepsilon x^2 - 1 = 0$ as a limit cycle. From Theorem 3.(i) and Lemma 5, for ε small enough, (18) presents limit cycles bifurcating from the circles of radius ρ where ρ are the simple zeros of the function

$$F_1(r) = -\frac{1}{2}r^4 + \frac{1}{2}r^2.$$

This function has only one positive zero, which is simple, at $\rho = 1$. The expression of $r(\theta; \rho, \varepsilon)$, from Theorem 1 and Theorem 3.(i), is

$$r(\theta; 1, \varepsilon) = 1 - \frac{1}{2}\varepsilon \cos^2\theta + \frac{3}{8}\varepsilon^2 \cos^4\theta - \frac{5}{16}\varepsilon^3 \cos^6\theta + \frac{35}{128}\varepsilon^4 \cos^8\theta + \cdots,$$

that agrees with the series development of the real solution $x^2 + y^2 + \varepsilon x^2 - 1 = r^2(1 + \varepsilon \cos^2\theta) - 1 = 0$, that is $r(\theta, \varepsilon) := r(\theta; 1, \varepsilon) = 1/\sqrt{1 + \varepsilon \cos^2\theta}$.

From the above expression of the radius and the differential equation in polar coordinates, the period of the limit cycle, see Corollary 4, can be computed as

$$T(\varepsilon; 1) = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon \sin\theta(\cos\theta + \sin\theta \cos\theta - r^2(\theta, \varepsilon)) - \varepsilon^2 \sin^3\theta \cos\theta r^2(\theta, \varepsilon)}$$

$$= \frac{2\pi}{\sqrt{1 + \varepsilon}} = 2\pi\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \frac{35}{128}\varepsilon^4 - \frac{63}{256}\varepsilon^5 + \cdots\right).$$

It can be checked that the method described in Section 2 gives the previous Taylor expansion up to any order.

3.2. **The van der Pol equation.** Let us consider van der Pol's equation

$$\begin{cases} \dot{x} & = & -y + \varepsilon x(y^2 - 1), \\ \dot{y} & = & x, \end{cases} \tag{19}$$

where ε is any real number. From Liénard's Theorem it is known that, for $\varepsilon \neq 0$, (19) has a unique hyperbolic limit cycle. Additionally it is known that when ε goes to 0, the limit cycle tends to the circle of radius 2.

According Theorem 3.(i) and Lemma 5, initial conditions for limit cycles of equation (19) are given by the simple zeros of

$$F_1(r) = \frac{1}{8}r^4 - \frac{1}{2}r^2.$$

The above polynomial only has a positive root, $\rho = 2$, which is simple. Hence, equation (19) has a limit cycle that tends to the circle of radius 2 when ε goes to zero. The shape of this limit cycle $r(\theta; \rho, \varepsilon)$, from Theorem 1 and Theorem 3.(i), is given by

$$r(\theta; 2, \varepsilon) = 2 + \varepsilon r_1(\theta) + \varepsilon^2 r_2(\theta) + \varepsilon^3 r_3(\theta) + \varepsilon^4 r_4(\theta) + \cdots$$

where the first values of $r_i(\theta)$ are,

$$r_1(\theta) = -\frac{1}{2}\sin(2\theta) - \frac{1}{4}\sin(4\theta),$$

$$r_2(\theta) = \frac{7}{128} + \frac{5}{32}\cos(2\theta) + \frac{3}{32}\cos(4\theta) - \frac{7}{96}\cos(6\theta) - \frac{7}{128}\cos(8\theta),$$

$$r_3(\theta) = \frac{19}{512}\sin(2\theta) + \frac{199}{6144}\sin(4\theta) - \frac{475}{9216}\sin(6\theta) - \frac{79}{1536}\sin(8\theta)$$
$$+ \frac{15}{1024}\sin(10\theta) + \frac{33}{2048}\sin(12\theta),$$

$$r_4(\theta) = -\frac{167}{1179648} - \frac{515}{49152}\cos(2\theta) - \frac{2743}{147456}\cos(4\theta)$$
$$+ \frac{9767}{442368}\cos(6\theta) + \frac{10505}{294912}\cos(8\theta) - \frac{1229}{81920}\cos(10\theta)$$
$$- \frac{3641}{147456}\cos(12\theta) + \frac{143}{49152}\cos(14\theta) + \frac{715}{131072}\cos(16\theta).$$

We note that $r_1(0) = r_3(0) = 0$ (in fact, by symmetry, $r_i(0) = 0$ for any odd natural number i) and $r_2(0) = 17/96$, $r_4(0) = -1577/552960$. More terms of the expression of $r(\theta; \rho, \varepsilon)$ can be easily computed.

From Corollary 4 and adding some extra terms to the above expression of $r(\theta; 2, \varepsilon)$ the first terms of the Taylor series, in ε, of the period of the limit cycle

can be obtained. Then

$$T(\varepsilon; 2) \approx 2\pi \left(1 + \frac{1}{16}\varepsilon^2 - \frac{5}{3072}\varepsilon^4 - \frac{431}{884736}\varepsilon^6 + \frac{557039}{5096079360}\varepsilon^8 + \frac{51720623}{9172942848000}\varepsilon^{10} \right.$$

$$- \frac{21232697921411}{3698530556313600000}\varepsilon^{12} + \frac{1954685779155368107}{5219366321069752320000000}\varepsilon^{14}$$

$$+ \frac{4076885097753254949526607}{1473113950458726894796800000000}\varepsilon^{16}$$

$$- \frac{2241455425314264055576930783643}{415771681377471078787448832000000000000}\varepsilon^{18}$$

$$\left. - \frac{41643131119097700402368554721725348469}{38724641786152554301399954253414400000000000000}\varepsilon^{20} \right).$$

The limit cycle in cartesian coordinates can be parametrized as

$$(x(\theta, \varepsilon), y(\theta, \varepsilon)) = (r(\theta; 2, \varepsilon) \cos\theta, r(\theta; 2, \varepsilon) \sin\theta).$$

