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Invariant curves on differential systems defined in $\mathbb{R}^n$, $n \geq 2$

Camila Aparecida Benedito Rodrigues de Lima

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Camila Aparecida Benedito Rodrigues de Lima

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Advisor: Profa. Dra. Regilene Delazari dos Santos Oliveira
Co-advisor: Prof. Dr. Jaume Llibre Saló

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Curvas invariantes em sistemas diferenciais definidos em $\mathbb{R}^n$, $n \geq 2$

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Aos meus pais, Valentina e João.

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"I've been reading books of old
The legends and the myths
Achilles and his gold
Hercules and his gifts
Spiderman's control
And Batman with his fists
And clearly I don't see myself upon that list
But she said, where'd you wanna go?
How much you wanna risk?
I'm not looking for somebody with some superhuman gifts."
Something Just Like This, Coldplay and The chainsmokers.

ABSTRACT

DE LIMA, C. A. B. R. **Invariant curves on differential systems defined in $\mathbb{R}^n$, $n \geq 2$.** 2018. 179 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Differential systems appear modelling many natural phenomena in different branches of science, in biological and physical applications among other areas. Differential systems usually have invariant curves and we can obtain a better description of the qualitative behaviour of their solutions using them. Such invariant curves can be algebraic or not and, in the case where they are closed and isolated, they are called limit cycles. There is a very famous problem, proposed by David Hilbert in 1900 what ask about the maximum number of limit cycle that a polynomial differential system could present. In this thesis we investigate the existence of some invariant curves in quadratic polynomial differential systems and in discontinuous piecewise differential systems (they are known as Filippov's systems).

Even after hundreds of studies on the topology of real planar quadratic vector fields the complete characterization of their phase portraits is a quite complex task, they depends on twelve parameters, after affine transformations and time rescaling, we have families with five parameters, which is still a large number. So many subclasses have been considered instead of the complete system. In this thesis we investigate conditions under the parameters of the system for a planar quadratic differential system present invariant algebraic curve of degree 3 (a cubic curve) and a Darboux invariant and obtain the topological classification of these systems.

The increasing interest in the theory of non–smooth vector fields has been mainly motivated by its strong relation with physics, engineering, biology, economy, and other branches of science. In the study of the Filippov's systems, we investigate the number of periodic orbits that they can present. In this study we apply the averaging theory. Such theory is used to study some classical models and we also present generalization of such technique for a class of non–smooth systems. In addition, we also show how the Lyapunov–Schmidt reduction can be used to consider cases where the averaging theory is not sufficient to study periodic solutions.

Keywords: Algebraic invariant curve, Darboux invariant, Averaging method, Filippov's systems, Lyapunov-Schmidt reduction.

RESUMO

DE LIMA, C. A. B. R. **Curvas invariantes em sistemas diferenciais definidos em $\mathbb{R}^n$, $n \geq 2$.** 2018. 179 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Sistemas diferenciais aparecem na modelagem de muitos fenômenos naturais em diferentes ramos da ciência, em aplicações biológicas e físicas, entre outras áreas. Sistemas diferenciais geralmente possuem curvas invariantes e podemos obter uma melhor descrição do comportamento qualitativo de suas soluções utilizando-as. Tais curvas invariantes podem ser algébricas ou não e, no caso de serem fechadas e isoladas, são chamadas de ciclos limites. Há um problema muito famoso, proposto por David Hilbert em 1900, que questiona o número máximo de ciclos limites que um sistema polinomial diferencial poderia apresentar. Nesta tese investigamos a existência de algumas curvas invariantes em sistemas diferenciais polinomiais quadráticos e em sistemas diferenciais contínuos por partes (eles são conhecidos como sistemas de Filippov).

Mesmo após centenas de estudos sobre a topologia dos campos vetoriais reais planares e quadráticos, a caracterização completa de seus retratos de fase é uma tarefa bastante complexa. Eles dependem de doze parâmetros e após transformações afins e reescalonamento de tempo, temos famílias com cinco parâmetros, o que ainda é um grande número. Assim muitas subclasses tem sido consideradas em vez do sistema completo. Nesta tese investigamos condições sob os parâmetros do sistema para que um sistema diferencial planar quadrático apresente uma curva algébrica invariante de grau 3 (curva cúbica) e um invariante de Darboux e obtemos a classificação topológica destes sistemas.

O crescente interesse na teoria dos campos de vetores suaves por partes tem sido motivado, principalmente, por sua forte relação com a física, engenharia, biologia, economia e outros ramos da ciência. No estudo dos sistemas de Filippov, investigamos o número de órbitas periódicas que eles podem apresentar. Para este estudo, aplicamos a teoria do averaging. Tal teoria é usada para estudar alguns modelos clássicos e também apresentamos a generalização de tal técnica para uma classe de sistemas suaves por partes. Além disso, mostramos também como a redução de Lyapunov - Schmidt pode ser usada para considerar casos em que a teoria do averaging sozinha não é suficiente para estudar soluções periódicas.

Palavras-chave: Curva algébrica invariante, Invariante de Darboux, Método do averaging, Sistemas de Filippov, Redução de Lyapunov-Schmidt.

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1

INTRODUCTION

The first ordinary differential equation (ODE) appeared in the works of Leibniz (1646–1716) and Newton (1642–1727) and since that it was used in the modelling of many natural phenomena and their laws, Astronomy, Mechanics, Physics, Chemistry, Engineering, Economicy, Ecology, Epidemiology, Neuroscience, among others. An ODE is an equation of the form

$$\frac{dx}{dt} = \dot{x} = f(t,x), \tag{1.1}$$

where $f : D \times \mathbb{R} \to \mathbb{R}^n$, D is a subset of $\mathbb{R}^n$ and the dot denotes the differentiation with respect to the independent variable t. A solution of (1.1) is a curve $x : I \subset \mathbb{R} \to D \subset \mathbb{R}^n$, where I is an interval, that satisfy the equation. Given the knowledge of other types of equations, the first plausible attempt would be to develop a method to obtain the solution of an ODE. However, if the function involved is a little more complicated, finding the solution curve in fact becomes a problem that can not be solved analytically. Indeed J. Liouville (1809-1882) proved that not all equations admit solutions that can be expressed terms of elementary functions $(\sin x,\ \cos x,\ e^x, \ldots)$. So at the end of 19th century, Poincaré inaugurates a new direction in the study and understanding of ODEs. Thanks to Poincaré perspective, the solutions began to be considered as geometric elements (orbits). Instead of finding the explicit solution of (1.1), qualitative theory studies the behavior of its solutions.

In other words, this new approach studies the dynamics of an ODE without to find an explicit expression of their solutions. The qualitative theory of differential equations leads to the description of the phase portraits of families of ODEs, introducing an equivalence relation between and the ODEs and their phase portraits. The analyze of the ODEs is done considering the description of the changes in these phase portraits which occur when there are changes in the parameters of the ODE. The phase portrait of an ODE is the union of all its orbits. Usually only a finite number of orbits of an ODE is enough to determine its phase portrait, these orbits are the singular points and some non–singular solutions. If and ODE does not depend explicitly of the parameter t it is known as autonomous ODE. In this case the invariant solutions are homeomorphic either to the straight line $\mathbb{R}$ or to the circle $\mathbb{S}^1$.

The local phase portrait a singular points can be described by important results from qualitative theory of differential equations, for instance Hartman–Grobman Theorem (HARTMAN, 1964) for hyperbolic singular points. But the complete description of global phase portraits of an ODE, even in the plane, is not a easy task. In order to solve such challenge many research have been developed recently.

Concerning about the investigation of algebraic invariant curves of ODEs in the plane we can mentioned the works (LLIBRE; MESSIAS; REINOL, 2014; LLIBRE; OLIVEIRA, 2015) and (LLIBRE; OLIVEIRA, 2018) where the authors classify planar polynomial differential systems having algebraic invariant curves of degree 2 and a Darboux invariant.

Another subject of great interesting related with invariant curves is the existence of isolated and closed solutions, known as limit cycles. In 1900, during a congress in Paris, the mathematician David Hilbert (HILBERT, 1900; HILBERT, 2000) proposed a list with 23 problems, which would guide the mathematics of the twentieth century. Among them, we highlight the sixteenth, which is divided in two parts: the first of them of interest of the algebraic geometry and the second one concerns about to determine the maximum number of limit cycles that a planar polynomial differential system of degree n can have and what the relative position between these cycles. Until recently this problem was open. Écalle (ÉCALLE, 1992) e Il'Yashenko (YASHENKO, 1991) showed that this number, denoted by $H(n)$ is finite, but proof of the result is not accessible, even for renowned mathematicians. In (LLIBRE; PEDREGAL, 2015), Llibre and Pedregal work to give an estimate for the Hilbert number $H(n)$. Such work has been improved over the lasy years. The averaging method has been used to provide lower bounds for the Hilbert number $H(n)$ see, for instance, (LLIBRE; MEREU; TEIXEIRA, 2010).

The interest on this topic extends to what we call discontinuous piecewise vector fields. The increasing interest in the theory of nonsmooth vector fields has been mainly motivated by its strong relation with Physics, Engineering, Biology, Economy, and other branches of science. In fact, their associated differential systems are very useful to model phenomena presenting abrupt switches such as electronic relays, mechanical impact, and neuronal networks, see for instance (BERNARDO *et al.*, 2008; VARIUS, 2012). The extension of the averaging theory to discontinuous piecewise vector field has been the central subject of investigation of the following works (ITIKAWA; LLIBRE; NOVAES, 2017; LLIBRE; MEREU; NOVAES, 2015; LLIBRE JAUME NOVAES, 2015; LLIBRE; NOVAES; TEIXEIRA, 2015).

The main objective of this thesis is to contribute to the investigation of the planar differential systems having algebraic invariant curves of degree 3 (from now on we only say invariant cubic) and the existence of limit cycles for some families of continuous and non–smooth piecewise differential systems defined in $\mathbb{R}^n, n \geq 2$.

The chapters of this thesis are organized as follows. In chapter 2 we classify the planar polynomial differential systems of degree 2 having an invariant cubic. We present their normal forms and draw the phase portrait in the Poincaré disc of such systems which has a Darboux invariant. The next result summarizes the investigation done in this chapter

Theorem 1.0.1. There exists 110 distinct and realizable phase portraits for planar polynomial differential systems of degree 2 having invariant cubic and a Darboux invariant. Such phase portraits are presented in Figures 1–7.

This is the main result of the preprint

J. LLIBRE, R. D. S. OLIVEIRA, C. A. B. RODRIGUES, *Quadratic systems with an invariant algebraic curve of degree* 3 *and a Darboux invariant*, Preprint 2018.

