Qualitative theory of differential equations in the plane and in the space, with emphasis on the center-focus problem and on the Lotka-Volterra systems

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Preface

In the theory of ordinary differential equations we can find two fundamental problems. The direct problem which consists in a broad sense in to find the solutions of a given ordinary differential equation, and the inverse problem. An inverse problem of ordinary differential equations is to find the more general differential system satisfying a set of given properties. For instance what are the differential systems in $\mathbb{R}^N$ having a given set of invariant hypersurfaces, or of first integrals?

Probably the first inverse problem appeared in Celestial Mechanics, it was stated and solved by Newton (1687) in Philosophie Naturalis Principia Mathematica, and it concerns with the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely the Kepler's laws.

The first statement of the inverse problem as the problem of finding the more general differential system of first order satisfying a set of given properties was stated by Erugin [21] and developed in [23, 33, 50].

The aim of the present thesis is to state and study the following three inverse problem.

(I) *The inverse approach to the center-focus problem for planar differential systems.*

Let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \tag{1}$$

be the real planar analytic or polynomial vector field associated to the real planar differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{2}$$

where the dot denotes derivative with respect to an independent variable t.

In what follows we assume that origin $O := (0, 0)$ is a singular point, i.e. $P(0, 0) = Q(0, 0) = 0$. The singular point O is a *center* if there exists an open neighborhood U of O where all the orbits contained in $U \setminus \{O\}$ are periodic.

The study of the centers of analytical or polynomial differential systems (2) has a long history. The first works are due to Poincaré [47] and Dulac [19] . Later on were developed by Bendixson [8], Frommer [22], Liapunov [30] and many others.

In the present thesis we shall study the differential system of the form

$$\dot{x} = -y + X, \quad \dot{y} = x + Y, \tag{3}$$

where $X = X(x, y)$ and $Y = Y(x, y)$ are real analytic or polynomials functions in an open neighborhood of O whose Taylor expansions at O do not contain constant and linear terms.

System (3) always has a center or a focus at the origin.

One of the classical problems in the qualitative theory of the differential system (3) is to characterize the local phase portrait in a sufficiently small neighborhood near the origin i.e. distinguishing between a center or focus. This problem is called the *center-focus problem* .

In the study of the center-focus problem the following theorems play a very important role (see for instance [30, 47, 44])

Theorem 1. *For the analytic differential system* (3) *there exists a formal power series* $H = \sum_{n=2}^{\infty} H_n := \frac{1}{2}(x^2 + y^2) + \sum_{n=3}^{\infty} H_n(x, y)$*, where* $H_j = H_j(x, y)$ *is a homogenous polynomial of degree* j *such that* $\frac{dH}{dt} = \sum_{j=1}^{\infty} v_{2k}(x^2 + y^2)^k$*, where* v_{2k} *are the Poincaré-Liapunov constants.*

Assume that the formal power series H converges. If the constants $v_j = 0$ for $j \in \mathbb{N}$ then there exists a local first integral $H := \frac{1}{2}(x^2+y^2)+\sum_{j=3}^{\infty} H_j$, and consequently the origin is a center. If there exists a first non–zero Liapunov constant v_{2k}, then the origin is a stable focus if $v_{2k} < 0$ and unstable if $v_{2k} > 0$. If $v_{2k} = 0$ for $k = 1, \ldots, n-1$ and $v_{2n} \neq 0$ then the system (3) has a *focus of order* n at the origin.

We recall the following definition. Let U be an open and dense set in $\mathbb{R}^2$. We say that a non-constant C^r with $r \geq 1$ function $F : U \to \mathbb{R}$ is a *first integral* of the analytic or polynomial vector field $\mathcal{X}$ on U, if $F(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of $\mathcal{X}$ is defined on U. Clearly F is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}F = 0$ on U.

Poincaré and Liapunov proved the next two results, see for instance [47, 30, 26, 44].

Theorem 2. *A planar polynomial differential system*

$$\dot{x} = -y + \sum_{j=2}^{m} X_j(x,y), \quad \dot{y} = x + \sum_{j=2}^{m} Y_j(x,y), \tag{4}$$

of degree m *has a center at the origin if and only if it has a first integral of the form*

$$H = \sum_{j=2}^{\infty} H_j(x,y) = \frac{1}{2}(x^2+y^2) + \sum_{j=3}^{\infty} H_j(x,y), \tag{5}$$

where X_j, Y_j *and* H_j *are homogenous polynomials of degree* j.

