The geometry of some tridimensional families of planar quadratic differential systems

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Abstract

Planar quadratic differential systems occur in many areas of applied mathematics. Although more than one thousand papers have been written on these systems, a complete understanding of this family is still missing. Classical problems, and in particular Hilbert's 16th problem, are still open for this family. One of the goals of recent researchers is the topological classification of quadratic systems. As this attempt is not possible in the whole class due to the large number of parameters (twelve, but, after affine transformations and time rescaling, we arrive at families with five parameters, which is still a large number), many subclasses are considered and studied. Specific characteristics are taken into account and this implies a decrease in the number of parameters, which makes possible the study. In this thesis we mainly study two subfamilies of quadratic systems: the first one possessing a finite semi–elemental triple node and the second one possessing a finite semi–elemental saddle–node and an infinite semi–elemental saddle–node formed by the collision of an infinite saddle with an infinite node. The bifurcation diagram for both families are tridimensional. The family having the triple node yields 28 topologically distinct phase portraits, whereas the closure of the family having the saddle–nodes within the bifurcation space of its normal form yields 417. Invariant polynomials are used to construct the bifurcation sets and the phase portraits are represented on the Poincaré disk. The bifurcation sets are the union of algebraic surfaces and surfaces whose presence was detected numerically. Moreover, we also present the analysis of a differential system known as SIS model (this kind of systems are easily found in applied mathematics) and the complete classification of quadratic systems possessing invariant hyperbolas.

Key words: quadratic differential systems; topological classification; affine invariant polynomials; semi–elemental triple node; semi–elemental saddle–node; global phase portrait; SIS model; invariant hyperbola

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Introduction

Closing problems is definitely the great pleasure of us mathematicians. We are also delighted when we have a long–time–ago theme concluded, when we write down the most famous quote *“quod erat demonstrandum”* in the end of the proof of a question formulated in the past. And this pleasure seems to be directly proportional to the time elapsed between the formulation of the question and the moment the answer is given.

With the advent of differential calculus, it became easy the possibility of solving many questions asked by ancient mathematicians, but at the same time some other questions were formulated any further. The searching for primitives for functions that could not be expressed algebraically or with a finite number of analytic terms has greatly complicated the future research and new areas of mathematics have even being created to answer these questions. And, besides the problem of searching for a primitive for a differential equation in one dimension we add more dimension, the problem became more complex.

As discussed above, the theory of ordinary differential equations became one of the basic tools of pure and applied mathematics. For instance, this theory makes it possible to study the population growth of species or the movement of a pendulum. If the derivation variable (well–known as the *time*) just plays an implicit role, the differential equation is said to be autonomous and, in this case, the systems can be considered as dynamical systems. The designation *time* for the derivation variable probably came from the evolution in time of a particle in space and is used in this sense since then. In addition, differential equations such as those used to solve real–life problems may not necessarily be directly solvable, i.e. their solutions do not have an explicit expression. Instead, solutions can be approximated using numerical methods.

Due to the complexity of solving generic differential systems and estimating their solutions, some strategies have been taken and developed in the attempt of “minimizing the problem”.

Firstly, the birth of the qualitative theory of the differential equations introduced by Poincaré [47] was a great breakthrough in the study of differential systems and, secondly, the restriction in the analysis of families of differential systems with specific properties.

We recall, for example, the Hilbert's 16th problem [33, 34]. It is the most investigated mathematical problem in the qualitative theory of dynamical systems in the plane. In short, this problem discusses on the number of limit cycles in polynomial systems in the plane. Although the proposed family (the polynomial case) is already a subfamily of the set of all differential equations, this problem is still difficult to solve. In view of this difficulty, many researchers have been improving and giving new statements to the problem.

We dare say that the complete study of the huge family of generic differential systems is impossible and, hence, researchers have been studying only particular classes of such family.

In this thesis we restrict ourselves to the study and the topological classification of planar quadratic differential systems. By quadratic we mean that the functions which define the systems are polynomials of degree two. However, this subfamily is also generic and we have some reasons to restrict more this class of differential systems.

The first reason is that each particular subclass provide interesting results. For example, Artés, Llibre and Schlomiuk [6] have classified topologically the quadratic systems possessing a weak focus of second order. This class is interesting itself because all phase portraits with limit cycles in it can be produced by perturbations of symmetric (reversible) quadratic systems with a center.