In chapter 3 using the classic averaging theory of first order for non–smooth differential systems, two classes of systems are investigated. The first class studied is the Michelson system. Such study has two parts, the investigation about the existence of periodic orbits for the continuous Michelson system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon(2d^2 - |x|), \end{aligned} \tag{1.2}$$

and the existence of periodic orbits in the discontinuous piecewise Michelson system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon(2d^2 - |x| - \operatorname{sign} x). \end{aligned} \tag{1.3}$$

The following results are proved

Theorem 1.0.2. For all $d > 0$ and $\varepsilon = \varepsilon(d) > 0$ sufficiently small the Michelson continuous piecewise linear differential system (1.2) has a periodic solution of the form

$$x(t) = -\pi d^2 + \mathscr{O}(\varepsilon), \quad y(t) = \pi d^2 \sin t + \mathscr{O}(\varepsilon), \quad z(t) = \pi d^2 \cos t + \mathscr{O}(\varepsilon).$$

Moreover this periodic solution is linearly stable.

Theorem 1.0.3. For $\varepsilon > 0$ sufficiently small the Michelson discontinuous piecewise linear differential system (1.3) satisfies the following statements.

(a) If $(-1+2d^2)\pi < 0$ then system (1.3) has two periodic solutions $(x(t,\varepsilon), r(t,\varepsilon), \theta(t,\varepsilon))$ of the form

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (x_0, r_0 \sin t, r_0 \cos t) + \mathscr{O}(\varepsilon), \tag{1.4}$$

where

$$r_0 = \frac{2\sqrt{1-a^2}}{a\sqrt{1-a^2} + \arcsin a}, \qquad x_0 = -r_0(1+a),$$

and a takes the value of the two unique zeros of the function

$$g(a) = \frac{2a^2 - 2 + \pi a\sqrt{1-a^2}d^2 + \arcsin a\left(\pi d^2 + \arcsin a - a\sqrt{1-a^2}\right)}{a\sqrt{1-a^2} + \arcsin a},$$

in the interval $(-1,1)$.

(b) If $(-1+2d^2)\pi > 0$, then system (1.3) has a periodic solution of the form (1.4) given by the unique zero of the function $g(a)$ in the interval $(-1,1)$.

These results are published in

J. LLIBRE, R. D. S. OLIVEIRA, C. A. B. RODRIGUES, *On the periodic solutions of the Michelson continuous and discontinuous piecewise linear differential system*, Computational and Applied Mathematics, 37 (2018): pp 1550–1561.

The second class studied using the classic averaging theory is

$$\dot{x} = A_0 x + \varepsilon\big(Ax + \varphi(x_1)b\big), \tag{1.5}$$

where $|\varepsilon| \neq 0$ a sufficiently small real parameter, A_0 is the $2n \times 2n$ matrix having on its principal diagonal the following 2×2 matrices

$$\begin{pmatrix} 0 & -(2k-1) \\ 2k-1 & 0 \end{pmatrix} \qquad \text{for } k = 1, \ldots, n,$$

and zeros in the complement, A is an arbitrary $2n \times 2n$ matrix and $b \in \mathbb{R}^{2n} \setminus \{0\}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is the continuous piecewise linear function

$$\varphi(x_1) = \begin{cases} -1 & \text{if } x_1 \in (-\infty, -1), \\ x_1 & \text{if } x_1 \in [-1, 1], \\ 1 & \text{if } x_1 \in (1, \infty), \end{cases} \tag{1.6}$$

where $x = (x_1, \ldots, x_m)^T$. Note that for $\varepsilon = 0$ system (1.5) becomes

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \quad \ldots \quad , \dot{x}_{2n-1} = -(2n-1)x_{2n}, \quad \dot{x}_{2n} = (2n-1)x_{2n-1}. \tag{1.7}$$

For such family we obtain the next result.

Theorem 1.0.4. For $|\varepsilon| > 0$ sufficiently small and if the conditions for applying the averaging theory of first order hold, then at most one limit cycle γ_ε of the continuous piecewise linear differential system (1.5) bifurcates from the periodic orbits of system (1.7), i.e. γ_ε tends to a periodic solution of system (1.7) when $\varepsilon \to 0$. Moreover there are systems (1.5) with $|\varepsilon| > 0$ sufficiently small having a such limit cycle.

In a similar way we consider the discontinuous piecewise linear differential systems

$$\dot{x} = A_0x + \varepsilon\big(Ax + \psi(x_1)b\big), \tag{1.8}$$

where

$$\psi(x_1) = \begin{cases} -1 & \text{if } x_1 \in (-\infty, 0), \\ 1 & \text{if } x_1 \in (0, \infty). \end{cases} \tag{1.9}$$

So for system (1.8) we have the following result

Theorem 1.0.5. For $|\varepsilon| > 0$ sufficiently small and if the conditions for applying the averaging theory of first order hold, then at most one limit cycle γ_ε of the discontinuous piecewise linear differential system (1.8) bifurcates from the periodic orbits of system (1.7). Moreover there are systems (1.8) with $|\varepsilon| > 0$ sufficiently small having a such limit cycle.

Theorems 1.0.4 and 1.0.5 are the main results of the following preprint

J. LLIBRE, R. D. S. OLIVEIRA, C. A. B. RODRIGUES, *Limit cycles for two classes of control piecewise linear differential*, arXiv:1804.08179.

Chapter 4 is dedicated to extend the averaging theory presented in chapter 3. We present the high order averaging for continuous systems and show that this tool can also be used to estimate the number of limit cycles in a big class of nonsmooth systems. Indeed the main result of the chapter concern about the existence of isolated periodic solutions of the following discontinuous nonautonomous 2π-periodic piecewise smooth differential equation

$$r'(\theta) = \sum_{i=0}^{k} \varepsilon^i F_i(\theta, r) + \varepsilon^{k+1} R(\theta, r, \varepsilon), \tag{1.10}$$

where

$$\begin{aligned} F_i(\theta, r) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) F_i^j(\theta, r),\ i = 0, 1, ..., k, \quad \text{and} \\ R(\theta, r, \varepsilon) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) R^j(\theta, r, \varepsilon). \end{aligned} \tag{1.11}$$

In this case the set of discontinuity is given by $\Sigma = (\{\theta = 0 \equiv 2\pi\} \cup \{\theta = \alpha_1\} \cup \cdots \cup \{\theta = \alpha_{n-1}\}) \cap \mathbb{S}^1 \times D$. Considering the averaged functions

$$f_i(\rho) = \frac{y_i(2\pi, \rho)}{i!}, \tag{1.12}$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}$ for $i = 1, 2, \ldots, k$, are defined recurrently by the following integral equations

$$
\begin{aligned}
y_1(\theta,\rho) = & \int_0^\theta \Big(F_1(\phi,\varphi(\phi,\rho)) + \partial F_0(\phi,\varphi(\phi,\rho))y_1(\phi,\rho)\Big)d\phi, \\
y_i(\theta,\rho) = & \; i!\int_0^\theta \Big(F_i(\phi,\varphi(\phi,\rho)) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \\
& \cdot\partial^L F_{i-l}(\phi,\varphi(\phi,\rho))\prod_{j=1}^{l} y_j(\phi,\rho)^{b_j}\Big)d\phi, \text{ for } i=2,\dots,k.
\end{aligned} \tag{1.13}
$$

and the hypothesis

(H1) For each $\rho\in D$ the solution $\varphi(\theta,\rho)$ is defined for every $\theta\in\mathbb{S}^1$, it reaches Σ only at crossing points, and it is 2π-periodic.

In such context we prove the following

Theorem 1.0.6. Assume that $(H1)$ holds and that for some $l\in\{1,2,\dots,k\}$ the functions defined in (1.12) satisfy $f_s=0$ for $s=1,2,\dots,l-1$ and $f_l\neq 0$. If there exists $\rho^*\in D$ such that $f_l(\rho^*)=0$ and $f_l'(\rho^*)\neq 0$, then for $|\varepsilon|\neq 0$ sufficiently small there exists a 2π-periodic solution $r(\theta,\varepsilon)$ of system (1.10) such that $r(0,\varepsilon)\to\rho^*$ when $\varepsilon\to 0$.

This result is in the following paper

J. LLIBRE, D. D. NOVAES, C. A. B. RODRIGUES, *Averaging theory of any order for computing limit cycles of discontinuous piecewise differential systems with many zones*, Physica D: Nonlinear Phenomena, 353/354 (2017): pp 1–10.

In chapter 5 we use the Lyapunov-Schmidt Reduction together with the averaging theory for estimate the number of limit cycles in a class of nonsmooth systems which can not be studied only using the tool developed in chapter 4. In short we define the bifurcation funcions that are given in terms of the averaged functions and we prove that their simple zeros control the number of limit cycles of system (1.10) when the hyphotesis (H1) is not valid. The main result of such study is Theorem 4.4.1 of chapter 5 and can be found in the preprint:

J. LLIBRE, D.D. NOVAES, C. A. B. RODRIGUES, *Branching of limit cyles from families of periodic solutions in piecewise differential systems*, arXiv:1804.08175.

Finally, in appendix A, we present the final version of the paper

A. C. MEREU, R. D. S. OLIVEIRA, C. A. B. RODRIGUES, *Limit cycles for a class of discontinuous piecewise generalized Kukles differential systems*, Nonlinear Dynamics, (2018): pp 1–12.

This work studies a particular discontinuous differential system, which we call the generalized Kukles polynomial differential system. The investigation of such family of systems started during my master degree (RODRIGUES, 2015) and it was improved after that. Here we present the final version.

CHAPTER

2

QUADRATIC SYSTEMS HAVING INVARIANT ALGEBRAIC CURVE OF DEGREE 3 AND DARBOUX INVARIANTS

Even after hundreds of studies on the topology of real planar quadratic vector fields the complete characterization of their phase portraits is a quite complex task. This family of systems depends on twelve parameters but, after affine transformations and time rescaling, we arrive at families with five parameters, which is still a big number of parameters. Many subclasses have been considered in order to reach such characterization. The main objetives of this chapter are to present a classification of the quadratic differential systems having invariant algebraic curves of degree 3 (invariant cubics) and to present the global phase portraits of such systems with a Darboux invariant. More precisely, using the normal forms of the quadratic system having invariant cubics we investigate which ones that admit a Darboux invariant of the form $e^{st} f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ if the cubic is the product of three straight lines $f_i = 0$ for $i = 1,2,3$, of the form $e^{st} f_1^{\lambda_1} f_2^{\lambda_2}$ if the cubic is the product of one straight line $f_1 = 0$ and an irreducible conic $f_2 = 0$, and of the form $e^{st} f_1^{\lambda_1}$ if $f_1 = 0$ is an irreducible cubic.

2.1 Basic concepts

Definition 2.1.1. A C^k vector field ($k \geq 0$ or $k = \infty$) is a C^k-map $F : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^{n+1}$ is an open subset of $\mathbb{R}^{n+1}$.

To the field F is associated the *ordinary differential equation (ODE)*

$$\dot{x} = F(t,x). \tag{2.1}$$

, where dot denotes derivative with respect to the *time* t. If $F = F(x)$ (it does not depends on t) system (2.1) is called an *autonomous system*.

A function $x : I \subset \mathbb{R} \to U$ satisfying

$$\frac{dx}{dt}(t) = F(t, x(t)) \quad \text{for all } t \in I,$$

is called a solution of the ODE (2.1). The largest interval I such that a solution is defined is called *maximal solution*.

The existence and uniqueness of a solution are determined by conditions imposed on the function F (the Existence and Uniqueness Theorem).