As in the proof of Corollary 4 we obtain the series of the time, $t(\theta; \varepsilon)$, as a function of θ. Using the inverse series we can compute $\theta(t; \varepsilon)$ and hence $y(t; \varepsilon)$, which satisfies

$$y'' - \varepsilon(y^2 - 1)y' + y = 0. \tag{20}$$

The Fourier series of $y(t; \varepsilon)$, using the frequency

$$\nu(\varepsilon; 2) = \frac{2\pi}{T(\varepsilon; 2)} \approx 1 - \frac{1}{16}\varepsilon^2 + \frac{17}{3072}\varepsilon^4 + \frac{35}{884736}\varepsilon^6 - \frac{678899}{5096079360}\varepsilon^8 + \frac{28160413}{2293235712000}\varepsilon^{10},$$

writes as

$$y(t; \varepsilon) \approx \left(2 + \frac{11}{4096}\varepsilon^4 \right) \sin(\nu t) + \left(-\frac{1}{4}\varepsilon + \frac{19}{768}\varepsilon^3 \right) \cos(\nu t)$$

$$+ \left(\frac{3}{16}\varepsilon^2 - \frac{29}{768}\varepsilon^4 \right) \sin(3\nu t) + \left(\frac{1}{4}\varepsilon - \frac{21}{256}\varepsilon^3 \right) \cos(3\nu t)$$

$$+ \left(-\frac{5}{96}\varepsilon^2 + \frac{1385}{27648}\varepsilon^4 \right) \sin(5\nu t) + \frac{5}{72}\varepsilon^3 \cos(5\nu t)$$

$$- \frac{2555}{110592}\varepsilon^4 \sin(7\nu t) - \frac{7}{576}\varepsilon^3 \cos(7\nu t) + \frac{61}{20480}\varepsilon^4 \sin(9\nu t).$$

The maximum amplitude of the periodic solution of (19) is reached for $\theta = \pi/2$ and it is

$$A(\varepsilon) = r\left(\frac{\pi}{2}; 2, \varepsilon \right) \approx 2 + \frac{1}{96}\varepsilon^2 - \frac{1033}{552960}\varepsilon^4 + \frac{1019689}{55738368000}\varepsilon^6$$

$$+ \frac{9835512276689}{1573159698432}\frac{\varepsilon^8}{10^5} - \frac{58533181813182818069}{732614178920988672}\frac{\varepsilon^{10}}{10^7}$$

$$- \frac{64080164704545321056982034833}{2927285727081047394311120896}\frac{\varepsilon^{12}}{10^9}$$

$$+ \frac{1032283355649797775460514246499330153 3597}{1520541448265573517964321797244452864}\frac{\varepsilon^{14}}{10^{11}}.$$

The previous expansions either agree or improve some previous works given in literature. In [1] and [3] the series of the period and frequency of the periodic solution of (20) are given. The Fourier series expansion of $y(t; \varepsilon)$ with a time

translation to eliminate the $\sin(\nu t)$ term is done in [20]. Finally, in [19] the maximum amplitude for large values of ε is computed numerically.

In Figure 1 we depict the period and maximum amplitude of the periodic solution of equation (20) both approximated up to order 20 in ε ($T_{20}(\varepsilon)$ and $A_{20}(\varepsilon)$) and numerically ($\overline{T}(\varepsilon)$ and $\overline{A}(\varepsilon)$). The differences are less than 10^{-6} when $\varepsilon \in [0, 1.18)$ and $\varepsilon \in [0, 1.34)$, respectively.

(a) $T_{20}(\varepsilon), \overline{T}(\varepsilon)$ (b)$T_{20}(\varepsilon) - \overline{T}(\varepsilon)$

(c) $A_{20}(\varepsilon), \overline{A}(\varepsilon)$ (d) $A_{20}(\varepsilon) - \overline{A}(\varepsilon)$
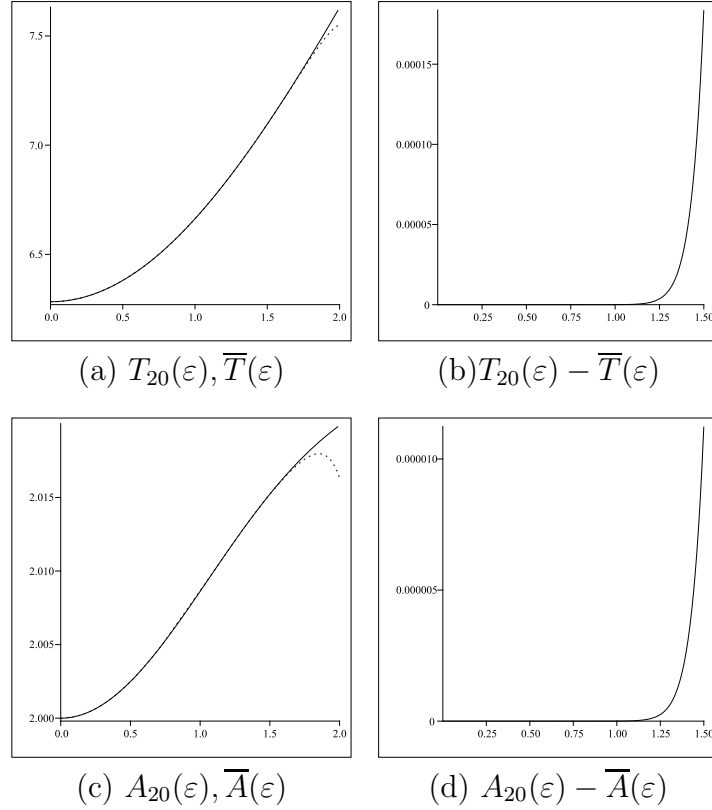
FIGURE 1. The period and the maximum amplitude up to order 20 with respect to ε, $T_{20}(\varepsilon)$ and $A_{20}(\varepsilon)$, versus the numerical approximation (dotted line), $\overline{T}(\varepsilon)$ and $\overline{A}(\varepsilon)$, of equation (19) with initial condition $\rho = 2$.

4. Liénard examples

This section is devoted to apply the results of this paper to two Liénard families of quintic differential equations. The first exhibits two simple limit cycles while in the second a double limit cycle is studied. Another example of this second case is also studied in [2].