Theorem 3. *An analytic planar differential system*

$$\dot{x} = -y + \sum_{j=2}^{\infty} X_j(x,y), \quad \dot{y} = x + \sum_{j=2}^{\infty} Y_j(x,y), \tag{6}$$

has a center at the origin if and only if it has a first integral of the form (5).

From Theorems 1, 2 and 3 it is clear that an analytic or polynomial differential system (3) has a center at the origin if and only if the Poincaré-Liapunov constants $v_k = 0$ for $k \geq 1$ (*Poincaré's criterion*). Moreover, the v_k's are polynomials over $\mathbb{Q}$ in the coefficients of the polynomial differential system. A necessary and sufficient condition to have a center is then the annihilation of all these constants. In view of the Hilbert's basis theorem this occurs if and only if for a finite number of k, $k < j$ and j sufficiently large, $v_k = 0$. Unfortunately, trying to solve the center problem computing the Poincaré-Liapunov constants is in general not possible due to the huge computations.

Although we have an algorithm for computing the Poincaré-Liapunov constants for linear type center, we have no algorithm to determine how many of them need to be zero to imply that all of them are zero for cubic or higher degree polynomial differential systems. Bautin [6] showed in 1939 that for a quadratic polynomial differential system, to annihilate all v_k's it suffices to have $v_k = 0$ for $i = 1, 2, 3$. So the problem of the center is solved for quadratic systems. This problem was solved for the cubic differential systems with homogenous nonlinearities (see for instance [43, 51, 52]).

The analytic function (5) is called the *Poincaré-Liapunov local first integral.*

Theorem 2 is due to Poincaré, and Theorem 3 is due to Liapunov.

Now we shall introduce another criterion for solving the center problem due to Reeb.

We will need the following definitions.

A function $V = V(x, y)$ is an *inverse integrating factor* of system (2) in an open subset $U \subset \mathbb{R}^2$ if $V \in C^1(U), V \not\equiv 0$ in U and

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right).$$

In short notation this equation can be rewritten as follows

$$\mathcal{Y}(V) := \frac{\partial\left(\frac{P}{V}\right)}{\partial x} + \frac{\partial\left(\frac{Q}{V}\right)}{\partial y} = 0 \Longleftrightarrow P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \mathrm{div}\mathcal{Y}\ V.$$

We note that $\{V = 0\}$ is formed by orbits of system (2). The function $1/V$ defines an integrating factor in $U\backslash\{V = 0\}$ of system (2) which allows to compute a first integral for (2) in $U\backslash\{V = 0\}$.

If exists an integrating factor then differential system (2) is topological equivalent to the Hamiltonian vector fields, i.e.

$$\frac{\mathcal{Y}}{V} = \frac{\partial H}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial y}.$$

Hence we get that

$$\frac{\partial H}{\partial x} = -\frac{Q}{V} \qquad \frac{\partial H}{\partial y} = \frac{P}{V},$$

thus the function H can be computed as follows (see for instance [29])

$$H = \int_{x_0}^{x}\left(-\frac{Q}{V}\right)dx + \int_{y_0}^{y}\left(\frac{P}{V}\right)\bigg|_{x=x_0} dy.$$

The following theorem holds (see for instance [49])

Theorem 4. *[Reeb 's criterion] (see for instance [49]). The analytic differential system* (6) *has a center at the origin if and only if there is a local nonzero analytic inverse integrating factor of the form* $V = 1 + h.o.t.$ *in a neighborhood of the origin.*

An analytic inverse integrating factor having the Taylor expansion at the origin $V = 1 + h.o.t.$ is called a *Reeb inverse integrating factor.*

Hence we get that to show that a singular point is a center for system (3) we have two basic mechanisms: we either apply Poincaré–Liapunov Theorem and we show that we have a local analytic first integral, or we apply the Reeb inverse integrating factor.

We observe that the difficulties of computations of the inverse integrating factor or Poincaré-Liapunov first integral for a given differential system are comparable to solving the system itself.

From this point of view it is interesting to state the problem about the determination of the structure of a differential system (3) knowing that it has a given first integral or an integral factor.The first chapter of my thesis is dedicated to the study these inverse problems.