Definition 2.1.2. Let $x : I_{x_0} \to U$ be a maximal solution of the initial value problem

$$\begin{cases} \dot{x} = F(t,x), \\ x(t_0) = x_0. \end{cases} \tag{2.2}$$

The image of the curve $\gamma_\varphi = \{x(t);\ t \in I_{x_0}\} \subset U$ with the induced orientation by x is called of *orbit*.

Definition 2.1.3. The phase portrait of the vector field $F : U \to \mathbb{R}^2$ is the set of oriented orbits. It consists of singular points and regular orbits, oriented according to the maximal solutions that described the system.

Let be

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{2.3}$$

an autonomous system where P and Q are polynomials on the real variables x and y. We define $m = \max\{\deg P, \deg Q\}$ as the *degree* of the system. Moreover when $m = 2$ we say that system (2.3) is a *quadratic polynomial differential system* or simply a *quadratic system*.The quadratic systems appear in the modeling of many natural phenomena described in different branches of science, in biological and physical applications. Besides the applications the quadratic systems became a matter of interest for the mathematicians.

In this thesis we assume that the polynomials P and Q are coprime, otherwise system (2.3) can be reduced to a linear or constant system doing a rescaling of the time variable.

2.1.1 Invariants

A nonconstant C^1 function $H : U \to \mathbb{R}$, defined in the open and dense set $U \subset \mathbb{R}^2$ is a *first integral* of system (2.3) on U if $H(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of system (2.3) contained in U. In other words H is a first integral of system (2.3) if and only if

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0, \tag{2.4}$$

for all $(x,y) \in U$

An *invariant* of system (2.3) on the open subset U of $\mathbb{R}^2$ is a nonconstant C^1 function I in the variables x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system (1) contained in U, i.e.

$$\frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial t} = 0, \tag{2.5}$$

for all $(x, y) \in U$. In short, I is a first integral of system (2.3) depending on the time t.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of system (2.3) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2.6}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, f is a polynomial first integral. Note that if a real polynomial differential system has a complex invariant algebraic curve then it has also its conjugate. It is important to consider the complex invariant algebraic curves of the real systems because sometimes these force the real integrability of the system. More details can be found in the subsection 2.1.2 or in Chapter 8 of (DUMORTIER; LLIBRE; ARTÉS, 2006).

Remark 2.1.4. Note that if f is an algebraic invariant curve and (x_0, y_0) is a point on the curve $f = 0$ so $P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y}(x_0, y_0) = 0$, i.e, if $F = (P, Q)$ then $\langle F, \nabla f\rangle(x_0, y_0) = 0$ and hence $F(x_0, y_0) \perp \nabla f(x_0, y_0)$. Therefore if the orbit of (x_0, y_0) intersects the curve $f = 0$ then it is contained on the curve. This justify the name “invariant”.

Let $f, g \in \mathbb{C}[x, y]$ and assume that f and g are relatively prime in the ring $\mathbb{C}[x, y]$, or that $g = 1$. Then the function $\exp(f/g)$ is called a *exponential factor* of system (2.3) if for some polynomial $L \in \mathbb{C}[x, y]$ of degree at most $m-1$ we have

$$P\frac{\partial \exp(f/g)}{\partial x} + Q\frac{\partial \exp(f/g)}{\partial y} = L\exp(f/g). \tag{2.7}$$

As previously we say that L is the *cofactor* of the exponential factor $\exp(f/g)$. We observe that in the definition of exponential factor $\exp(f/g)$ if $f, g \in \mathbb{C}[x, y]$ then the exponential factor is a complex function. Again when we look for a complex exponential factor of a real polynomial system we are thinking the real polynomial system as a complex polynomial system.

2.1.2 Darboux invariants

An invariant I is called a *Darboux invariant* if it can be written into the form

$$I(x, y, t) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{st}, \tag{2.8}$$

where $f_i = 0$ are invariant algebraic curves of system (2.3) for $i = 1, \ldots p$, and F_j are exponential factors of system (2.3) for $j = 1, \ldots, q$, $\lambda_i, \mu_j \in \mathbb{C}$ and $s \in \mathbb{R} \setminus \{0\}$.

Observe that if among the invariant algebraic curves a complex conjugate pair $f = \mathrm{Re}(f)+\mathrm{Im}(f)i = 0$ and $\bar{f} = \mathrm{Re}(f)-\mathrm{Im}(f)i = 0$ occurs, then the Darboux invariant has a factor of the form $f^{\lambda}\bar{f}^{\bar{\lambda}}$, which is the real multi-valued function

$$\left((\mathrm{Re}(f))^2+(\mathrm{Im}(f))^2\right)^{\mathrm{Re}(\lambda)} e^{-2\mathrm{Im}(\lambda)\arctan(\mathrm{Im}(f)/\mathrm{Re}(f))}.$$

So if system (2.3) is real then the Darboux invariant is also real, independently of the fact of having complex invariant curves or complex exponential factors.

The next result is proved in Proposition 8.4 of (DUMORTIER; LLIBRE; ARTÉS, 2006).

Proposition 2.1.5. Suppose that $f \in \mathbb{C}[x,y]$ and let $f = f_1^{n_1}\ldots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x,y]$. Then for a polynomial differential system (2.3), $f = 0$ is an invariant algebraic curve with cofactor k_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1,\ldots,r$ with cofactor k_{f_i}. Moreover $k_f = n_1k_{f_1}+\ldots+n_rk_{f_r}$.

The next result, proved in (CHRISTOPHER; LLIBRE, 2000), explain how to obtain a Darboux invariant using the algebraic invariant curves of a polynomial differential system.

Proposition 2.1.6. Suppose that a polynomial system (2.3) of degree m admits p invariant algebraic curves $f_i = 0$ with cofactors k_i for $i = 1,...,p$, q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1,...,q$, then, if there exist λ_i and $\mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p}\lambda_ik_i+\sum_{j=1}^{q}\mu_jL_j = -s, \tag{2.9}$$

for some $s \in \mathbb{R}\backslash\{0\}$, then substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$, the real (multi-valued) function

$$f_1^{\lambda_1}\ldots f_p^{\lambda_p}\left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1}\ldots\left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}e^{st}$$

is a Darboux invariant of system (2.3).

The search of first integrals is a classic tool in order to describe the phase portraits of a 2–dimensional differential system. As usual the phase portrait of a system is the decomposition of the domain of definition of this system as union of all its orbits.

It is well known that the existence of a first integral or an a invariant for a planar differential system allow to draw its phase portrait. Here we investigate the existence of invariants of the form $f(x,y)e^{st}$, called Darboux invariants, see section 2.1.2 for details. Such invariants describe the asymptotic behavior of the solutions of the system.

Indeed let $\phi_p(t)$ be the solution of system (2.3) passing through the point $p \in \mathbb{R}^2$, defined on its maximal interval (α_p,ω_p) such that $\phi_p(0) = p$. If $\omega_p = \infty$ we define the *ω-limit set* of p as

$$\omega(p) = \{q \in \mathbb{R}^2 : \exists\{t_n\} \text{ with } t_n = \infty \text{ and } \phi_p(t_n) = q \text{ when } n = \infty\}.$$

In the same way, if $\alpha_p = -\infty$ we define the α-*limit set* of p as

$$\alpha(p) = \{q \in \mathbb{R}^2 : \exists\{t_n\} \text{ with } t_n = -\infty \text{ and } \phi_p(t_n) = q \text{ when } n = \infty\}.$$

For more details on the ω– and α–limit sets see for instance section 1.4 of (DUMORTIER; LLIBRE; ARTÉS, 2006).

The existence of a Darboux invariant of system (2.3) provides information about the ω– and α–limit sets of all orbits of system (2.3). More precisely, we have the following result, where the definition of Poincaré compactification and Poincaré disc is given in subsection 2.2. Its proof can be found in (LLIBRE; OLIVEIRA, 2015).

Proposition 2.1.7. Let $I(x,y,t) = f(x,y)e^{st}$ be a Darboux invariant of system (2.3). Let $p \in \mathbb{R}^2$ and $\phi_p(t)$ the solution of system (2.3) with maximal interval (α_p, ω_p) such that $\phi_p(0) = p$.

1. If $\omega_p = \infty$ then ω(p)$\subset \{f(x,y) = 0\} \cup \mathbb{S}^1$,
2. If $\alpha_p = -\infty$ then α(p)$\subset \{f(x,y) = 0\} \cup \mathbb{S}^1$.

Here $\mathbb{S}^1$ denotes the infinity of the Poincaré disc.

2.2 Poincaré compactification

Let $\mathscr{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ be the planar polynomial vector field of degree m associated to the polynomial differential system (2.3). The Poincaré compactified vector field $\pi(\mathscr{X})$ corresponding to $\mathscr{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows (for more details, see (DUMORTIER; LLIBRE; ARTÉS, 2006)).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3;\ y_1^2 + y_2^2 + y_3^2 = 1\}$ and $T_y\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at point y. We identify $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$ and we consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 = \mathbb{S}^2$. The map f defines two copies of $\mathscr{X}$ on $\mathbb{S}^2$, one in the southern hemisphere and the other in the northern hemisphere. Denote by $\mathscr{X}'$ the vector field $D(f \circ \mathscr{X})$ defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$, where $\mathbb{S}^1 = \{y \in \mathbb{S}^2;\ y_3 = 0\}$ is identified with the infinity of $\mathbb{R}^2$.

For extending $\mathscr{X}'$ to a vector field on $\mathbb{S}^2$, including $\mathbb{S}^1$, $\mathscr{X}$ must satisfy convenient conditions. Since the degree of $\mathscr{X}$ is m, $\pi(\mathscr{X})$ is the unique analytic extension of $y_3^{m-1}\mathscr{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1$ there is two symmetric copies of $\mathscr{X}$, and once we know the behavior of $\pi(\mathscr{X})$ near $\mathbb{S}^1$, we know the behavior of $\mathscr{X}$ in a neighborhood of the infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $\pi(\mathscr{X})$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*, and its boundary is $\mathbb{S}^1$.

Two polynomial vector fields $\mathscr{X}$ and $\mathscr{Y}$ on $\mathbb{R}^2$ are topologically equivalent if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $\pi(\mathscr{X})$ into orbits of the flow induced by $\pi(\mathscr{Y})$ preserving or not the orientation of all the orbits.