4.1. **An example with two simple limit cycles.** Given equation

$$\begin{cases} \dot{x} &= -y + \varepsilon(a_1 x + a_3 x^3 + a_5 x^5), \\ \dot{y} &= x, \end{cases} \tag{21}$$

using the first Poincaré-Pontryagin-Melnikov function, it is known that, up to first order, it can have at most two limit cycles, in Perko [21, Th. 6, §3.8] it is also

proved that equation (21), when ε is small enough, has exactly two limit cycles for some choice of the parameters.

Proposition 10. *For ε small enough, the system*

$$\begin{cases} \dot{x} & = & -y + \dfrac{\varepsilon}{6}\left(15x - 25x^3 + 6x^5\right), \\ \dot{y} & = & x, \end{cases} \tag{22}$$

has two limit cycles that bifurcate from the circles $x^2 + y^2 = \rho_i^2$, for $\rho_1 = 1$ and $\rho_2 = 2$. Moreover these limit cycles can be written as

$$r^{(i)}(\theta; \rho_i, \varepsilon) = \rho_i + r_1^{(i)}(\theta)\varepsilon + r_2^{(i)}(\theta)\varepsilon^2 + \cdots,$$

where

$$r_1^{(1)}(\theta) = -\frac{35}{192}\sin(2\theta) - \frac{1}{12}\sin(4\theta) + \frac{1}{192}\sin(6\theta),$$

$$r_2^{(1)}(\theta) = \frac{43577}{221184} - \frac{1231}{4608}\cos(2\theta) - \frac{12085}{147456}\cos(4\theta) - \frac{65}{2304}\cos(6\theta)$$
$$- \frac{653}{73728}\cos(8\theta) + \frac{1}{512}\cos(10\theta) - \frac{11}{147456}\cos(12\theta)$$

and

$$r_1^{(2)}(\theta) = \frac{5}{12}\sin(2\theta) + \frac{11}{24}\sin(4\theta) + \frac{1}{6}\sin(6\theta),$$

$$r_2^{(2)}(\theta) = \frac{39389}{13824} - \frac{3143}{1152}\cos(2\theta) - \frac{2065}{1152}\cos(4\theta) - \frac{1115}{1152}\cos(6\theta)$$
$$- \frac{2071}{4608}\cos(8\theta) - \frac{11}{64}\cos(10\theta) - \frac{11}{288}\cos(12\theta).$$

In addition $r_1^{(i)}(0) = r_3^{(i)}(0) = r_5^{(i)}(0) = 0$, $i = 1, 2$, and

$$r_2^{(1)}(0) = -\frac{20711}{110592}, \qquad\qquad r_2^{(2)}(0) = -\frac{11401}{3456},$$

$$r_4^{(1)}(0) = \frac{695992493381}{7191587192832}, \qquad\qquad r_4^{(2)}(0) = \frac{413104715543}{7023034368}.$$

Finally, the period of the corresponding periodic orbits are

$$T^{(1)}(\varepsilon) \approx 2\pi\left(1 + \frac{925}{3072}\varepsilon^2 - \frac{2169875}{169869312}\varepsilon^4 - \frac{20854739099125}{394509926006784}\varepsilon^6 \right.$$
$$\left. + \frac{248214516630214166787875}{1026169173652881864130056}\varepsilon^8\right)$$

and

$$T^{(2)}(\varepsilon) \approx 2\pi\left(1 + \frac{175}{192}\varepsilon^2 - \frac{860375}{1327104}\varepsilon^4 - \frac{1245515960375}{96315899904}\varepsilon^6 \right.$$
$$\left. + \frac{33587358157317806375}{195726237040115712}\varepsilon^8\right).$$

Proof. According Theorem 3.(i) and Lemma 5, for ε small enough, equation (22) has two limit cycles emerging from the circles of radii which are the simple zeros of the function

$$F_1(r) = -\frac{5}{16}r^2(r^4 - 5r^2 + 4) = -\frac{5}{16}r^2(r^2 - 1)(r^2 - 4).$$

In this case all the non null zeros, $\rho_1 = 1$ and $\rho_2 = 2$, are simple since $F_1'(1) = 15/8$ and $F_1'(2) = -15$. Then, from Theorem 1 and Theorem 3.(i), we obtain the expression of $r^{(i)}(\theta; \rho_i, \varepsilon)$ for $i = 1, 2$, that appear in the statement. From Corollary 4 we get the corresponding periods. □

The above result only shows some of the explicit expressions, due to the size of them, of the power expansion of $r^{(i)}(\theta; \rho_i, \varepsilon)$ in ε. Nevertheless some extra terms are necessary to provide all the terms that appear for $r^{(i)}(0; \rho_i, \varepsilon)$ and $T^{(i)}(\varepsilon)$. Finally it can be proved that by symmetry all the odd terms, $r_{2k+1}^{(i)}(0)$, vanish at the origin.

4.2. **An example with a double limit cycle.** Here we present an example of a Liénard system having two limit cycles that emerge, for ε small enough, from a double zero of F_1. This problem is also studied in [14]. The series expansion of the period function of a semistable limit cycle is considered in [9], where it is proved that, in general, the first non constant term depends on $\sqrt{\varepsilon}$. In this example, we observe that, the first non constant term depends on ε^2.

Proposition 11. *For ε small enough, system*

$$\begin{cases} \dot{x} &= -y + \varepsilon\left(10x - \dfrac{80}{3}x^3 + 16x^5\right) + \varepsilon^2\left(\dfrac{905}{72}x - \dfrac{475}{27}x^3 + x^5\right), \\ \dot{y} &= x, \end{cases} \quad (23)$$

has two limit cycles that bifurcate form the circle $x^2 + y^2 = 1$. Moreover these limit cycles can be expressed as

$$r^{(i)}(\theta; 1, \varepsilon) = 1 + r_1^{(i)}(\theta)\varepsilon + r_2^{(i)}(\theta)\varepsilon^2 + \cdots,$$

for $i = 1, 2$, where

$$r_1^{(1)}(\theta) = \frac{1}{4} - \frac{5}{12}\sin(2\theta) - \frac{1}{12}\sin(4\theta) + \frac{1}{12}\sin(6\theta),$$