Another reason is the existence of algebraic tools to deal with the problem of classifying topologically quadratic systems with peculiarities. Concerning this issue, in 1966, Coppel [22] believed that the classification of the quadratic systems could be completed purely algebraically, i.e. by means of algebraic equalities and inequalities, it would be possible to find the phase portrait of a quadratic system. At that time, his thoughts were not easy to be refuted. It is known that the finite singular points of a quadratic system can be found as the zeroes of a resultant of degree four, and its solutions can be calculated algebraically, as well as the infinite singular points. Additionally, limit cycles could be generated by Hopf bifurcation whose conditions were also determined algebraically.

However, as it is so often in mathematics that everything which is not perfectly proved may be completely false, Dumortier and Fiddelaers [26] proved in 1991 that starting with the quadratic systems (and following all subsequent systems) there exist geometric and topological phenomena in their phase portrait whose determination cannot be fixed by means of algebraic relations. Specifically, most of the connections between separatrices and occurrence of double or semi–stable limit cycles is not determinable algebraically. This shows us that it is at least interesting and challenging the study of quadratic systems and the attempt to classify topologically all their phase portraits.

And the last reason (but not the less important nor, in fact, the last one) is the desire to classify all the codimension–one unstable quadratic systems. We explain a little more about it. Artés, Kooij and Llibre [4] have studied the structurally stable quadratic systems, modulo limit cycles. In their book, they proposed the determination of how many and which phase portraits a quadratic system can have after its coefficients suffer small perturbations. To obtain a structurally stable system modulo limit cycle we need few conditions. Simply, the existence of multiple singular points and separatrices connections are not allowed. Centers, weak foci, semi–stable limit cycles and all other unstable elements are "eliminated" by the quotient *modulo limit cycles*. The main result in the book [4] is that there exist exactly 44 topologically distinct phase portrait in the family of the structurally stable quadratic systems, modulo limit cycles.

As the main goal is the complete classification of the family of quadratic systems, and having classified topologically all the structurally stable quadratic systems, the natural continuation is to study the quadratic systems with a degree of degeneracy one higher, i.e. the codimension–one unstable quadratic systems, modulo limit cycles. We now allow the existence of multiple singular points and separatrices connections. Following a methodical and systematic study like the one conducted in [4], we can generate a family of topologically possible cases for this codimension. Moreover, we have the advantage that not all the topologically possible phase portraits can be realizable (fact that was learnt by constructing the 44 topological classes of structurally stable quadratic systems).

Following this methodology and other similar ones already applied in [4], we hope it will be possible to point out the candidates which are non–realizable and, using the extensive bibliography, it may be possible to find many of those realizable ones, either because they have previously appeared, or by some perturbation of them.

The state of the research is well advanced, remaining a few cases that refuse to find their example (or to prove their impossibility) and providing nearly 200 phase portraits to the collection of the 44. Again this is a very topological process with traces of qualitative theory.

Once we have completed the classification of the unstable quadratic systems of codimension one, it would be the turn of the codimension–two systems. Although the entire process can be exhaustive, it can be subdivided into sections, and we also have the advantage that the higher degeneration the system possesses, the greater the existing bibliography is. Furthermore, the degree of codimension to further study is limited and, therefore, realizable.

Even if there exists a large literature from which we can take new examples of phase portraits still unknown, new families of quadratic systems must be studied in order to contribute to this systematic process.

The mainly used technique has been to produce a normal form for such family which fixes the position of two finite singular points, allowing the identification of the two other finite singular points (real or complex) by means of a quadratic equation. The study of singular points at infinity, even involving the study of a simple cubic, has become easier when assuming a single variable.

Sometimes, instead, this technique has forced a normal form which has behaved in a more complicated way in determining the bifurcation curves or surfaces, or simply which could have not be extended continuously to the boundaries of the parameter space. The alternative of fixing only one finite singular point is even more impracticable as it requires the use of a cubic to determine the other three finite singular points.