As $\mathbb{S}^2$ is a differentiable manifold, in order to compute the explicit expression of $\pi(\mathscr{X})$, we consider six local charts $U_i = \{y \in \mathbb{S}^2;\ y_i > 0\}$ and $V_i = \{y \in \mathbb{S}^2;\ y_i < 0\}$, where $i = 1,2,3$, and the diffeomorphisms $F_i : U_i = \mathbb{R}^2$ and $G_i : V_i = \mathbb{R}^2$, for $i = 1,2,3$, which are the inverses of the central projections from the tangent planes at the points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$ and $(0,0,-1)$, respectively. We denote by $z = (u,v)$ the value of $F_i(y)$ and $G_i(y)$, for any $i = 1,2,3$, therefore z means different things depending on the local charts where we are working. So after some computations $\pi(\mathscr{X})$ is given by:

$$v^m \Delta(z) \left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right), -vP\left(\frac{1}{v}, \frac{u}{v}\right) \right) \text{ in } U_1, \tag{2.10}$$

$$v^m \Delta(z) \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right), -vQ\left(\frac{u}{v}, \frac{1}{v}\right) \right) \text{ in } U_2, \tag{2.11}$$

$$\Delta(z)(P(u,v), Q(u,v)) \text{ in } U_3, \tag{2.12}$$

where $\Delta(z) = (u^2 + v^2 + 1)^{-(m-1)/2}$. The expressions for V_i's are the same as that for U_i's but multiplied by the factor $(-1)^{m-1}$. In these coordinates $v = 0$ always denotes the points of the infinity $\mathbb{S}^1$.

2.3 Invariant algebraic curves of degree 3

As mentioned in the introduction, one of the objective of this thesis is to study vector fields having invariant algebraic curves of degree 3. So it is necessary to classify all the cubics $f : \mathbb{R}^2 \to \mathbb{R}$. If $f(x,y)$ is a polynomial function of degree 3 it can be irreducible on the ring $\mathbb{R}[x,y]$ or it can be reducible and consequently written either as $f = f_1 f_2$, with f_1 an irreducible conic and f_2 a polynomial of degree one, or as $f = f_1 f_2 f_3$, with f_i has degree one, $i = 1,2,3$.

Since the cubic curves can be classified as reducible and irreducible curves (according to the polynomial defining the curve admits factorization or not), we split the obtained results in two subsections. In the first one we consider planar quadratic systems having irreducible cubics and in the second one, the reducible ones.

2.3.1 *Irreducible invariant cubics*

According to (BIX, 2006) we can classify an irreducible cubic using its *flex points*. Let p be a nonsingular point of the algebraic curve $f = 0$ and l the tangent straight line of $f = 0$ at p. The point p is called a *flex point* or *inflexion* of f if the contact between $f = 0$ and l at p is greater or equal to 3. For example the curve $y = x^3$ has a flex point at the origin.

The next results characterize all irreducible cubics, their proofs can be found in (BIX, 2006).

Proposition 2.3.1. A cubic is non–singular and irreducible and has a flex point if and only if it can be transformed with affine transformations into either

$$y^2 = x(x-1)(x-r) \qquad \text{with } r > 1,$$

or

$$y^2 = x(x^2+sx+1) \qquad \text{with } -2 < s < 2.$$

Proposition 2.3.2. A cubic is singular and irreducible if and only if it can be transformed with affine transformations into one of the forms

$$y^2 = x^3, \qquad y^2 = x^2(x+1), \qquad y^2 = x^2(x-1).$$

Moreover in (BIX, 2006) it is proved that every non-singular and irreducible curve has a flex point. So we have the complete characterization of the irreducible cubics.

2.3.2 Reducible invariant cubics

Proposition 2.3.3. A real quadratic system having an invariant conic after an affine change of coordinates can be written in one of the following forms

$$\begin{array}{rl}
\text{(real ellipse)} & \dot{x} = (A/2)(x^2+y^2-1)+2y(p+qx+ry),\\
& \dot{y} = (B/2)(x^2+y^2-1)-2x(p+qx+ry),\\
\text{(complex ellipse)} & \dot{x} = (A/2)(x^2+y^2+1)+2y(p+qx+ry),\\
& \dot{y} = (B/2)(x^2+y^2+1)-2x(p+qx+ry),\\
\text{(hyperbola)} & \dot{x} = (A/2)(x^2-y^2-1)-2y(p+qx+ry),\\
& \dot{y} = -(B/2)(x^2-y^2-1)-2x(p+qx+ry),\\
\text{(parabola)} & \dot{x} = A(y-x^2)-(p+qx+ry),\\
& \dot{y} = B(y-x^2)-2x(p+qx+ry),\\
\text{(Lotka-Volterra)} & \dot{x} = x(p_1+q_1x+r_1y)\\
& \dot{y} = y(p_2+q_2x+r_2y),\\
\text{(two parallel real lines)} & \dot{x} = x^2-1\\
& \dot{y} = Q(x,y),\\
\text{(double line)} & \dot{x} = x^2\\
& \dot{y} = Q(x,y),\\
\text{(two parallel complex lines)} & \dot{x} = x^2+1\\
& \dot{y} = Q(x,y),\\
\text{(two non-parallel complex lines)} & \dot{x} = (A/2)(x^2+y^2)+(C/2)x+2y(p+qx+ry),\\
& \dot{y} = (B/2)(x^2+y^2)+(C/2)y-2x(p+qx+ry).
\end{array}$$

Here $Q(x,y)$ denotes an arbitrary polynomial of degree 2.

The proof of the previous result can be found in (CAIRÓ; LLIBRE, 2002), except to the normal form of the system with a parabola that is proved in (LLIBRE; MESSIAS; REINOL, 2014). The next result is due to Christopher, Llibre, Pantazi, Zhang and Zholadek, see (CHRISTOPHER, 1994; CHRISTOPHER *et al.*, 2002; ŻOŁĄDEK, 1995). An algebraic proof of it also can be found in (CHRISTOPHER *et al.*, 2002).

Theorem 2.3.4. Let $f_i = 0$ for $i = 1,\dots,q$ be q irreducible algebraic curves in $\mathbb{C}^2$, and let $k = \sum_{i=1}^{q} \deg f_i$. We assume

(i) there are no points at which f_i and its first derivatives all vanish,

(ii) the highest order terms of f_i have no repeated factors,

(iii) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(iv) no two curves have a common factor in their highest order terms, then any polynomial vector field X of degree m tangent to all $f_i = 0$ is of the form describe bellow.

(a) If $m > k-1$ then

$$X = Y\left(\prod_{i=1}^{q} f_i\right) + \sum_{i=1}^{q}\left(\prod_{j=1, j\neq i}^{q} f_j\right) X_{f_i}, \tag{2.13}$$

where $X_{f_i} = (-\partial f_i/\partial y, \partial f_i/\partial x)$ is a Hamiltonian vector field, the h_i are polynomials of degree $\leq m-k+1$ and Y is a polynomial vector field of degree $\leq m-k$.

(b) If $m = k-1$ then

$$X = \sum_{i=1}^{q} \alpha_i \left(\prod_{j=1, j\neq i}^{q} f_j\right) X_{f_i}, \tag{2.14}$$

where $\alpha_i \in \mathbb{C}$. In this case a Darboux first integral exists.

(c) If $m < k-1$ then $X \equiv 0$.

Theorem 2.3.5. [Lemma 7 of (CHRISTOPHER *et al.*, 2002)] Assume that $f = 0$ and $g = 0$ are different irreducible invariant algebraic curves of system (2.3) of degree m, and that they satisfy conditions (i) and (iii) of Theorem 2.3.4. If $gcd(f_x, f_y) = 1$ and $gcd(g_x, g_y) = 1$, then system (2.3) has the normal form

$$\dot{x} = Afg - h_1 f_y g - h_2 f g_y \qquad \dot{y} = Bfg + h_1 f_x g + h_2 f g_x, \tag{2.15}$$

where A, B and h_j are polynomials, for $i = 1, 2$.

2.4 Classification of quadratic systems having irreducible invariant cubics

2.4.1 Normal forms

The objective of this subsection is to describe the differential systems as in Proposition 2.3.3 and Theorems 2.3.4, 2.3.5, in terms of normal forms. In this sense we present the first result of this thesis

Theorem 2.4.1. Each quadratic system admitting an irreducible invariant cubic after an affine change of coordinates and a rescaling of the time variable can be written as one of the following systems.

$$
\begin{aligned}
(i) \quad & \dot{x} = 2(ax + by + dxy + cx^2), \\
& \dot{y} = 3(ay + bx^2 + cxy + dy^2), \\
(ii) \quad & \dot{x} = 2(ax + by + (3b - 2c)xy + ax^2), \\
& \dot{y} = 2bx + 2ay + 2cx^2 + 3axy + (9b - 6c)y^2, \\
(iii) \quad & \dot{x} = 2(ax - by + (3b + 2c)xy - ax^2), \\
& \dot{y} = 2bx + 2ay + 2cx^2 - 3axy + (9b + 6c)y^2, \\
(iv) \quad & \dot{x} = 2y(a + bx), \\
& \dot{y} = ar - 2(ar + a + br)x + (3a + br + b)x^2 + 3by^2, \\
(v) \quad & \dot{x} = 2y(b + cx), \\
& \dot{y} = b + 2(br - c)x + (3b - cr)x^2 + 3cy^2.
\end{aligned}
$$

Proof. First of all we write

$$P(x,y) = a_{00} + a_{01}y + a_{02}y^2 + a_{10}x + a_{11}xy + a_{20}x^2,$$

$$Q(x,y) = b_{00} + b_{01}y + b_{02}y^2 + b_{10}x + b_{11}xy + b_{20}x^2.$$

If a quadratic system (2.3) has a singular irreducible invariant cubic $f(x,y) = 0$ by Proposition 2.3.2 the function f can be written as $f(x,y) = y^2 - x^3$ or $f(x,y) = y^2 - x^2(x+1)$ or $f(x,y) = y^2 - x^2(x-1)$. The curve $f(x,y) = y^2 - x^3 = 0$ is an invariant cubic for system (2.3) if and only if equation (2.6) is satisfied. The solution of this equation in terms of the parameters of the system is

$$a_{00} = a_{02} = b_{00} = b_{10} = 0, \quad b_{01} = 3a_{10}/2, \quad b_{02} = 3a_{11}/2, \quad b_{11} = 3a_{20}/2, \quad b_{20} = 3a_{01}/2.$$

So the cofactor of f is $K = 3(a_{10} + a_{20}x + a_{11}y)$. Doing $a_{10} = a$, $a_{20} = b$, $a_{01} = c$, $a_{11} = d$ and a rescaling of the time we obtain system (i) of Theorem 2.4.1.

When $f(x,y) = y^2 - x^2(x \pm 1)$ we obtain the normal forms given in (ii) and (iii) of the theorem following similar steps.

Now if a quadratic system (2.3) has an invariant non–singular irreducible cubic $f(x,y) = 0$ then by Proposition 2.3.1 we can write $f(x,y) = y^2 - x(x-1)(x-r)$ with $r > 1$ or $f(x,y) = y^2 - x(x^2 + sx + 1)$ with $-2 < s < 2$. In the first case solving equation (2.6) we obtain three solution but fixing $r > 1$ only one solution can hold $a_{00} = a_{02} = a_{10} = a_{20} = b_{01} = b_{11} = 0,\ b_{00} = a_{01}r/2,\ b_{02} = 3a_{11}/2,\ b_{10} = -a_{01}(r+1) - a_{11}r,\ b_{20} = (3a_{01} + a_{11}r + a_{11})/2$. It corresponds to system (iv) of Theorem 2.4.1.