$$r_2^{(1)}(\theta) = \frac{8461}{2016} - \frac{1225}{1728}\sin(2\theta) - \frac{79}{72}\cos(2\theta) - \frac{329}{1728}\sin(4\theta) + \frac{95}{576}\cos(4\theta)$$
$$+ \frac{7}{64}\sin(6\theta) + \frac{35}{288}\cos(6\theta) + \frac{29}{576}\cos(8\theta) + \frac{1}{32}\cos(10\theta) - \frac{11}{576}\cos(12\theta)$$

and

$$r_1^{(2)}(\theta) = \frac{25}{72} - \frac{5}{12}\sin(2\theta) - \frac{1}{12}\sin(4\theta) + \frac{1}{12}\sin(6\theta),$$

$$r_2^{(2)}(\theta) = -\frac{3043}{1134} - \frac{1015}{1728}\sin(2\theta) - \frac{79}{72}\cos(2\theta) - \frac{119}{1728}\sin(4\theta) + \frac{95}{576}\cos(4\theta)$$
$$+ \frac{259}{1728}\sin(6\theta) + \frac{35}{288}\cos(6\theta) + \frac{29}{576}\cos(8\theta) + \frac{1}{32}\cos(10\theta) - \frac{11}{576}\cos(12\theta).$$

In addition

$$r_1^{(1)}(0) = \frac{1}{4}, \qquad\qquad r_1^{(2)}(0) = \frac{25}{72},$$

$$r_2^{(1)}(0) = \frac{1545}{448}, \qquad\qquad r_2^{(2)}(0) = -12452936288,$$

$$r_3^{(1)}(0) = \frac{44204981}{351232}, \qquad\qquad r_3^{(2)}(0) = -\frac{64048647121}{512096256},$$

$$r_4^{(1)}(0) = \frac{1794546526490287}{200741732352}, \qquad r_4^{(2)}(0) = -\frac{32301462867376505}{3613351182336}.$$

Finally, the corresponding periods are

$$T^{(1)}(\varepsilon) \approx 2\pi\left(1 + \frac{25}{12}\varepsilon^2 + \frac{1625}{288}\varepsilon^3 - \frac{11501975}{580608}\varepsilon^4\right)$$

and

$$T^{(2)}(\varepsilon) \approx 2\pi\left(1 + \frac{25}{12}\varepsilon^2 + \frac{4175}{864}\varepsilon^3 + \frac{6915275}{193536}\varepsilon^4\right).$$

Proof. Applying Lemma 5 to system (23) we obtain

$$\begin{aligned}
F_1(r) &= 5r^2(r^2 - 1)^2, \\
F_2(r) &= \frac{5}{144}r^2(r^2 - 1)(9r^2 - 181).
\end{aligned}$$

We note that F_1 has a double zero at $\rho = 1$ and $F_2(1) = 0$. Then from Theorem 3.(ii) equation (23) presents two limit cycles bifurcating from $\rho = 1$ and $r_1^{(i)}(0)$, $i = 1, 2$, are the two roots of the quadratic polynomial

$$-\frac{125}{72} + \frac{215}{18}\lambda - 20\lambda^2 = -20\left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{25}{72}\right) = 0.$$

Hence Theorem 1 provides the expressions for $r_1^{(i)}(\theta)$ and $r_1^{(i)}(0)$, $i = 1, 2$ given in the statement. The proof ends computing, from Corollary 4, their corresponding periods. □

5. Perturbing linear-like systems

The method described in Section 2 can be applied also for studying the shape of limit cycles that appear, by small perturbations, from isochronous centers. For these systems there exists a change of variables, the linearization, $(u, v) = \varphi(x, y)$ such that system

$$\begin{cases} \dot{x} &= P_0(x, y) + \varepsilon P_1(x, y), \\ \dot{y} &= Q_0(x, y) + \varepsilon Q_1(x, y), \end{cases} \tag{24}$$

linearizes to

$$\begin{cases} \dot{u} &= -v + \varepsilon \widetilde{P}_1(u, v), \\ \dot{v} &= u + \varepsilon \widetilde{Q}_1(u, v). \end{cases} \tag{25}$$

In concrete examples the computations can be very intricate for most φ functions, but in some cases we can do it in an explicit way, as we can show in the families of this section.

An easier case is when φ is a polynomial function whose inverse is also polynomial. These functions are called Jacobian changes and satisfy that $\det D\varphi$ is constant. As an illustration we can consider the system

$$\begin{cases} \dot{x} &= -y-7x^2+8xy+12y^2+20x^3-120x^2y+240xy^2-160y^3+\varepsilon P(x,y), \\ \dot{y} &= x-6x^2+14xy-4y^2+10x^3-60x^2y+120xy^2-80y^3+\varepsilon Q(x,y). \end{cases} \tag{26}$$

The change of variables

$$(u,v) = \left(x - 2(x - 2y)^2, y - (x - 2y)^2\right) \tag{27}$$

whose inverse, which is also polynomial, is given by

$$(x,y) = \left(u + 2(u - 2v)^2, v + (u - 2v)^2\right) \tag{28}$$

writes (26) as

$$\begin{cases} \dot{u} &= -v+\varepsilon\big((1 - 4u + 8v)P(u + 8v^2 - 8uv + 2u^2, v + 4v^2 - 4uv + u^2) \\ &\quad +(8u - 16v)Q(u + 8v^2 - 8uv + 2u^2, v + 4v^2 - 4uv + u^2)\big), \\ \dot{v} &= u+\varepsilon\big((-2u + 4v)P(u + 8v^2 - 8uv + 2u^2, v + 4v^2 - 4uv + u^2) \\ &\quad +(1 + 4u - 8v)Q(u + 8v^2 - 8uv + 2u^2, v + 4v^2 - 4uv + u^2)\big). \end{cases} \tag{29}$$

Clearly, we could apply to previous system all the theory described along the paper and we could obtain, for every k, the shape of the limit cycles up to order k. This is so because the perturbation, for the new system, remains polynomial although the degree increases up to $2n + 1$ if P and Q are polynomials of degree n.