However, there comes to us in recent time great advances in the theory of invariants, mainly from the Sibirsky's school [57] in Moldova by the hand of one of his main students, N. Vulpe. The idea of the invariants is very simple and we will explain it with an example. We suppose that a quadratic system has generically four singular points (real or complex), but in some hypersurface of the parameter space a finite singular point goes to infinity and in another hypersurface another singular point also goes to infinity. The way to calculate them is to obtain the resultant of degree four in one of the variables. We denote by $\mu_0 x^4 + \mu_1 x^3 + \mu_2 x^2 + \mu_3 x + \mu_4$ such resultant. Therefore, in the union of the hypersurfaces in where a finite point has gone to infinity we must have $\mu_0 = 0$, so that the resultant gives exactly three (real or complex) solutions to the problem. And likewise, if two finite singular points have gone to infinity, then $\mu_0 = \mu_1 = 0$. Furthermore, the fact that a finite point collides with a point at infinity is an invariant under any affine transformation. We can modify the normal form as we desire, and the fact that $\mu_0 = 0$ will be maintained. In fact, we would have to be able (and we are) to obtain such expression not in terms of a certain normal form, but in terms of the general quadratic system with twelve coefficients. And so, having obtained these expressions, we can address classifications in wider parameter spaces since the position of the singular points has no longer any influence in calculating the bifurcations.

The great result of Sibirsky's school has been obtaining the "bricks" of these invariants, the tools to manipulate them, and obtaining a basis of elements which make up the ideal of invariant polynomials up to degree twelve. By now, these basic elements have proven to be sufficient to set all that can be determined algebraically in a quadratic system. Not only if a point goes to infinity or not, but also if two (or more) points collide, if a system has a certain degenerate singular point or not, if there exist invariant lines, if there exist centers and which global portrait they have, if there exist certain types of first integrals, if there exist weak points (foci or saddles) which are important to determine the possibility of creation of limit cycles. In short, everything that has been studied in some particular normal form can now be viewed in terms of invariants and we can obtain its bifurcations independently of the choice of the normal form.

From the papers of Artés, Llibre, Schlomiuk and Vulpe we obtain the classification of all possible combinations of finite singular points [8], of infinite singular points [53], systems with 6, 5 or 4 invariant straight lines, systems with weak focus or weak saddle, systems with polynomial first integral, systems with rational first integral of second or third order, and it is in progress the refinements of these works in terms of the tangential equivalence of singular points, i.e. in sense of distinguishing a generic node with two directions from non–generic nodes with one or infinite directions, or distinguishing the qualitative way a degenerate singular point is located at infinity. Likewise, it is also under construction a comprehensive classification of all quadratic systems in terms of their singular points. This still would not be the complete classification of phase portraits, but we would get very close to its completeness, besides being an essential step to achieve this.

Using these algebraic tools, together with numerical tools to determine the nonalgebraic bifurcations, in the recent years researchers have managed to classify families of quadratic systems that depend on four parameters. Turning to the projective space $\mathbb{RP}^3$ and by foliating it, it is possible to complete in a reasonable time studies which involve partitions of the parameter space of about 400 parts, which include about 125 different phase portraits. Many of these portraits provide new examples, which will be included in the great encyclopedia of the quadratic systems and are the first found representatives of a certain structurally stable configuration with a concrete number of limit cycles. Among these studies, we include the classification of quadratic systems possessing a focus of second order [6].

Recalling the last reason discussed above for restricting the family of quadratic systems to subfamilies with some specific characteristic, our purpose is to contribute to the classification of the structurally unstable quadratic systems of codimension one. One way to obtain codimension–one phase portraits is considering a perturbation of known phase portraits of quadratic systems of higher codimension. This perturbation would decrease the codimension of the system and we may find a representative for a topological equivalence class in the family of the codimension–one systems and add it to the existing classification.

With this intention, we propose the study of two classes of quadratic systems. The first one possessing a finite semi–elemental triple node, and the other possessing a finite semi–elemental saddle–node and an infinite semi–elemental saddle–node formed by the collision of an infinite saddle with an infinite node. It is worth mentioning that this last class was divided into three subclasses according to the position of the infinite saddle–node.