For $f(x,y) = y^2 - x(x^2 + sx + 1)$ we obtain only one solution corresponding to system (v) of the theorem. □

2.4.2 Darboux invariants and phase portrait on the Poincaré disc

Using the normal forms described in Theorem 2.4.1 we investigate when these systems admit a Darboux invariant of the form $I(x,y,t) = e^{st} f(x,y)$.

Theorem 2.4.2. Each quadratic system admitting an irreducible invariant cubic having a Darboux invariant can be written after an affine change of coordinates and a rescaling of the time variable as

$$\dot{x} = x + y, \qquad \dot{y} = \frac{3}{2}y + x^2. \tag{2.16}$$

In this case $y^2 = x^3$ is the invariant algebraic curve and the Darboux invariant is given by $I_1(x,y,t) = e^{-6t}(y^2 - x^3)$. The global phase portrait of such system is given in Figure 1.

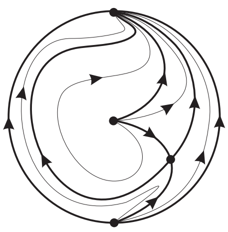

Figure 1 – Phase portrait of system (2.16).

Proof. First of all is easy to see that the cofactor K of f in systems $(ii) - (v)$ of Theorem 2.4.1 has no constant terms. Then equation (2.9) becomes $\lambda K + s = 0$ which never holds if $s \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore we conclude that systems $(ii) - (v)$ do not admit a Darboux invariant of such form.

Now considering system (i) of Theorem 2.4.1 we have $f(x,y) = y^2 - x^3 = 0$ as invariant curve with cofactor $K = 6(a + cx + dy)$. In this case the solution of equation (2.9) is given by $\{c = 0,\ d = 0,\ s = -6a\lambda\}$. Taking $\lambda = -s/(6a)$ we obtain the system

$$\dot{x} = 2(ax + by), \qquad \dot{y} = 3(ay + bx^2),$$

with Darboux invariant $I_1(x,y,t) = e^{-6at}(y^2 - x^3)$.

The normal form described in Theorem 2.4.2 is obtained doing the following change of coordinates and rescaling of the time $x = \frac{2a^2}{3b^2}X$, $y = \frac{2a^3}{3b^3}Y$, $t = \frac{1}{2a}T$.

Now it remains to study the phase portrait of system (2.16). This system has two singular points, namely $z_1 = (0,0)$ yperbolic unstable node, and $z_2 = (3/2,-3/2)$ a hyperbolic saddle. Applying the Poincaré compactification in the local chart U_1 and on the line $v = 0$ the compactified system has no singular points. However in the local chart U_2 the origin $(0,0)$ is a nilpotent singularity. With the notation of Theorem 3.5 of (DUMORTIER; LLIBRE; ARTÉS, 2006) the compactified system has $F(u) = -u^5 - (3/2)u^6$ and $G(u) = -4u^2 - (7/2)u^3$. Hence the origin of U_2 is a nilpotent stable node. By the previous statements it follows that the phase portrait of system (2.16) is the one described in Figure 1. □

2.5 Classification of quadratic systems having reducible invariant cubics

2.5.1 Normal forms

Each reducible cubic can be written as the product of two polynomials one of degree two and the other of degree one (i.e, a conic and a straight line respectively). The conics can be classified in ellipses (E), complex ellipses (CE), hyperbolas (H), parabolas (P), two real straight lines intersecting in a point , two real parallel straight lines (PL), one double invariant real straight line (DL), two complex straight lines intersecting in a real point (p), and two complex parallel straight lines (CL). So the normal forms of the reducible cubics, except to an affine transformation, are

(E) $(x^2+y^2-1)(ax+by+c) = 0$,

(CE) $(x^2+y^2+1)(ax+by+c) = 0$,

(H) $(x^2-y^2-1)(ax+by+c) = 0$,

(P) $(y-x^2)(ax+by+c) = 0$,

(LV) $xy(ax+by+c) = 0$,

(PL) $(x^2-1)(ax+by+c) = 0$,

(DL) $x^2(ax+by+c) = 0$,

(CL) $(x^2+1)(ax+by+c) = 0$,

(p) $(x^2+y^2)(ax+by+c) = 0$.

We shall say that a quadratic system is of type (E) if it has a real ellipse and a straight line as invariant irreducible algebraic curves; of type (CE) if it has a complex ellipse and a straight line as invariant irreducible algebraic curves, and respectively with all the nine types of conics described above.

Again we first classify the systems with respect to their invariant cubics.

Theorem 2.5.1. If a quadratic system (2.3) has a reducible invariant cubic then it can be written, after an affine change of coordinates, into one of the following forms

$$
\begin{array}{rl}
(CE) & \dot{x} = -(x^2+y^2+1) - 2\alpha_1\, y(y+ax+c),\\
& \dot{y} = a(x^2+y^2+1) + 2\alpha_1\, x(y+ax+c),\\
(E.1) & \dot{x} = -(x^2+y^2-1) - 2\alpha_1 y(y+ax+c),\\
& \dot{y} = a(x^2+y^2-1) + 2\alpha_1 x(y+ax+c),\\
(E.2) & \dot{x} = (\beta_1/2)(x^2+y^2-1) - y(\beta_2\, y - \alpha_2\, x + c\beta_2),\\
& \dot{y} = (y+c)(\alpha_2\, y + \beta_2 c x + \alpha_2), \qquad \text{with } \alpha_2(c+1) = 0,\\
(H.1) & \dot{x} = (\beta_1/2)(x^2-y^2-1) + \beta_2\, y(y+c),\\
& \dot{y} = \beta_2\, y(y+c),\\
(H.2) & \dot{x} = (x+c)(\alpha_2 x + \gamma_2 y + \alpha_2),\\
& \dot{y} = -(\gamma_1/2)(x^2-y^2-1) + x(\gamma_2 x + \alpha_2 y + c\gamma_2), \qquad \text{with } \alpha_2(c+1) = 0,\\
(H.3) & \dot{x} = (A/2)(x^2-y^2-1) - y(\alpha - c\beta + x(\beta - c\alpha) + y(\gamma - c\alpha)),\\
& \dot{y} = (A/2)(x^2-y^2-1) - x(\alpha - c\beta + \beta x + y(\gamma - c\alpha)) + c\alpha(y^2+1), \ \text{with } c(\gamma+\beta) = 0,\\
(H.4) & \dot{x} = (A/2)(x^2-y^2-1) + y(a\alpha - \beta\sqrt{d} + x(a\beta - \alpha\sqrt{d}) + \beta y),\\
& \dot{y} = (-Aa/2)(x^2-y^2-1) + x(a\alpha - \beta\sqrt{d} + a\beta x + \beta y) - \alpha\sqrt{d}(y^2+1), \text{ with } d = a^2-1,\\
(H.5) & \dot{x} = -(x^2-y^2-1) + 2\alpha_1 y(y+ax+c),\\
& \dot{y} = a(x^2-y^2-1) + 2\alpha_1 x(y+ax+c), \qquad \text{with } c^2 \neq a^2-1,\\
(P.1) & \dot{x} = x(\alpha_2 + \beta_2 x + \gamma_2 y),\\
& \dot{y} = \alpha_1(y-x^2) + 2\alpha_2 x^2 + 2y(\beta_2 x + \gamma_2 y),\\
(P.2) & \dot{x} = -\beta_1(y-x^2) + y(\beta_2 + \gamma_2 x) + (\alpha_2 + \gamma_2 c)x + c\beta_2,\\
& \dot{y} = 2(y+c)(\alpha_2 + \beta_2 x + \gamma_2 y), \qquad \text{with } c\,\alpha_2 = 0,\\
(P.3) & \dot{x} = -(y-x^2) - \alpha(y+ax+c),\\
& \dot{y} = a(y-x^2) - 2\alpha x(y+ax+c), \qquad \text{with } c \neq a^2/4,\\
(LV.1) & \dot{x} = x(\alpha + ry + \beta x),\\
& \dot{y} = y(\alpha + (r-q+\beta)y + qx),\\
(LV.2) & \dot{x} = x(p + qx + ry),\\
& \dot{y} = y(y+c), \qquad \text{with } c(c+1) = 0,
\end{array}
$$

$$
\begin{aligned}
(LV.3) \quad & \dot{x} = -x(y+\alpha(y+ax+c)),\\
& \dot{y} = y(ax+\beta(y+ax+c)), \qquad \text{with } ac\neq 0,\\
(RPL) \quad & \dot{x} = x^2-1,\\
& \dot{y} = y(\alpha+\beta x+\gamma y),\\
(DL) \quad & \dot{x} = x^2,\\
& \dot{y} = y(\alpha+\beta x+\gamma y),\\
(CPL) \quad & \dot{x} = x^2+1,\\
& \dot{y} = y(\alpha+\beta x+\gamma y),\\
(p.1) \quad & \dot{x} = (\beta/2)(x^2+y^2)-\beta_3 y^2+x(\alpha_3+\gamma_3 y),\\
& \dot{y} = y(\alpha_3+\beta_3 x+\gamma_3 y),\\
(p.2) \quad & \dot{x} = -(x^2+y^2)+(\beta x-\alpha y)(y+ax+c),\\
& \dot{y} = a(x^2+y^2)+(\beta y+\alpha x)(y+ax+c), \text{with } c\neq 0,
\end{aligned}
$$

where $a,c,A,p,q,r,\alpha,\beta,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1$ and γ_2 are the parameters of the system.

Proof. The proof is done according to the conic that appears in the expression of the reducible cubic.

Systems of type (E)

If system (2.3) has an invariant cubic of the form $f(x,y)=f_1(x,y)f_2(x,y)$ with $f_1=x^2+y^2-1$ and $f_2=ax+by+c$, then applying a rotation we can assume $b=1$. Therefore it follows from Proposition 2.1.5 that f_j is an invariant curve with cofactor $k_j=\alpha_j+\beta_j x+\gamma_j y$, $j=1,2$. Consider two cases: $a=0$ and $a\neq 0$.

If $a=0$ then using equation (2.6) we have $Q=k_2f_2$ and $P=(k_1f_1-2yk_2f_2)/(2x)$. As P is a polynomial the parameters of the system must satisfy on the of following conditions

$$
\begin{aligned}
s_1 &= \{c=-1,\ \alpha_1=0,\ \gamma_1=2\alpha_2,\ \gamma_2=\alpha_2\},\\
s_2 &= \{c=1,\ \alpha_1=0,\ \gamma_1=-2\alpha_2,\ \gamma_2=-\alpha_2\},\\
s_3 &= \{\alpha_1=0,\ \gamma_1=0,\ \gamma_2=0\}.
\end{aligned}
$$

Moreover the solutions s_1 and s_2 provide equivalent systems, and we can summarize the solutions s_1 and s_3 writing the system

$$
\begin{aligned}
\dot{x} = &\quad (\beta_1/2)(x^2+y^2-1)-y(\beta_2 y-\alpha_2 x+c\beta_2),\\
\dot{y} = &\quad (y+c)(\alpha_2 y+\beta_2 c x+\alpha_2),
\end{aligned} \tag{2.17}
$$

with $\alpha_2(c+1)=0$. This is exactly system $(E.1)$ of Theorem 2.5.1.