Proposition 12. *The system*

$$\begin{cases} \dot{x} &= -y-7x^2+8xy+12y^2+20x^3-120x^2y+240xy^2-160y^3+\varepsilon(x^2-5x), \\ \dot{y} &= x-6x^2+14xy-4y^2+10x^3-60x^2y+120xy^2-80y^3 \end{cases} \tag{30}$$

has a parametrized limit cycle

$$x(\theta;\varepsilon) \approx 5 + \cos\theta - 4\sin(2\theta) - 3\cos(2\theta)$$
$$+ \frac{6101}{240}\left(-10 - \cos\theta + 8\sin(2\theta) + 6\cos(2\theta)\right)\varepsilon,$$
$$y(\theta;\varepsilon) \approx \frac{5}{2} + \sin\theta - 2\sin(2\theta) - \frac{3}{2}\cos(2\theta)$$
$$+ \frac{6101}{240}\left(-5 - \sin\theta + 4\sin(2\theta) + 3\cos(2\theta)\right)\varepsilon,$$

where $\theta \in [0, 2\pi]$ and passing through the point $(x(\varepsilon), y(\varepsilon))$ where

$$x(\varepsilon) \approx 3 - \frac{6101}{48}\varepsilon - \frac{13216604857}{806400}\varepsilon^2 - \frac{228947349374103607}{44706816000}\varepsilon^3,$$
$$y(\varepsilon) \approx 1 - \frac{6101}{120}\varepsilon - \frac{28791221}{4480}e^2 - \frac{2249282681363173}{1117670400}\varepsilon^3,$$

and the period satisfies

$$T(\varepsilon) \approx 2\pi\left(1 + \frac{499}{4}\varepsilon + \frac{9526313}{192}\varepsilon^2 + \frac{590345743447}{23040}\varepsilon^3 + \frac{233137962146049461}{15482880}\varepsilon^4\right).$$

Proof. With the change of variables (27), system (30) becomes (29), with $P(x, y) = x^2 - 5x$ and $Q(x, y) = 0$. This last differential equation has a limit cycle which emerges from $\rho = 1$, because Lemma 5 give us $F_1(r) = 5r^2(r^2 - 1)/2$. Moreover, from Theorem 1 and Theorem 3, we have the explicit expression of the radius up to any order. For example, when $\theta = 0$,

$$r(0; 1, \varepsilon) \approx 1 - \frac{6101}{240}\varepsilon - \frac{2851765297}{806400}\varepsilon^2 - \frac{49004734865049767}{44706816000}\varepsilon^3$$
$$- \frac{12112729537967232398473411}{25776877142016000}\varepsilon^4.$$

Then the limit cycles can be parametrized by θ as $(u, v) = r(\theta; 1, \varepsilon)(\cos\theta, \sin\theta)$. The expressions of the statement follow recovering the original coordinates, using (28). Finally, the period is obtained from Corollary 4 because the change of variables (27) does not modify the time.   □

The change of variables chosen to generate the previous example belongs to a particular Jacobian changes: the Tame transformations, see [28]. Now we briefly introduce them and then we study the perturbation of a system, from [27], which is isochronous and that linearizes through a Tame transformation. See Proposition 13.

We recall that a Jacobian change is a polynomial transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ with an inverse which is also polynomial. The name comes from the relation of these changes with the Jacobian conjecture, whose statement says that all the polynomial changes φ with constant $\det D\varphi$ have a polynomial inverse, see [28]. It is not restrictive consider only the case $\det D\varphi \equiv 1$.

First, we introduce the transformations

$$\begin{aligned}
(u, v) &= \varphi_n^u(x, y) = (x + f_n(y), y), \\
(u, v) &= \varphi_n^l(x, y) = (x, y + g_n(x)),
\end{aligned}$$

with $f_n(x)$ and $g_n(x)$ polynomials of degree n. We use them as upper and lower Tame generators. We recall now that a Tame transformation is an element of the transformation group, $\mathcal{T}$, generated by φ_n^u, φ_n^l and the linear transformations with determinant 1.

If we consider a collection of upper and lower transformations of different degrees and we compose them we obtain a Jacobian change φ. Moreover for any $\varphi \in \mathcal{T}$, the linear part of φ is the identity and the corresponding system (24) has a linear part of type linear center. We can also compose with some linear transformations as the change of variables (27). It is a Tame change because can be written as $L_2 \circ \varphi_2^u \circ L_1$, where $\varphi_2^u(x, y) = (x - y^2, y)$, $L_1(x, y) = (y, -x + 2y)$ and $L_2(x, y) = L_1^{-1}(x, y) = (2x - y, x)$.

Proposition 13. *The system*

$$\begin{cases}
\dot{x} &= -y - Cx^2, \\
\dot{y} &= x(1 + 2Cy + 2C^2x^2) + \dfrac{\varepsilon}{2}(1 + 2y - 2x^2 + 2Cx^2)(y + Cx^2 - x^2),
\end{cases} \quad (31)$$

has a limit cycle that can be written, for ε small enough, as

$$x(\theta; \varepsilon) \approx \cos(\theta) + \varepsilon\left(-\frac{3}{8} - \frac{1}{16}\sin(\theta) - \frac{11}{32}\cos(2\theta) - \frac{1}{32}\sin(3\theta)\right.$$

$$\left. + \frac{1}{40}\cos(4\theta) + \frac{1}{32}\sin(5\theta) - \frac{1}{160}\cos(6\theta)\right),$$

$$y(\theta; \varepsilon) \approx -\frac{C}{2} + \sin(\theta) - \frac{C}{2}\cos(2\theta) + \varepsilon\left(\frac{35C - 2}{32}\cos(\theta)\right.$$

$$+ \frac{3C - 13}{32}\sin(2\theta) + \frac{51C + 15}{160}\cos(3\theta) + \frac{3}{80}\sin(4\theta)$$

$$\left. - \frac{3C + 5}{160}\cos(5\theta) - \frac{5C + 1}{160}\sin(6\theta) + \frac{C}{160}\cos(7\theta)\right),$$

for $\theta \in [0, 2\pi]$ and passes through the point

$$(1, -C) + \sum_{i=1}^{\infty} \varepsilon^i(x_i, Cy_i)$$

where

$$x_1 = -\frac{7}{10}, \qquad\qquad y_1 = \frac{7}{5},$$

$$x_2 = \frac{677}{4800}, \qquad\qquad y_2 = -\frac{1853}{2400},$$

$$x_3 = -\frac{87203}{705600}, \qquad\qquad y_3 = \frac{1568663}{3528000},$$

$$x_4 = \frac{26159870617}{54190080000}, \qquad\qquad y_4 = -\frac{31386898801}{27095040000},$$

$$x_5 = -\frac{1563866274535521169}{2443433145753600000}, \qquad\qquad y_5 = \frac{4864284760285373849}{2443433145753600000}.$$