(II) *Weak centers.*

The second chapter of the thesis is dedicated to the determination of differential equations (3) under the condition that they have a Poincaré-Liapunov first integral of the form $H = \dfrac{1}{2}(x^2 + y^2)\,(1 + h.o.t.)\,.$

We find a new class of centers which we call weak centers. We say that a center at the origin of an analytic differential system is a *weak center* if in a neighborhood of the origin it has an analytic first integral of the form $H = \dfrac{1}{2}(x^2 + y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j\right)$, where Υ_j is a homogenous polynomial of degree j.

We have characterized the expression of an analytic or polynomial differential system having a weak center at the origin. We prove that the following statements are equivalent.

(a) If an analytic (or polynomial) differential systems has a weak center at the origin then it can be written as

$$\dot{x} = -y(1 + \Lambda) + x\Omega, \qquad \dot{y} = x(1 + \Lambda) + y\Omega, \tag{7}$$

(see Theorem 9).

(b) Let $\dot{z} = iz + R(z, \bar{z})$ be the system (4) in complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. Then if this system has a weak center at the origin then $R(z, \bar{z}) = z\Phi(z, \bar{z})$ (see Proposition 30). This is a very simple criterium to determine the non existence of a weak center.

(c) If an analytic (or polynomial) differential system (3.30) (or (6)) has a weak center, then the first integral (5) satisfies that its homogenous polynomial H_j for $j = 3, \ldots, k \le \infty$ is $H_j = H_2\Upsilon_{j-2}$, for $j = 3, \ldots, k \le \infty$, where Υ_i is a homogenous polynomial of degree i.

Moreover we prove that the uniform isochronous centers and the isochronous holomorphic centers are weak centers.

It is well known the following result (see for instance [7]).

Let $\mathcal{X}$ be an analytic vector field associated to differential system (3). Then $\mathcal{X}$ has either a focus or a center at the origin, and under a formal change of coordinates the differential system associated to $\mathcal{X}$ can be written into the Kirchhoff normal form

$$\begin{aligned}\dot{x} &= \ -y(1+S_2(x^2+y^2)) + xS_1(x^2+y^2),\\ \dot{y} &= \ x(1+S_2(x^2+y^2)) + yS_1(x^2+y^2),\end{aligned}$$

where $S_j = S_j(x^2+y^2)$ for $j = 1, 2$ are formal series in the variable x^2+y^2 . Clearly these differential equations are particular case of (7).

We have extended the weak conditions of a center given by Alwash and Lloyd in [3] for linear centers with homogenous polynomial nonlinearities (see Proposition 24), to a general analytic or polynomial differential system see Theorem 12. Furthermore the centers satisfying the generalized weak conditions of a center, introduced in Theorem 12, are weak centers.

(III) *Construction of the generalized 3-dimensional Lotka-Volterra systems having a Darboux invariant. Final Evolutions*

The Lotka-Volterra systems in $\mathbb{R}^3$ are the differential systems (see for instance [25])

$$\dot{x}_j = x_j\left(d_j + a_j x_1 + b_j x_2 + c_j x_3\right), \quad \text{for} \quad j = 1, 2, 3, \tag{8}$$

The state space of this system is the set

$$\mathbb{R}^3_+ = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_j \geq 0, \quad \text{for} \quad j = 1, 2, 3.\right\}.$$

Models of Lotka-Volterra systems in $\mathbb{R}^3$ occur frequently in the physical and engineering sciences, as well in the biological. Differential system (8) was introduced independently by Lotka-Volterra in 1920s to model the interaction among the species (see [39, 54, 25])

The applications of Lotka-Volterra model in the population biology is well-known. For example for two dimensional Lotka-Volterra model we have the predator-prey model and for the three dimensional Lotka-Volterra model we have the symmetric and non-symmetric May-Leonard model (see for instance [13, 25, 32]) describing the competitions between three species.

