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Qualitative Theory of Dynamical Systems

The Infinitesimal Hilbert's 16th Problem in the Real and Complex Planes

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Abstract. In this paper we study the infinitesimal Hilbert's 16th problem on $\mathbb{R}^2$ and $\mathbb{C}^2$, describing some techniques used for its research in the last few years. We make an alternative proof of a theorem of Gavrilov and Ilyashenko which allows the study of higher order Poincaré–Pontryagin functions. We apply these techniques to the research of limit cycles of systems of Liénard type obtained by special perturbations of the harmonic oscillator. For this class of systems, we obtain an upper bound, depending on the degree of the perturbation, for the number of limit cycles that can be generated from cycles of the harmonic oscillator. Explicit examples are given.

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1. Introduction

In the early 1880's, in an article of four parts (see [54]), Poincaré created the qualitative theory of differential equations. The most important goal of this theory is to describe the phase portraits of autonomous differential equations. To accomplish this goal in the planar case, it is essential to know the configuration of their limit cycles, i.e. the number and relative position of the limit cycles of each autonomous planar differential equation. In [54] Poincaré studied autonomous planar polynomial differential equations and for the first time provided the notion of limit cycle (see [54] part II, p. 261). In addition he proved that a planar polynomial differential equation without saddle connections has only a finite number of limit cycles. Actually, the most famous problem in this direction, is the second part of Hilbert's 16th problem stated in 1900 (see [28]) which corresponds to the study of planar polynomial differential equations. Classically this problem has been divided in parts and can be stated as follows.

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