Systems possessing a finite triple node depend on 3 parameters (and, then, their bifurcation space has dimension three — it is $\mathbb{R}^3$) and yields a partition in the parameter space of 63 parts, generating 28 topologically distinct phase portraits. The results on this family are contained in:

J. C. ARTÉS, A. C. REZENDE, R. D. S. OLIVEIRA, *Global phase portraits of quadratic polynomial differential systems with a semi–elemental triple node*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **23** (2013), 21pp.

Systems possessing the saddle–nodes as described above are divided into three subclasses according to the position of the infinite saddle–node, namely: (A) with the infinite saddle–node in the horizontal axis, (B) with the infinite saddle–node in the vertical axis and (C) with the infinite saddle–node in the bisector of the first and third quadrants. These systems are 4−parametric, but, after affine transformations and time rescaling, one of these parameters can be fixed as 1 and, hence, their bifurcation spaces have dimension three — they are $\mathbb{RP}^3$). Doing this, we are able to provide the classification of the closure of each one of these families within the set of their representatives in the parameter space of the adopted normal forms for each family.

The parameter space of the closure of family (A) is partitioned in 85 parts, yielding 38 topologically distinct phase portraits; the parameter space of the closure of family (B) is partitioned in 43 parts, yielding 25 topologically distinct phase portraits; and the parameter space of the closure of family (C) is partitioned in 1034 parts, yielding 371 topologically distinct phase portraits. The results on this families are contained in:

J. C. ARTÉS, A. C. REZENDE, R. D. S. OLIVEIRA, *The geometry of quadratic polynomial differential systems with a finite and an infinite saddle–node* (A,B), Internat. J. Bifur. Chaos Appl.

Sci. Engrg. **24** (2014), 30pp.

and

J. C. ARTÉS, A. C. REZENDE, R. D. S. OLIVEIRA, *The geometry of quadratic polynomial differential systems with a finite and an infinite saddle–node* (C), Preprint, 2014.

For the analysis of the systems described above we have used the theory of invariant polynomials proposed by Sibirsky and his pupils. In addition to this algebraic tool, we have used the softwares *Mathematica*, *P4* and also an implementation in *Fortran*.

Besides these three works, we dare to go a little beyond. While we were studying the preliminaries for this thesis, we faced the problem of classifying topologically a quadratic system of type *SIS model*. Until that time, we had not had contact with the theory developed by the Sibirsky's school, so we had to use the classical results on qualitative theory of differential equations. It is a 4−parametric family which yields 3 topologically distinct phase portraits. The results on this family are contained in:

R. D. S. OLIVEIRA, A. C. REZENDE, *Global phase portraits of a SIS model*. Appl. Math. Comput. **219** (2013), 4924–4930.

Finally, but not less important, we also present in this thesis a joint work with Vulpe, which was done during the Brazilian summer of 2014 at ICMC–USP. In this work, we use the invariant polynomials to classify all the quadratic systems possessing a nondegenerate hyperbola given necessary and sufficient conditions by means of the invariant polynomials for these systems to possess at least one invariant hyperbola. Moreover, we provide their number and their multiplicity. The results on these family are contained in:

R. D. S. OLIVEIRA, A. C. REZENDE, N. VULPE, *Family of quadratic differential systems with invariant hyperbolas: a complete classification in the space* $\mathbb{R}^{12}$. Cadernos de Matemática. **15** (2014), 19–75.

This thesis is divided as follows. In Chapter 1 we provide basic concepts on the qualitative theory of differential equations and we also give an emphasis for the quadratic systems; the reader which is familiar to these concepts can skip this chapter. In Chapter 2 we present all the nomenclature concerning the singular points; they refer to "new" definitions more deeply related to the geometry of the singular points, their multiplicity and, especially, their Jacobian matrices.

Chapter 3 presents the notions of blow–up and Poincaré's compactification; in this chapter we discuss the results of the SIS model.

In Chapter 4 we describe the theory of invariant polynomials stated by Sibirsky and his pupils. This is the most important tool used in this thesis.

The results of the systems possessing a triple node are demonstrated in Chapter 5, while Chapters 6 and 7 discuss the systems having the two saddle–nodes, one finite and the other infinite.

In Chapter 8 we classify all the quadratic systems possessing an invariant nondegenerate hyperbola and, finally, in Chapter 9 we describe the further works to be done and ideas for the future.

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Have a good reading!