Recently it has become important the generalization of 3-dimensional Lotka-Volterra systems of the form

$$\begin{aligned}\dot{x} &= \ x(a_0 + a_1x + a_2y + a_3z) = X_1,\\ \dot{y} &= \ y(b_0 + b_1x + b_2y + b_3z) = X_2,\\ \dot{z} &= \ c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2\\ &= \ X_3.\end{aligned} \tag{9}$$

In particular to study the resistant viral and bacterial strains, and the treatment on their proliferation (see for instance [9]). One framework for studying such systems is the multistrain model study by Castillo-Chavez and Feng [10]. The model, which is an approximation of the full system discussed in [10] was proposed in [53]. The model has only a single susceptible compartment and two infectious compartments corresponding to the two infectious agents. The model equations are

$$\begin{aligned}\dot{x} =& \quad x(-b_1 - \gamma_1 + \nu\, y + \beta_1\, z),\\ \dot{y} =& \quad y(-b_1 - \gamma_2 - \nu\, x + \beta_2\, y),\\ \dot{z} =& \quad z(-b_1 - \beta_1 x - \beta_2 y) + b_1 + \gamma_1 x + \gamma_2 y.\end{aligned}$$

The state space of this system is the set

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 : \quad \text{and} \quad x \geq 0,\, y \geq 0 \quad z \geq 0\right\}.$$

In [32] the authors characterized all the final evolution of this model under generic assumptions.

The main objective of this section is to study the global dynamics of the generalized Lotka -Volterra model (9) with the Darboux invariant $I = (x + y + z - 1)e^{-at}$ (see definition below) defined in the positive quadrant of $\mathbb{R}^3$

$$\mathbb{M} = \{x \geq 0, \quad y \geq 0, \quad \forall z \in \mathbb{R}\}. \tag{10}$$

We observe that this differential system does not correspond in general to a biological model, in view of (10), as the z variable can have in this case a negative value for the population.

The dynamics of the obtained differential equations on the invariant plane $\Pi : x + y + z = 1$ is described by the two dimensional Lotka Volterra system. Dynamics of the obtained differential system at infinity produces a cubic planar Kolmogorov system. To solve the center focus problem for this system we apply the results given in the solution of the problem I and II. We characterize all the final evolutions of this model.

0.1 Statement of the main results

0.1.1 Main results for the inverse center problem

One of the main objectives of the present thesis is to analyze the center problem from the inverse point of view. We state and solve the following two inverse problems of analytic and polynomial vector fields.

Problem 5. Inverse Poincaré-Liapunov's Problem *Determine the analytic (polynomial) planar differential systems* (3)*, or the associated vector fields*

$$\mathcal{X} = \sum_{j=1}^{k} \left(X_j \frac{\partial}{\partial x} + Y_j \frac{\partial}{\partial y} \right), \qquad \textit{for} \quad k \leq \infty, \tag{11}$$

where $X_j = X_j(x, y)$*,* $Y_j = Y_j(x, y)$ *for* $j \geq 2$ *are homogenous polynomials of degree* j *for which the given function* (5) *is a local analytic first integral.*

Problem 6. Inverse Reeb Problem *Determine the analytic (polynomial) planar vector fields* (11) *for which the*

$$V = 1 + \sum_{j=1}^{\infty} V_j,$$

where $V_j = V_j(x, y)$ *for* $j \geq 2$ *are homogenous polynomials of degree* j*, is a Reeb inverse integrating factor, i.e.*

$$\mathcal{X}(x) \frac{\partial V}{\partial x} + \mathcal{X}(y) \frac{\partial V}{\partial y} = V \left(\frac{\partial \mathcal{X}(x)}{\partial x} + \frac{\partial \mathcal{X}(y)}{\partial y} \right). \tag{12}$$

The solutions of the problems 5 and 6 for analytic and polynomial vector fields are given in the following theorems.

We will denote by $\{f, g\}$ the Poisson bracket, i.e.

$$\{f, g\} := \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}.$$

Theorem 7. *Consider the analytic vector field* $\mathcal{X}$ *associated to the differential system* (6) *Then this vector field has a Poincaré-Liapunov local first integral if and only if it has a Reeb inverse integrating factor. Moreover,*

(i) the analytic differential system (6) *for which* $H = (x^2 + y^2)/2 + h.o.t.$ *is a local first integral can be written as*

$$\dot{x} = \left(1 + \sum_{j=1}^{\infty} g_j \right) \{H, x\}, \qquad \dot{y} = \left(1 + \sum_{j=1}^{\infty} g_j \right) \{H, y\} \tag{13}$$

where $g_j = g_j(x, y)$ *is an arbitrary homogenous polynomial of degree* j *which we choose in such a way that the series* $\sum_{j=1}^{\infty} g_j$ *converge in the neighborhood of the origin.*