Moreover, the corresponding period is

$$T(\varepsilon) \approx 2\pi\left(1 + \frac{251}{320}\varepsilon^2 + \frac{23683041}{14336000}\varepsilon^4 + \frac{117449035916903}{20230963200000}\varepsilon^6\right.$$

$$\left. + \frac{370726500803682522263539}{16330562973204480000000}\varepsilon^8\right).$$

Proof. With the change $(u, v) = (x, y + Cx^2)$ system (31) writes as

$$\begin{cases} \dot{u} &= -v, \\ \dot{v} &= u + \varepsilon\left(\frac{1}{2}v + v^2 - \frac{1}{2}u^2 - 2u^2v + u^4\right), \end{cases} \tag{32}$$

and using Lemma 5 we obtain that

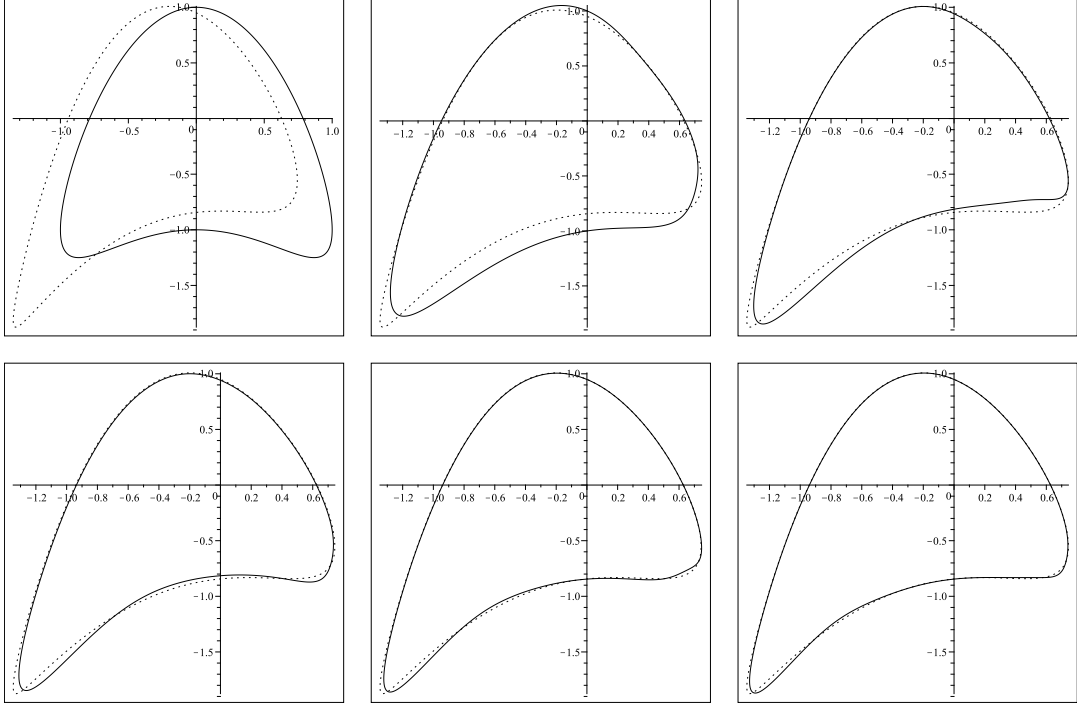$$F_1(r) = \frac{1}{4}r^2(r^2 - 1).$$

FIGURE 2. Approximations of degrees 1 to 6 and the numerical limit cycle (dotted line) for $\varepsilon = 0.4$ and $C = 1$ of system (31)

Then system (32), from Theorem 3, has a limit cycle which emerges from $\rho = 1$, for ε small enough. Moreover, using Theorem 1, we can also compute

$$r(\theta; 1, \varepsilon) = 1 + r_1(\theta)\varepsilon + r_2(\theta)\varepsilon^2 + \cdots,$$

where

$$
\begin{aligned}
r_1(\theta) &= -\frac{3}{4}\cos\theta - \frac{1}{8}\sin(2\theta) + \frac{1}{16}\cos(3\theta) + \frac{1}{16}\sin(4\theta) - \frac{1}{80}\cos(5\theta), \\
r_2(\theta) &= -\frac{307}{1024} + \frac{39}{80}\sin\theta + \frac{213}{512}\cos(2\theta) - \frac{17}{960}\sin(3\theta) - \frac{1}{1280}\cos(4\theta) \\
&\quad - \frac{153}{1600}\sin(5\theta) + \frac{523}{15360}\cos(6\theta) + \frac{23}{2240}\sin(7\theta) - \frac{9}{1024}\cos(8\theta) \\
&\quad - \frac{1}{320}\sin(9\theta) + \frac{9}{25600}\cos(10\theta),
\end{aligned}
$$

and

$$r_1(0) = -\frac{7}{10}, \quad r_2(0) = \frac{677}{4800}, \quad r_3(0) = -\frac{87203}{705600}, \quad r_4(0) = \frac{26159870617}{54190080000},$$

$$r_5(0) = -\frac{1563866274535521169}{2443433145753600000}, \quad r_6(0) = \frac{14164506595172902414493}{12510377706258432000000}.$$

Then recovering the original coordinates with the inverse changes of variables, $(x, y) = (u, v - Cu^2)$ and computing the Taylor expansion in ε we obtain the statement. Finally, Corollary 4 provided the expression for the period. $\qquad\square$