When $a \neq 0$ we check when the hypotheses of Theorem 2.3.4 are satisfied. Clearly f_1 and f_2 satisfies $(i),(ii)$ and (iv). Condition (iii) is not satisfied when $c^2 = a^2+1$ because the line $f_2 = 0$ is tangent to the real ellipse $f_1 = 0$. Indeed if the straight line $f_2 = y+ax+c = 0$ is tangent to the real ellipse $f_1 = x^2+y^2-1 = 0$ at the point (x_0, y_0), then their gradients are parallel in such point, what means that $x_0 - ay_0 = 0$. Replacing $y_0 = x_0/a$ in the ellipse we conclude that $x_0 = \pm a/\sqrt{a^2+1}$. From $f_2 = 0$ we get $c = \mp\sqrt{a^2+1}$. Therefore the condition for the tangency is $c^2 = a^2+1$. In this case applying a rotation we can get $f_2 = y-1$. Again we are in system (2.17) with $c = -1$.

Now assuming $c^2 \neq a^2+1$ it follows from Theorem 2.3.4 that our system is given by

$$\dot{x} = -\alpha_2(x^2+y^2-1) - 2\alpha_1 y(y+ax+c), \quad \dot{y} = a\alpha_2(x^2+y^2-1) + 2\alpha_1 x(y+ax+c), \tag{2.18}$$

where α_1, $\alpha_2 \in \mathbb{C}$ and a, $c \in \mathbb{R}$. As we are looking for a real system, then α_1, $\alpha_2 \in \mathbb{R}$, and doing a rescaling of the time we can assume $\alpha_2 = 1$. Note that system (2.18) is exactly system $(E.2)$ of Theorem 2.5.1.

Systems of type (CE)

In this case we can follow the same steps applied previously. If system (2.3) has an invariant cubic of the form $f = f_1 f_2$ with $f_1 = x^2+y^2+1$ and $f_2 = ax+by+c$ we suppose, without loss of generality, $b = 1$. Since the coefficients a, b and c are real numbers the straight line $f_2 = 0$ cannot be tangent to the complex ellipse $f_1 = 0$. So we get

$$\dot{x} = -\alpha_2(x^2+y^2+1) - 2\alpha_1 y(y+ax+c), \quad \dot{y} = a\,\alpha_2(x^2+y^2+1) + 2\alpha_1 x(y+ax+c), \tag{2.19}$$

where α_1, $\alpha_2 \in \mathbb{C}$ and a, $c \in \mathbb{R}$. Applying a rescaling we have $\alpha_2 = 1$ in (2.19), and we get the normal form for the systems of type (CE).

Systems of type (H)

Let $f_1 = x^2-y^2-1$ and $f_2 = ax+by+c$ be two real algebraic invariant curves of system (2.3), so $a^2+b^2 \neq 0$. Proceeding as before if $a = 0$ then we can assume $b = 1$ and the system can be written in the form

$$\dot{x} = (\beta_1/2)(x^2-y^2-1) + \beta_2 y(y+c), \quad \dot{y} = \beta_2 y(y+c), \tag{2.20}$$

with $\beta_1\beta_2 \neq 0$. This is system $(H.1)$ of Theorem 2.5.1.

If $a \neq 0$ and $b = 0$ we take $a = 1$ and system (2.3) satisfies $P = k_2 f_2$ and $2yQ = 2xP - k_1 f_1$, where $k_j = \alpha_j + \beta_j x + \gamma_j y$, for $j = 1, 2$. Since Q is a polynomial in the parameters of the system it must satisfy one of the following conditions

$$s_1 = \{c = -1,\ \alpha_1 = 0,\ \beta_1 = 2\alpha_2,\ \beta_2 = \alpha_2\},$$
$$s_2 = \{c = 1,\ \alpha_1 = 0,\ \beta_1 = -2\alpha_2,\ \beta_2 = -\alpha_2\},$$
$$s_3 = \{\alpha_1 = 0,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = 0\}.$$

Applying the change of coordinates $x = -X, y = Y$ we conclude that case s_1 and s_2 provide equivalent systems. Moreover we can summarize solutions s_1 and s_3 in the unique system

$$\dot{x} = (x+c)(\alpha_2 x + \gamma_2 y + \alpha_2), \quad \dot{y} = -(\gamma_1/2)(x^2 - y^2 - 1) + x(\gamma_2 x + \alpha_2 y + c\gamma_2), \tag{2.21}$$

with $\alpha_2(c+1) = 0$. System (2.21) corresponds to system $(H.2)$ of Theorem 2.5.1.

If $ab \neq 0$ we assume $b = 1$ and consider three cases, according to the conditions of Theorem 2.3.4. Note that condition (i) of Theorem 2.3.4 holds because $\nabla f_1(x,y) = (2x, -2y)$ and $\nabla f_2(x,y) = (a,1)$, where ∇ indicates the gradient. Condition (ii) also holds. However condition (iv) is not verified when $a^2 - 1 = 0$. Indeed in this case $f_1 = (x+y)(x-y) - 1$ and $f_2 = (y \pm x) + c$. Condition (iii) does not hold when $c^2 = a^2 - 1$ since the straight line $f_2 = y + ax + c = 0$ is tangent to the hyperbola. The proof of this last statement can be done analogously as for the systems of type (E). Hence when $a^2 - 1 = 0$ or $c^2 = a^2 - 1$ Theorem 2.3.4 does not hold and we split the study of systems of type (H) for $ab \neq 0$ in three cases: $a^2 - 1 = 0$, $c^2 = a^2 - 1$ and $(a^2 - 1)(c^2 - a^2 + 1) \neq 0$.

For the first two cases we apply Propositions 2.1.5 and 2.3.3 to conclude that f_1 is an algebraic invariant curve of a quadratic system (2.3) and it can be written as

$$\dot{x} = (A/2)(x^2 - y^2 - 1) - 2y(p + qx + ry), \quad \dot{y} = -(B/2)(x^2 - y^2 - 1) - 2x(p + qx + ry), \tag{2.22}$$

where $A, B, p, q, r \in \mathbb{R}$. Fixing the cofactor of $f_2 = 0$ as $k_2 = \alpha + \beta x + \gamma y$, where $\alpha, \beta, \gamma \in \mathbb{R}$ and using system (2.22) we solve (2.6). First considering $a = -1$ (the case $a = 1$ is analogous except by a reflection) equation (2.6) has two possible solutions

$$s_1 = \{B = -A,\, c = 0, p = \alpha/2,\, q = \beta/2,\, r = \gamma/2\},$$
$$s_2 = \{B = -A + 2c\alpha,\, p = (\alpha c - \beta)/2,\, q = (\beta - c\alpha)/2, r = -(\beta + c\alpha)/2,\, \gamma = -\beta\}.$$

Using the two above solutions we get the system

$$\begin{aligned} \dot{x} = &\quad (A/2)(x^2 - y^2 - 1) - y(\alpha - c\beta + x(\beta - c\alpha) + y(\gamma - c\alpha)), \\ \dot{y} = &\quad (A/2)(x^2 - y^2 - 1) - x(\alpha - c\beta + \beta x + y(\gamma - c\alpha)) + c\alpha(y^2 + 1), \end{aligned}$$

with $c(\gamma + \beta) = 0$. This is system $(H.3)$ of Theorem 2.5.1.

Now considering $c^2 = a^2 - 1$ we investigate the conditions that must be satisfied by the parameters of system (2.22) in order that $f_2 = y + ax \pm \sqrt{a^2 - 1}$ be an invariant curve. Without loss of generality we can assume $c = \sqrt{a^2 - 1}$. Equation (2.6) has one solution, namely

$$B = aA - 2\alpha\sqrt{d}, p = (\beta\sqrt{d} - a\alpha)/2, r = -\beta/2, q = (\alpha\sqrt{d} - a\beta)/2, \gamma = a\beta - \alpha\sqrt{d},$$

where $d=a^2-1$. Replacing it in (2.22) we get

$$
\begin{aligned}
\dot{x} = &\quad (A/2)(x^2-y^2-1)+y(a\alpha-\beta\sqrt{d}+x(a\beta-\alpha\sqrt{d})+\beta y),\\
\dot{y} = &\quad (-Aa/2)(x^2-y^2-1)+x(a\alpha-\beta\sqrt{d}+a\beta x+\beta y)-\alpha\sqrt{d}(y^2+1),
\end{aligned}
\tag{2.23}
$$

where $d=a^2-1$, and this systems corresponds to system $(H.4)$ of Theorem 2.5.1.

Finally if $(a^2-1)(c^2-a^2+1)\neq 0$ applying Theorem 2.3.4 we obtain the system

$$
\begin{aligned}
\dot{x} = &\quad -\alpha_2(x^2-y^2-1)+2\alpha_1 y(y+ax+c),\\
\dot{y} = &\quad a\alpha_2(x^2-y^2-1)+2\alpha_1 x(y+ax+c),
\end{aligned}
\tag{2.24}
$$

which is system $(H.5)$ of Theorem 2.5.1.

Systems of type (P)

Let $f=(y-x^2)(ax+by+c)=0$ be an invariant cubic of system (2.3). When $b=0$ we can assume $f=x(y-x^2)$. Indeed if $b=0$ we take $a=1$ and do the change of coordinates $x=X-c$, $y=Y-2cX+c^2$. Using that $f_2=x=0$ is an invariant straight line we have $P=k_2f_2$ with $k_2=\alpha_2+\beta_2x+\gamma_2y$, and a quadratic system (2.3) can be written as

$$
\dot{x}=x(\alpha_2+\beta_2x+\gamma_2y),\quad \dot{y}=\alpha_1(y-x^2)+2\alpha_2x^2+2y(\beta_2x+\gamma_2y). \tag{2.25}
$$

If $b\neq 0$ and $a=0$ we can take $b=1$ and proceed as in systems of type (H) and (E), then we get the system

$$
\dot{x}=-\beta_1(y-x^2)+y(\beta_2+\gamma_2x)+(\alpha_2+\gamma_2c)x+c\beta_2,\quad \dot{y}=2(y+c)(\alpha_2+\beta_2x+\gamma_2y), \tag{2.26}
$$

with $c\,\alpha_2=0$. Observe that when $c=0$ the invariant line is $y=0$ and when $\alpha_2=0$ it is $y+c=0$.

If $ab\neq 0$ and $f_2=y\pm ax+a^2/4$, $f_2=0$ is tangent to the parabola. In this case we can assume $f_2=y+ax+a^2/4$ (the other case is a reflection). Applying the change of coordinates $x=-X-a/2$ and $y=Y+aX+a^2/4$ the cubic $f=(y-x^2)(y+ax+a^2/4)$ becomes $f=(Y-X^2)Y$, which already has been studied above. Indeed it corresponds to system (2.25) with $c=0$.

Otherwise there is no tangency between the straight line and the parabola, and we apply Theorem 2.3.4 to get the differential system

$$
\dot{x}=-(y-x^2)-\alpha(y+ax+c),\quad \dot{y}=a(y-x^2)-2\alpha x(y+ax+c). \tag{2.27}
$$

Systems (2.25), (2.26) and (2.27) correspond to systems $(P.1)$, $(P.2)$ and $(P.3)$ of Theorem 2.5.1, respectively.