(ii) The differential system (4) *for which* $V = 1 + \sum_{j=1}^{\infty} V_j$ *is a Reeb integrating factor can be written as*

$$\dot{x} = \left(1 + \sum_{j=1}^{\infty} V_j\right) \{F, x\}, \qquad \dot{y} = \left(1 + \sum_{j=1}^{\infty} V_j\right) \{F, y\},$$

where $F = \sum_{j=2}^{\infty} F_j$ *and* $F_2 = (x^2 + y^2)/2$, $F_j = F_j(x, y)$ *for* $j > 2$ *is an arbitrary homogenous polynomial of degree* j *which we choose in such a way that* $\sum_{j=2}^{\infty} F_j$ *converges, i.e.* F *is an arbitrary Poincaré-Liapunov local first integral.*

Theorem 8. *Consider the polynomial vector field* $\mathcal{X}$ *associated to the differential system* (4). *Then this polynomial vector field has a Poincaré-Liapunov local first integral if and only if it has a Reeb inverse integrating factor. Moreover, the differential system associated to the vector field* $\mathcal{X}$ *for which* $H = (x^2 + y^2)/2 + h.o.t.$ *is a local first integral can be written as*

$$\begin{aligned} \dot{x} &= \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, x\} \\ &= \{H_{m+1}, x\} + (1 + g_1)\{H_m, x\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, x\}, \\ \dot{y} &= \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, y\} \\ &= \{H_{m+1}, y\} + (1 + g_1)\{H_m, y\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, y\}. \end{aligned} \tag{14}$$

where g_j *are homogenous polynomial of degree* j *for* $j \geq 1$ *such that* $\sum_{j=1}^{\infty} g_j$ *is an analytic function in the neighborhood of the origin and the function*

$$\Psi = \left(1 + \sum_{j=1}^{\infty} g_j\right)^{-1} = 1 - \sum_{k=1}^{\infty} \left(g_k - \sum_{n=1}^{k-1} g_n G_{k-n\ m+1}\right) := 1 - \sum_{k=1}^{\infty} G_{k\ m+1},$$

satisfies the following conditions

$$\{H_{m+1}, \Psi\} + \{H_m, (1 + g_1)\Psi\} + \ldots + \{H_2, (1 + g_1 + \ldots + g_{m-1})\Psi\} = 0. \tag{15}$$

The first integral H becomes

$$H = \frac{1}{2}(x^2 + y^2) + \sum_{j=2}^{\infty} H_j = \tau_1 H_{m+1} + \tau_2 H_m + \ldots + \tau_m H_2, \tag{16}$$

with

$$\tau_s = 1 - s \sum_{j=1}^{\infty} \frac{G_{j\,s}}{m+1+j}. \tag{17}$$

for $s = 2, \ldots, m$ are an analytic function in the neighborhood of the origin, where

$$G_{k\,j} := g_{k+m+1-j} - \sum_{j=1}^{k-1} g_n G_{k-n\ j}, for \quad k \geq 1, \quad j = 2, \ldots, m+1. \tag{18}$$

Theorem 7 and 8 are proved in section 1.3.

0.1.2 Main results for the inverse weak center problem

Differential system (3) has a weak center at the origin if it has a local analytic first integral of the form $H = \frac{1}{2}(x^2 + y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j(x, y)\right) = H_2 \Phi(x, y)$, where Υ_j is a homogenous polynomial of degree j.

One of the important characterization of the weak center consists in that the equation (3) can be written as

$$\dot{x} = -y(1 + \Lambda) + x\Omega, \qquad \dot{y} = x(1 + \Lambda) + y\Omega, \tag{19}$$

where $\Lambda = \Lambda(x, y)$ and $\Omega = \Omega(x, y)$ are convenient analytic (polynomial) functions. These differential systems are called $\Lambda - \Omega$ *differential systems*. This results is given in the following theorem.

Theorem 9. *A center at the origin of an analytic differential system* (6) *is a weak* *center at the origin if and only if this system can be written as*

$$
\begin{aligned}
\dot{x} = \quad & -y\left(1+\sum_{j=2}^{\infty}\left(\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}\right)\right)\\
& +\frac{x}{2}\sum_{j=2}^{\infty}\Big(\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\}\Big)\\
& := -y(1+\Lambda)+x\Omega,\\
\dot{y} = \quad & x\left(1+\sum_{j=2}^{\infty}\left(\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}\right)\right)\\
& +\frac{y}{2}\sum_{j=2}^{\infty}\Big(\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\}\Big)\\
& := x(1+\Lambda)+y\Omega,
\end{aligned}
\tag{20}
$$