We point out that for sake of brevity in the statement of the proposition we only give the first order approximation in ε of the limit cycle but in Figure 2 we

show approximations up to order six together with the numerical. Moreover, as it can be seen in Figure 3, the approximation is not uniform with respect to ε, for the angle θ.
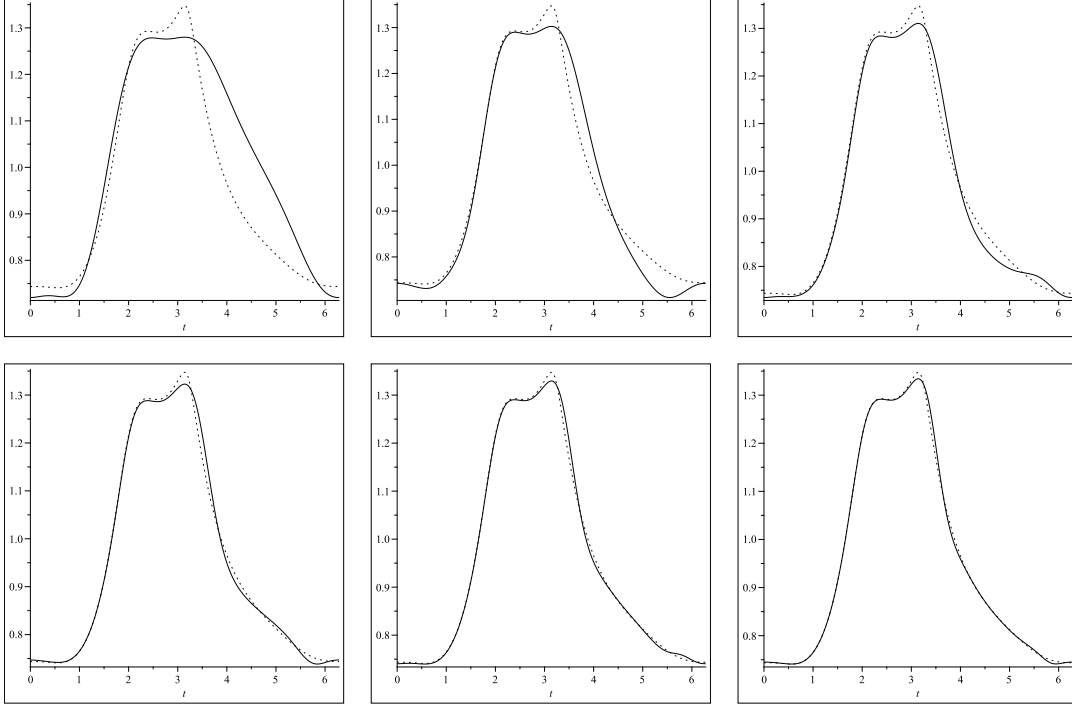

FIGURE 3. Approximations of degrees 1 to 6 and the numerical (dotted line) limit cycle for $\varepsilon = 0.4$ of system (32) in polar coordinates. The non-uniformity in the variable θ can be observed.

6. Radial but not linear Hamiltonians

If we consider a Hamiltonian $H = H(r^2)$, instead of $H = H(r)$ as in the previous sections, with $r^2 = x^2 + y^2$ then system

$$\begin{cases} \dot{x} &= -\frac{\partial H}{\partial y} + \varepsilon P(x,y), \\ \dot{y} &= \frac{\partial H}{\partial x} + \varepsilon Q(x,y), \end{cases} \tag{33}$$

is equivalent to the reparametrized system

$$\begin{cases} x' &= -y + \varepsilon \frac{r}{\frac{\partial H}{\partial r}} P(x,y), \\ y' &= x + \varepsilon \frac{r}{\frac{\partial H}{\partial r}} Q(x,y), \end{cases} \tag{34}$$

because $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial r}\frac{y}{r}$ and $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial r}\frac{x}{r}$. For these systems we can apply the results of this paper directly to system (33) or in its equivalent form (34). After the change of time, the expression of the limit cycles in polar coordinates remains the same but the period function changes. When the main objective is the period function the first form is more convenient. Next example is an illustration of this fact.

Proposition 14. *For ε small enough, system*

$$\begin{cases} \dot{x} & = & -y\left(1+x^2+y^2\right)+\dfrac{\varepsilon}{3}\left(3x^2-4x^3\right), \\ \dot{y} & = & x\left(1+x^2+y^2\right)+\varepsilon\left(1+y\right), \end{cases} \tag{35}$$

has a limit cycle close to the curve $2r^2+r^4=3$, in polar coordinates, that writes as

$$r(\theta;1,\varepsilon)=1+r_1(\theta)\varepsilon+r_2(\theta)\varepsilon^2+\cdots,$$

where

$$\begin{aligned} r_1(\theta) & = & -\frac{1}{2}+\frac{3}{8}\sin\theta-\frac{1}{2}\cos\theta-\frac{7}{24}\sin(2\theta)+\frac{1}{24}\sin(3\theta)-\frac{1}{48}\sin(4\theta), \\ r_2(\theta) & = & -\frac{2863}{3072}+\frac{11}{32}\sin\theta-\frac{67}{768}\cos\theta+\frac{1}{12}\sin(2\theta)-\frac{11}{96}\cos(2\theta) \\ & & +\frac{7}{64}\sin(3\theta)+\frac{229}{2304}\cos(3\theta)-\frac{7}{96}\cos(4\theta)+\frac{1}{64}\sin(5\theta) \\ & & +\frac{347}{11520}\cos(5\theta)-\frac{1}{72}\cos(6\theta)+\frac{5}{2304}\cos(7\theta)-\frac{1}{1536}\cos(8\theta). \end{aligned}$$

Additionally,

$$r_1(0)=-1,\quad r_2(0)=-\frac{50207}{46080},\quad r_3(0)=-\frac{219337}{129024},$$

$$r_4(0)=-\frac{11972387810004169}{2678117105664000},\quad r_5(0)=-\frac{4223772604811883923}{412430034272256000}$$

and the corresponding period of the limit cycle satisfies

$$T(\varepsilon;1)\approx\pi\left(1+\frac{1}{2}\varepsilon+\frac{4301}{4608}\varepsilon^2+\frac{7679}{4608}\varepsilon^3+\frac{14597509901}{4246732800}\varepsilon^4+\frac{225290628911}{29727129600}\varepsilon^5\right).$$

