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Complete Abelian integrals for polynomials whose generic fiber is biholomorphic to $\mathbb{C}^*$

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ABSTRACT

Let H be a polynomial of degree $m+1$ on $\mathbb{C}^2$ such that its generic fiber is biholomorphic to $\mathbb{C}^*$, and let ω be an arbitrary polynomial 1-form of degree n on $\mathbb{C}^2$. We give an upper bound depending only on m and n for the number of isolated zeros of the complete Abelian integral defined by H and ω.

1. Introduction and statement of the results

Let $H : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial whose generic fiber is irreducible, and let ω be a polynomial 1-form on $\mathbb{C}^2$. By the *complete Abelian integral* defined by H and ω, we mean the function

$$I(c) = \int_{[\gamma_c]} \omega,$$

where the parameter c varies over the set of generic values of H, and $[\gamma_c]$ is a *cycle* of H: $[\gamma_c]$ is the homology class of a loop $\gamma_c \subset H^{-1}(c)$, and $[\gamma_c]$ is non-trivial in the first homology group $H_1(H^{-1}(c), \mathbb{Z})$ of the generic fiber $H^{-1}(c)$ of H.

From the classical Poincaré–Pontryagin–Andronov criterion we know that the isolated zeros of $I(c)$ are related to the limit cycles of the infinitesimal *perturbed Hamiltonian system*

$$dH - \varepsilon\omega = 0 \quad \text{with } 0 \neq \varepsilon \in (\mathbb{C}, 0) \text{ fixed},$$

that arise from the cycles of the *Hamiltonian system* $dH = 0$, which are precisely the cycles of H. In this sense, the problem of finding the upper bound $Z(m, n) \in \mathbb{N}$, depending on $m = \deg(H) - 1$ and $n = \deg(\omega)$ for the number of isolated zeros of $I(c)$, counting multiplicities, is referred to as the *weak infinitesimal Hilbert's* 16th *problem* (see [1]). Of course, in this problem we must consider all polynomials H of degree $m+1$ and all the 1-forms ω of degree n.

Khovanskiĭ [2] and Varchenko [3] proved that $Z(m, n)$ is finite. Petrov and Khovanskiĭ claimed that $Z(m, n) \leq A(m)n + B(H)$, where $A(m)$ is an explicit constant depending only on m while $B(H)$ is independent of ω but depends on H. The proof of this assertion was given by Żołądek [4, Theorem 6.26]. Recently Binyamini et al. [5] proved that $Z(n, n) \leq 2^{2^{\mathrm{Po}(n)}}$, where $\mathrm{Po}(n) = O(n^p)$ stands for an explicit polynomially growing term with the exponent p not exceeding 61.

A difficulty in finding an explicit upper bound for $Z(m, n)$ is that even though $I(c)$ is a locally single-valued function, globally it can be multi-valued since its analytic continuation depends on the monodromy of the polynomial H (see Section 2).

If $\dim H_1(H^{-1}(c), \mathbb{Z}) = 1$ for a generic value c of H, then the generic fiber of H is irreducible and biholomorphic to $\mathbb{C}^*$; therefore, H is called a primitive polynomial of type $\mathbb{C}^*$. This is the simplest non-trivial case for studying $I(c)$ because there

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