where $\Upsilon_0 = 1$, $g_0 = 1$, g_j *and* Υ_j *are homogenous polynomials of degree* j *for* $j \geq 1$ *has the first integral* $H = H_2\left(1+\sum_{j=2}^{\infty}\Upsilon_j\right)$. *Moreover assuming that*

$$
\begin{aligned}
\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1} &= 0,\\
\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\} &= 0,
\end{aligned}
\tag{21}
$$

for $j \geq m+1$, *we obtain necessary and sufficient conditions under which the polynomial differential system* (20) *of degree* m *and has the first integral*

$$
H = H_2(1+\mu_1\Upsilon_1+\ldots+\mu_{m-1}\Upsilon_{m-1}), \tag{22}
$$

where $\mu_j = \mu_j(x,y)$ *is a convenient analytic function in the neighborhood of the origin for* $j = 1,\ldots,m-1$.

Corollary 10. *Define*

$$
\Lambda_{j-1} = \frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}, \quad \textit{for} \quad j>1,
$$

a homogenous polynomial of degree $j-1$, *then differential system* (20) *and condi-*

tions (21) *can be rewritten as follows*

$$\begin{aligned}\dot{x} = & -y(1+\sum_{j=1}^{\infty}\Lambda_j) + x\sum_{j=2}^{\infty}\frac{1}{j+1}\left(\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}-\Lambda_{j-1}\}\right)\\ = & \ P,\\ \dot{y} = & \ x(1+\sum_{j=1}^{\infty}\Lambda_j) + y\sum_{j=2}^{\infty}\frac{1}{j+1}\left(\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}-\Lambda_{j-1}\}\right)\\ = & \ Q,\end{aligned}$$

and

$$\begin{aligned}(j+1)H_{j+1}+jg_1H_j+\ldots+3g_{j-2}H_3+2g_{j-1}H_2 = & \ 0,\\ \{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\} = & \ 0,\end{aligned}$$

for $j > m$ *respectively. Moreover the following relation holds*

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sum_{j=1}^{\infty}\left(\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\}\right).$$

A center O of system (2) is a *uniform isochronous center* if the equality $x\dot{y}-y\dot{x}=\kappa(x^2+y^2)$ holds for a nonzero constant κ; or equivalently in polar coordinates (r,θ) such that $x=r\cos\theta$, $y=r\sin\theta$, we have that $\dot{\theta}=\kappa$.

Corollary 11. *An analytic differential system* (19) *has a uniform isochronous center at the origin if and only if*

$$\begin{aligned}\dot{x} = & -y+x\sum_{j=2}^{\infty}\frac{1}{j+1}\Big(\{H_j,g_1\}+\ldots+\{H_2,g_{j-1}\}\Big)=-y+x\Omega,\\ \dot{y} = & \ x+y\sum_{j=2}^{\infty}\frac{1}{j+1}\Big(\{H_j,g_1\}+\ldots+\{H_2,g_{j-1}\}\Big)=x+y\Omega,\end{aligned} \tag{23}$$

and

$$0 = (j+1)H_{j+1}+jg_1H_j+\ldots+3g_{j-2}H_3+2g_{j-1}H_2, \quad \textit{for} \quad j\geq 2.$$

Moreover a polynomial differential system of degree m (19) *has a uniform isochronous center at the origin if and only if* (23) *holds and*

$$\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\}=0, \quad \textit{for} \quad j\geq m+1.$$

The inverse approach to study the uniform isochronous center was given in [38].

The proofs of Theorem 9 and Corollaries 10 and 11 are given in section 2.1

Theorem 12. *[Generalized weak condition of a center of an analytic (polynomial) differential systems] We consider an analytic (polynomial) differential system (6). Then the origin is a weak center if there exists $\mu \in \mathbb{R}\backslash\{0\}$ such that the following relations hold*

$$\begin{aligned}(x^2+y^2)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right) &= \mu\,(xX+yY)\,,\\ \int_0^{2\pi}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)|_{x=\cos t,\,y=\sin t}dt &= 0,\end{aligned} \tag{24}$$

where $\mu \in \mathbb{R}\backslash\{0\}$. Moreover this differential system can be written as