Systems of type (LV)

In this case $f = xy(ax+by+c) = 0$ is the invariant curve and except by a rotation we can assume $b = 1$. We consider different cases according to $ac = 0$ or $ac \neq 0$. Note that if $c = 0$ hypothesis (iii) of Theorem 2.3.4 is not valid, whereas $a = 0$ breaks the hypothesis (iv).

When $c = 0$ and $a \neq 0$, doing the change of coordinates $x = -\frac{Y}{\sqrt[3]{a^2}}$, $y = \sqrt[3]{a}X$ the cubic becomes $F = XY(Y-X)$. So using Proposition 2.3.3 the differential system can be written as

$$\dot{x} = x(p_1 + q_1 x + r_1 y) \quad \dot{y} = y(p_2 + q_2 x + r_2 y). \tag{2.28}$$

If (2.28) has $f_3 = y - x$ as an invariant curve with cofactor $k = \alpha + \beta x + \gamma y$, then equation (2.6) must be satisfied. Solving it we get

$$s_1 = \{p_2 = \alpha, r_2 = \beta - q_2 + r_1, q_1 = \beta, p_1 = \alpha, \gamma = \beta - q_2 + r_1\}.$$

Replacing in (2.28) and writing $q = q_2$, $r = r_1$ we obtain system $(LV.1)$ of Theorem 2.5.1.

Now if $c = a = 0$ then $f_2 = y = 0$ is a double line, and it is not difficult to see that we can write the system as

$$\dot{x} = x(p + qx + ry), \quad \dot{y} = y^2. \tag{2.29}$$

Finally, when $a = 0$ and $c \neq 0$, doing the change of coordinates $x = X/c^2$, $y = cY - c$ the cubic $f = 0$ becomes $F = XY(Y-1)$. So without loss of generality we can work with $f_3 = y - 1$. Again the idea is to write the system as in (2.28), and see what are the conditions in order that $f_3 = 0$ to be an invariant curve for such system. Solving equation (2.6) and replacing the solutions in (2.28) we get

$$\dot{x} = x(p + qx + ry), \quad \dot{y} = y(y-1). \tag{2.30}$$

Systems (2.29) and (2.30) can be summarized as

$$\dot{x} = x(p + qx + ry), \quad \dot{y} = y(y + c),$$

with $c = 0$ or $c = -1$. This is exactly system $(LV.2)$ of Theorem 2.5.1.

In the last case, $ac \neq 0$ the invariant cubic is $f = xy(y + ax + c) = 0$ and by the geometry to the curves we can assume $a < 0$ and $c < 0$. Applying Theorem 2.3.4 we get the system

$$\dot{x} = -\alpha_2 x(y + ax + c) - \alpha_3 xy, \quad \dot{y} = \alpha_1 y(y + ax + c) + a\alpha_3 xy.$$

Note that we can take $\alpha_3 = 1$. Doing $\alpha = \alpha_2$, $\beta = \alpha_1$ we obtain system $(LV.3)$ of Theorem 2.5.1.

Systems of type (RPL)

Here the invariant cubic is $f = f_1 f_2 f_3 = 0$ where $f_1 = x + 1$, $f_2 = x - 1$ and $f_3 = ax + by + c$. When $b = 0$ we apply Proposition 2.3.3 (case *(RPL)*), then it is easy to see that the corresponding normal form has one additional invariant curve $f_3 = 0$ as invariant straight line if and only if it is a multiple of f_1 or f_2. However we cannot consider any of these cases because if the system has f_2 as an invariant double straight line for example, then there would be a change of coordinates so that the system would be written as

$$\dot{x} = (x-1)(x+1)^2, \quad \dot{y} = Q(x,y),$$

then having degree 3 instead of 2.

When $b \neq 0$ we can fix $b = 1$. In this case the cubic $f = (x^2-1)(y+ax+c) = 0$ can be reduced to $F = y(x^2-1)$ by change of coordinates $x = X, y = Y - aX - c$. If the quadratic differential system (2.3) has the invariant curve $f = y(x^2-1) = 0$, then $f_1 = 0$ and $f_2 = 0$ are invariant curves and by Proposition 2.3.3 such system can be written as

$$\dot{x} = x^2 - 1, \quad \dot{y} = Q(x,y), \tag{2.31}$$

where $Q(x,y)$ is an arbitrary polynomial of degree 2. Imposing that $f_3 = y = 0$ is an additional invariant curve with cofactor $k_3 = \alpha + \beta x + \gamma y$, the above system must satisfy $Q(x,y) = y(\alpha + \beta x + \gamma y)$. This expression justify the normal form given in (RPL) of Theorem 2.5.1.

Systems of type (DL)

These systems have a double straight line as invariant curve which can be taken as $f_1 = x$. We write $f_2 = ax + by + c$ and use the normal form of a system having $f = f_1 f_2 = 0$ as an invariant cubic. For such normal form, if $b = 0$ then $f_2 = 0$ is an invariant straight line if and only if $c = 0$ but in this case the system cannot have a triple invariant straight line.

If $b \neq 0$ we can take $b = 1$ and $f = x^2(y+ax+c)$. Doing the change $x = X, y = Y - aX - c$ the function f can be written as $F = X^2 Y$. Hence it is enough to consider $f_2 = y$. By Proposition 2.3.3 a quadratic system (2.3) can be written as

$$\dot{x} = x^2, \quad \dot{y} = Q(x,y),$$

where $Q(x,y)$ is an arbitrary polynomial of degree 2. Imposing that $f_2 = 0$ is an additional invariant curve with cofactor $k_2 = \alpha + \beta x + \gamma y$, we conclude that $Q(x,y) = y(\alpha + \beta x + \gamma y)$. This expression justify the normal form given in (DL) of Theorem 2.5.1.

Systems of type (CPL)

The proof for this case is analogous to the case (DL) so we will omit some details. In short the cubic is given by $f = f_1 f_2 f_3 = 0$ where $f_1 = x + i$, $f_2 = x - i$ and $f_3 = ax + by + c$. In order to $f_3 = 0$ to be an invariant curve with $b = 0$ it is necessary that $c = \pm i$. So $b \neq 0$ and we assume $b = 1$. This reduce f to the cubic $F = y(x^2+1)$ and then we get the normal form (CPL) described in Theorem 2.5.1.

Systems of type (p)

In this case the cubic is given by $f=(x^2+y^2)(ax+by+c)=0$ and except by a rotation we can assume $b=1$. When $c=0$ the three curves intersect at the same point and the conditions of Theorem 2.3.4 are not satisfied. But if $c=0$ doing the change of coordinates

$$x=-\frac{X}{\sqrt[3]{(a^2+1)^2}}+\frac{aY}{\sqrt[3]{(a^2+1)^2}}, \quad y=\frac{aX}{\sqrt[3]{(a^2+1)^2}}+\frac{Y}{\sqrt[3]{(a^2+1)^2}},$$

the cubic $f=(x^2+y^2)(y+ax)=0$ is reduced to $f=Y(X^2+Y^2)$. Now using that system (2.3) has $f_3=y=0$ as a third invariant curve it follows that $Q(x,y)=k_3f_3$ where $k_3=\alpha_3+\beta_3x+\gamma_3y$ is the cofactor of f_3. Moreover $f_1f_2=0$ is also an invariant curve then we must have

$$2xP(x,y)+2yQ(x,y)=k(x,y)(x^2+y^2),$$

with $k(x,y)=\alpha+\beta x+\gamma y$ being the sum of the cofactors of f_1 and f_2. So a quadratic system (2.3) can be written as

$$\dot{x}=(\beta/2)(x^2+y^2)-\beta_3y^2+x(\alpha_3+\gamma_3y), \quad \dot{y}=y(\alpha_3+\beta_3x+\gamma_3y),$$

which is exactly system $(p.1)$ of Theorem 2.5.1.

When $c\neq 0$ we apply Theorem 2.3.4 and conclude that a quadratic system (2.3) can be written as

$$\begin{aligned}\dot{x}&=-\alpha_3(x^2+y^2)-((\alpha_2+\alpha_1)y-i(\alpha_2-\alpha_1)x)(y+ax+c),\\ \dot{y}&=a\alpha_3(x^2+y^2)+((\alpha_2+\alpha_1)x-i(\alpha_2-\alpha_1)y)(y+ax+c),\end{aligned} \tag{2.32}$$

with α_1,α_2 and $\alpha_3\in\mathbb{C}$. Writing $\alpha_j=m_j+in_j$ with $m_j,n_j\in\mathbb{R}$ and using that such system have real parameters we conclude that $m_2=m_1$, $n_2=-n_1$ and $n_3=0$. Replacing this conditions in (2.32) we get the system

$$\dot{x}=-m_3(x^2+y^2)+2(n_1x-m_1y)(y+ax+c), \quad \dot{y}=am_3(x^2+y^2)+2(m_1x+n_1y)(y+ax+c). \tag{2.33}$$

Note that if $m_3=0$ then the system has a common factor, so we can take $m_3=2$. Applying a rescaling of the time and writing $\alpha=m_1$, $\beta=n_1$ we obtain system $(p.2)$ of Theorem 2.5.1. It follows from the previous study the proof of Theorem 2.5.1 □

2.5.2 Darboux invariants and phase portrait on the Poincaré disc

Using the normal forms described in Theorem 2.5.1 we can make a topological classfica- tion of the systems, that is, to draw its phase portraits on the Poincaré disc when the system has a Darboux invariant.

Theorem 2.5.2. The global phase portrait in the Poincaré disc of each quadratic differential system admitting a reducible invariant cubic $f(x,y) = 0$ and having a Darboux invariant of the form $I(x,y,t) = e^{-st} f(x,y)$ is topologically equivalent to one of the phase portraits presented in Figures 2-7. Their normal forms according to Theorem 2.5.1 is labelled in the corresponding figure.

The proof of Theorem 2.5.2 will be done using nine Propositions (2.5.3–2.5.12).