Proof. System (35) is a perturbation of the Hamiltonian $H(r)=(2r^2+r^4)/4$, and using Lemma 5 we obtain

$$F_1(r)=\frac{1}{2}r^2\left(r^2-1\right). \tag{36}$$

Then for ε small enough, following Theorem 3.(i), there is one simple limit cycle that emerges from $H(r)=H(1)=3/4$, that corresponds to $\rho=1$, which satisfies $F_1(1)=0$ and $F_1'(1)=1$. For this case the limit cycle, using Theorem 1, can be written as the expression given in the statement. The proof ends computing the period using Corollary 4. □

Remark 15. *As we have mentioned above, another way to do the computations for system* (35) *is to consider the reparametrized rational system, in form* (34),

$$\begin{cases} x' & = & -y+\varepsilon\dfrac{3x^2-4x^3}{3(1+x^2+y^2)}, \\ y' & = & x+\varepsilon\dfrac{1+y}{1+x^2+y^2}. \end{cases}$$

Now the Hamiltonian is $\widehat{H}(r)=r^2/2$, and the functions involved in the computations of the shape of the limit cycles are not the same, for example, using Lemma 5,

$$\widehat{F}_1(r)=\frac{r^2(r^2-1)}{2(r^2+1)}.$$

Note that while the zeros of F_1, given by (36), and $\widehat{F}_1$ coincide, the period of the periodic solutions is different.

7. Acknowledgements

References

[1] C. M. Andersen and J. F. Geer. Power series expansions for the frequency and period of the limit cycle of the van der Pol equation. *SIAM J. Appl. Math.*, 42(3):678–693, 1982.

[2] T. R. Blows and L. M. Perko. Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems. *SIAM Rev.*, 36(3):341–376, 1994.

[3] A. Buonomo. The periodic solution of van der Pol's equation. *SIAM J. Appl. Math.*, 59(1):156–171, 1998.

[4] H. Cartan. *Differential forms.* Paris: Hermann, Publishers in Arts and Science; London: Kershaw Publishing Company, Ltd., 1970.

[5] M. B. Dadfar, J. Geer, and C. M. Andersen. Perturbation analysis of the limit cycle of the free van der Pol equation. *SIAM J. Appl. Math.*, 44(5):881–895, 1984.

[6] J.-P. Françoise. Successive derivatives of a first return map, application to the study of quadratic vector fields. *Ergodic Theory Dyn. Syst.*, 16(1):87–96, 1996.

[7] J.-P. Françoise. The first derivative of the period function of a plane vector field. *Publ. Mat.*, 41(1):127–134, 1997.

[8] J.-P. Françoise. The successive derivatives of the period function of a plane vector field. *J. Differential Equations*, 146(2):320–333, 1998.

[9] A. Gasull, V. Mañosa, and J. Villadelprat. On the period of the limit cycles appearing in one-parameter bifurcations. *J. Differential Equations*, 213:255–288, 2005.

[10] A. Gasull and J. Torregrosa. A relation between small amplitude and big limit cycles. *Rocky Mountain J. Math.*, 31(4):1277–1303, 2001.

[11] A. Gasull and J. Torregrosa. Center-focus problem for discontinuous planar differential equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(7):1755–1765, 2003.

[12] H. Giacomini, J. Llibre, and M. Viano. On the nonexistence, existence and uniqueness of limit cycles. *Nonlinearity*, 9(2):501–516, 1996.

[13] H. Giacomini, J. Llibre, and M. Viano. On the shape of limit cycles that bifurcate from Hamiltonian centers. *Nonlinear Anal.*, 41(3-4):523–537, 2000.

[14] H. Giacomini, J. Llibre, and M. Viano. Semistable limit cycles that bifurcate from centers. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(11):3489–3498, 2003.

[15] H. Giacomini and M. Viano. Determination of limit cycles for two-dimensional dynamical systems. *Phys. Rev. E (3)*, 52(1):222–228, 1995.

[16] I. D. Iliev. On second order bifurcations of limit cycles. *J. London Math. Soc. (2)*, 58(2):353–366, 1998.

[17] I. D. Iliev and L. M. Perko. Higher order bifurcations of limit cycles. *J. Differential Equations*, 154(2):339–363, 1999.

[18] N. Minorsky. *Nonlinear oscillations.* D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1962.

[19] K. Odani. On the limit cycle of the Liénard equation. *Arch. Math. (Brno)*, 36(1):25–31, 2000.

[20] M. S. Padn, F. I. Robbio, J. L. Moiola, and G. Chen. On limit cycle approximations in the van der Pol oscillator. *Chaos, Solitons & Fractals*, 23(1):207–220, 2005.

[21] L. Perko. *Differential equations and dynamical systems.* Texts in Applied Mathematics. 7. New York, NY: Springer, 1991.

[22] J. C. Poggiale. *Applications des variétés invariantes a la modélisation de lhétérogénéite en dynamique des populations.* Ph.D. Thesis, Université de Bourgogne, 1994.

[23] L. S. Pontryagin. Über Autoschwingungssysteme, die den Hamiltonschen nahe liegen. *Physikal. Z. Sowjetunion*, 6:25–28, 1935.

[24] R. Prohens and J. Torregrosa. Number, shape and period of some perturbed periodic solutions via rational trigonometric integrals. Preprint, 2012.

[25] R. Roussarie. *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, volume 164 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1998.

[26] S. Sun. The equations of the closed trajectories in rotated vector fields. *Chinese Math.*, 5:54–62, 1964.

[27] B. Toni. Higher order branching of periodic orbits from polynomial isochrones. *Electron. J. Differential Equations*, 35:15 pp., 1999.

[28] A. van den Essen. *Polynomial automorphisms and the Jacobian conjecture*, volume 190 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2000.

[1]Dept. de Matemàtiques i Informàtica, Escola Politècnica Superior, Universitat de les Illes Balears, 07122, Palma de Mallorca. Spain
E-mail address: `rafel.prohens@uib.cat`

[2]Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193 Bellaterra, Barcelona. Spain
E-mail address: `torre@mat.uab.cat`