$$\begin{aligned}\dot{x} &= -y\left(1+q(H_2)+(1-1/\lambda)\Upsilon+\frac{1}{2}\left(x\frac{\partial\,\Upsilon}{\partial\,x}+y\frac{\partial\,\Upsilon}{\partial\,y}\right)\right)+\frac{x}{2}\{\Upsilon,H_2\},\\ \dot{y} &= x\left(1+q(H_2)+(1-1/\lambda)\Upsilon+\frac{1}{2}\left(x\frac{\partial\,\Upsilon}{\partial\,x}+y\frac{\partial\,\Upsilon}{\partial\,y}\right)\right)+\frac{y}{2}\{\Upsilon,H_2\},\end{aligned} \tag{25}$$

with $\lambda = 2/\mu$, and $\Upsilon = \Upsilon(x,y)$ and $q = q(H_2)$ are convenient analytic functions.

If differential system (25) is a polynomial differential system of degree m, i.e. $\Upsilon = \Upsilon(x,y)$ is a polynomial of degree $m-1$ and $q(H_2)) = \sum_{j=1}^{[(m-1)/2]} \alpha_j H_2^{j-1}$, here $[(m-1)/2]$ is the integer part of $(m-1)/2$, α_j is a constant for $j = 1,\ldots,[(m-1)/2]$ such that $1+\alpha_1+\dfrac{\lambda-1}{\lambda}\Upsilon(0,0) \neq 0$, then the system (25) is quasi Darboux-integrable with the first integral F which is given in what follows

(i) If $\lambda \neq 1$ and $\prod_{n=2}^{[m/2]}(n-1/\lambda) \neq 0$, then

$$F = \frac{H_2}{\left(\Upsilon+\dfrac{1+\alpha_1}{1-1/\lambda}+\dfrac{\alpha_2 H_2}{2-1/\lambda}+\ldots+\dfrac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2]-1/\lambda}\right)^{\lambda/(\lambda-1)}}. \tag{26}$$

The algebraic curves $H_2 = 0$ and

$$g = \Upsilon+\frac{1+\alpha_1}{1-1/\lambda}+\frac{\alpha_2 H_2}{2-1/\lambda}+\ldots+\frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2]-1/\lambda} = 0,$$

are invariant curve with cofactors $\{g,H_2\}$ and $(1-1/\lambda)\{g,H_2\}$, respectively.

(ii) If $\lambda = 1$ and $1+\alpha_1 \neq 0$, then

$$F = H_2 e^{-\left(\Upsilon+\alpha_2 H_2+\alpha_3 H_2^2/2+\ldots+\alpha_m H_2^{[(m-1)/2]-1}/([(m-1)/2]-1)\right)/(1+\alpha_1)}. \tag{27}$$

The algebraic curves $H_2 = 0$ is invariant with cofactor $\{H_2,\Upsilon\}$.

(iii) *If* $1/[m/2] \le \lambda = 1/k < 1$ *and* $\left(\alpha_k \prod_{n=2, n\neq k}^{[(m-1)/2]} (n-k)\right) \neq 0$, *then*

$$F = \frac{H_2}{\left(\Upsilon + \frac{1+\alpha_1}{1-k} + \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1} + \alpha_k H_2^{k-1} \log H_2\right)^{1/(k-1)}}. \quad (28)$$

The algebraic curve $H_2 = 0$ *and the non-polynomial curve*

$$f = \Upsilon + \frac{1+\alpha_1}{1-k} + \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1} + \alpha_k H_2^{k-1} \log H_2 = 0,$$

are invariant curves with cofactors $\{\Upsilon, H_2\}$ *and* $(1-k)\{\Upsilon, H_2\}$, *respectively. We observe that* $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(iv) *If* $1/[(m-1)/2] \le \lambda = 1/k < 1$, $\alpha_k = 0$ *and* $\prod_{n=2, n\neq k}^{[(m-1)/2]} (n-k) \neq 0$, *then*

$$F = \frac{H_2}{\left(\Upsilon + \frac{1+\alpha_1}{1-k} + \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1}\right)^{1/(k-1)}}. \quad (29)$$

The algebraic curves $H_2 = 0$ *and*

$$g = \Upsilon + \frac{1+\alpha_1}{1-k} + \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1} = 0$$

are invariant algebraic curves with cofactors $\{g, H_2\}$ *and* $(1-k)\{g, H_2\}$ *respectively.*

The given first integrals has the following Taylor extension at the origin $F = H_2(1 + h.o.t.)$. *Consequently the origin is a weak center. In an analogous way we can study the analytic case.*

Theorem 12 is proved in section 2.2.

For the $\Lambda - \Omega$ differential systems we get the following conjectures.

Conjecture 13. *The polynomial differential system of degree m*

$$\begin{aligned} \dot{x} &= -y(1 + \mu\,(a_2 x - a_1 y)) + x(a_1 x + a_2 y + \Omega_{m-1}), \\ \dot{y} &= x(1 + \mu\,(a_2 x - a_1 y)) + y(a_1 x + a_2 y + \Omega_{m-1}), \end{aligned} \quad (30)$$

where $(\mu + m - 2)(a_1^2 + a_2^2) \neq 0$, *and* $\Omega_{m-1} = \Omega_{m-1}(x, y)$ *is a homogenous polynomial of degree* $m - 1$, *has a weak center at the origin if and only if system* (30) *after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$.

Theorem 14. *Conjecture 13 holds for* $m = 2, 3, 4, 5, 6$.

The only difficulty for proving Conjectures 13 for the Λ–Ω systems of degree m with $m > 6$ is the huge number of computations for obtaining the conditions that characterize the centers.

The proofs of Theorem 14 is given in subsection 2.4.3.

Conjecture 15. *The polynomial differential system of degree* m

$$\begin{aligned} \dot{x} = & \quad -y(1 + \mu(a_2 x - a_1 y)) + x(a_1 x + a_2 y + \sum_{j=2}^{m-1} \Omega_j(x, y)), \\ \dot{y} = & \quad x(1 + \mu(a_2 x - a_1 y)) + y(a_1 x + a_2 y + \sum_{j=2}^{m-1} \Omega_j(x, y)), \end{aligned} \tag{31}$$

under the assumptions $(\mu + (m - 2))(a_1^2 + a_2^2) \neq 0$ *and* $\sum_{j=2}^{m-2} \Omega_j \neq 0$, *where* $\Omega_j = \Omega_j(x, y)$ *is a homogenous polynomial of degree* j *for* $j = 2, \ldots, m - 1$, *has a weak center at the origin if and only if system* (31) *after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$. *Moreover differential system* (31) *in the variables* X, Y *becomes*

$$\begin{aligned} \dot{X} = & \quad -Y(1 + \mu Y) + X^2 \Theta(X^2, Y) = -Y(1 + \mu Y) + X\{H_2, \Phi\}, \\ \dot{Y} = & \quad X(1 + \mu Y) + XY\Theta(X^2, Y) = X(1 + \mu Y) + Y\{H_2, \Phi\}, \end{aligned}$$

where $\Theta(X^2, Y)$ *is a polynomial of degree* $m - 2$, *and* Φ *is a polynomial of degree* $m - 1$ *such that* $\{H_2, \Phi\} = X\Theta(X^2, Y)$.

Theorem 16. *Conjecture 15 holds for* $m = 2, 3$ *and for* $m = 4$ *with* $\mu = 0$.

The proof of Theorem 16 for $\mu = 0$ and $m = 2$ goes back to Loud [40]. The proof of Theorem 16 for $\mu = 0$ and $m = 3$ was done by Collins [15]. Finally the proof of Theorem 16 for $\mu = 0$ and $m = 4$ goes back to [2, 1, 11]. But in the proof of this last result there is some mistakes. The phase portraits of these systems are classified in [4, 27, 28]. The proof that these centers are weak centers has been done in Theorem 9.

The proof of Theorem 16 is given in section 2.6.

0.1.3 Main results on the generalized 3-dimensional Lotka-Volterra systems having a Darboux invarian

The main result in the study of the differential systems

$$\begin{aligned}\dot{x} &= x(a-b-cx-(d+e)y-fz) = P(x,y,z),\\ \dot{y} &= y(a-g+dx-hy-iz) = Q(x,y,z),\\ \dot{z} &= z(fx+iy+a)+x(cx+ey+b)+y(hy+g)-a = R(x,y,z)\end{aligned} \tag{32}$$

is given in the following theorem.

Theorem 17. *The generalized Lotka-Volterra differential systems* (32) *has* 7920 *different phase portraits.*

We have also solved the center-focus problem for the two dimensional Lotka-Volterra systems (see Proposition 66) and for the two dimensional cubic Kolmogorov systems (see Proposition 68).

Theorem 17 is proved in subsection 3.6.3.

We note that the results of the first two chapters have been published in the papers [36, 35, 37].