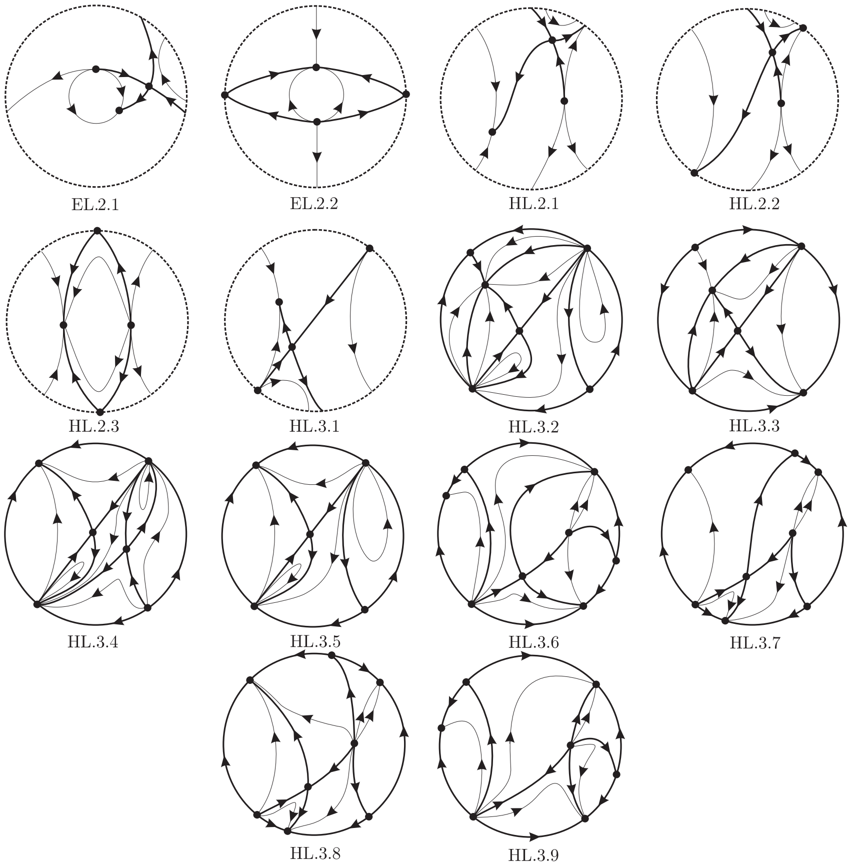

Figure 2 – Phase portraits of systems of type (E) and (H) when they have a Darboux invariant.

Proposition 2.5.3 (E)**.** Each real planar quadratic differential system with a real ellipse and a straight line having a Darboux invariant can be written, after an affine change of coordinates, as system $(E.2)$ with $c = -1$, $\alpha_2 \neq 0$. Moreover, such system has the Darboux invariant

$$I_2(x,y,t) = e^{-t}(y-1)^{1/\alpha_2}(x^2+y^2-1)^{-\frac{1}{2\alpha_2}}.$$

and, these systems have only 2 non–equivalent phase portraits, see phase portraits EL.2.1 and EL.2.2 of Figure 2.

Proof. If follows from the reducible cubic classification that we can fix $f_1 = x^2 + y^2 - 1 = 0$ as the real ellipse and by Theorem 2.5.1 there are only two families of systems having $f_1 = 0$ and a straight line as invariant curves $(E.1)$ and $(E.2)$. We shall prove later that $(E.1)$ does not

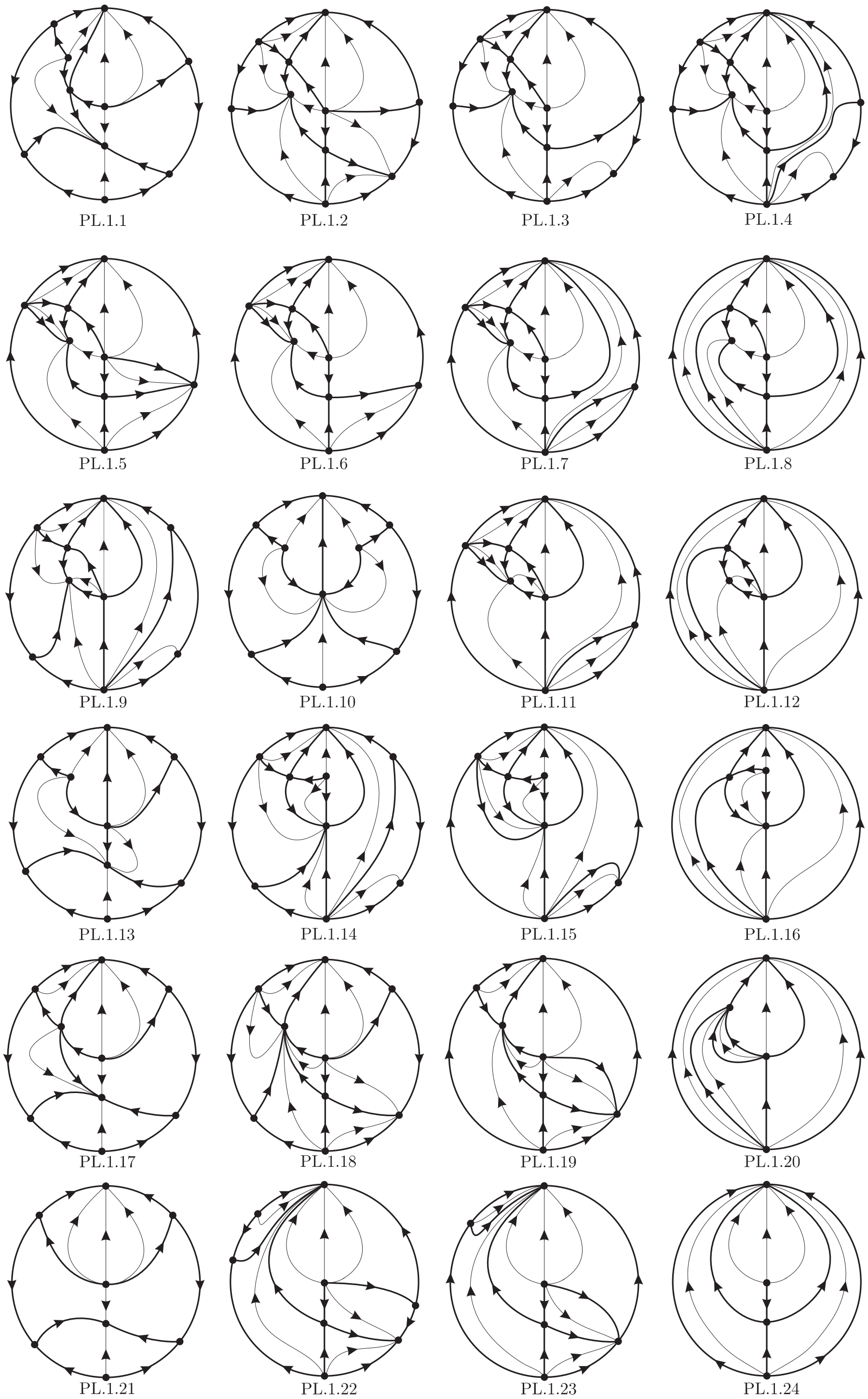

Figure 3 – Phase portraits of systems of type (P) when they have a Darboux invariant.

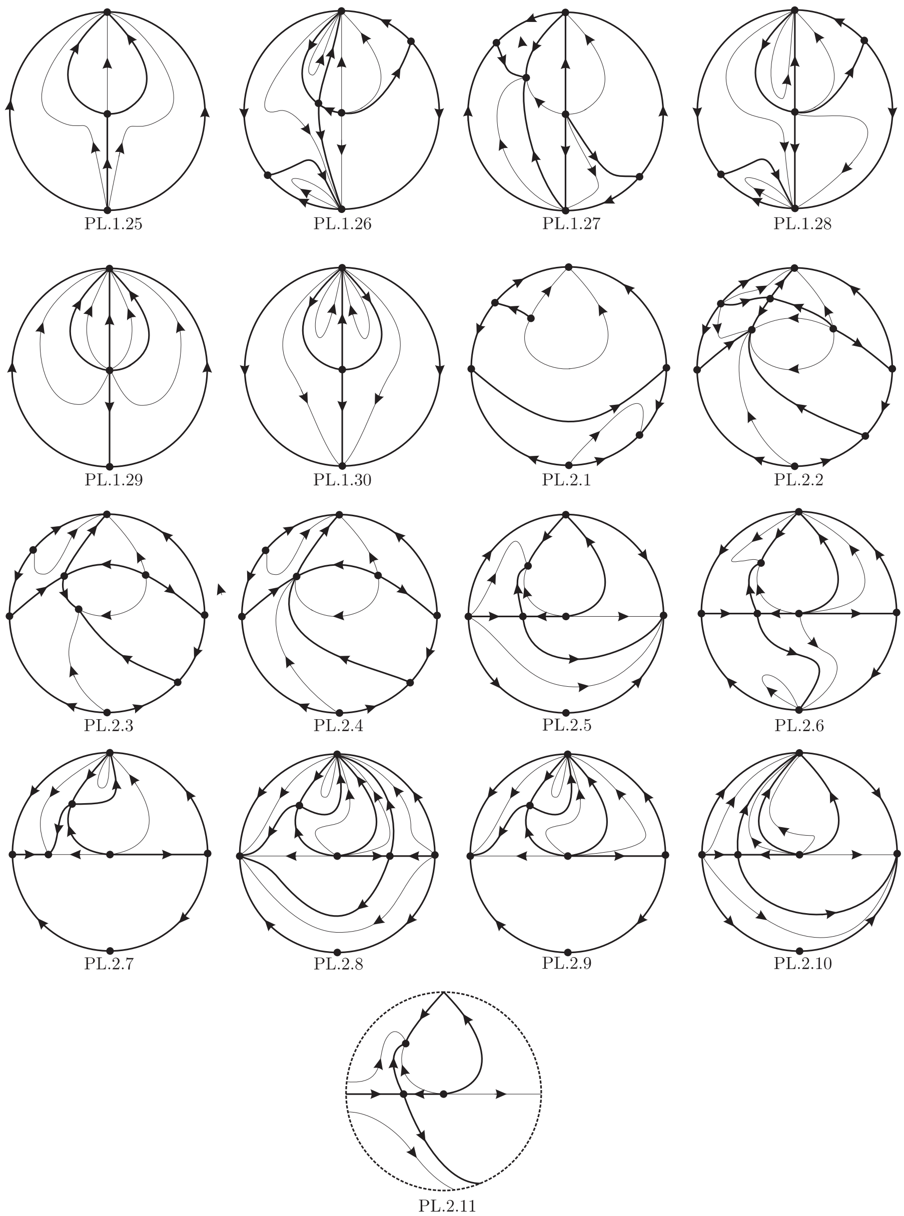

Figure 4 – Phase portraits of systems of type (P) when they have a Darboux invariant.

admit a Darboux invariant. Now we study system $(E.2)$. By Proposition 2.1.6 system $(E.2)$ has a Darboux invariant if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ not both equal to zero such that (2.9) holds with $s \in \mathbb{R} \setminus \{0\}$ and k_1, k_2 being the cofactors of $f_1 = 0$ and $f_2 = y + c = 0$, respectively. But for system $(E.2)$ we must have $\alpha_2 = 0$ or $c = -1$. If $\alpha_2 = 0$ the cofactors are $k_1 = \beta_1 x$ and $k_2 = \beta_2 x$ and the equation $\lambda_1 k_1 + \lambda_2 k_2 + s = 0$ has no solution for $s \neq 0$. Hence if $\alpha_2 = 0$ system $(E.2)$ has no Darboux invariant.

If $\alpha_2 \neq 0$ and $c = -1$ then the cofactors are $k_1 = \beta_1 x + 2\alpha_2 y$ and $k_2 = \alpha_2 + \beta_2 x + \alpha_2 y$ and the unique solution of (2.9), with $s \neq 0$ is

$$\beta_1 = 2\beta_2,\, s = -\alpha_2 \lambda_2,\, \lambda_1 = -\lambda_2/2. \tag{2.34}$$

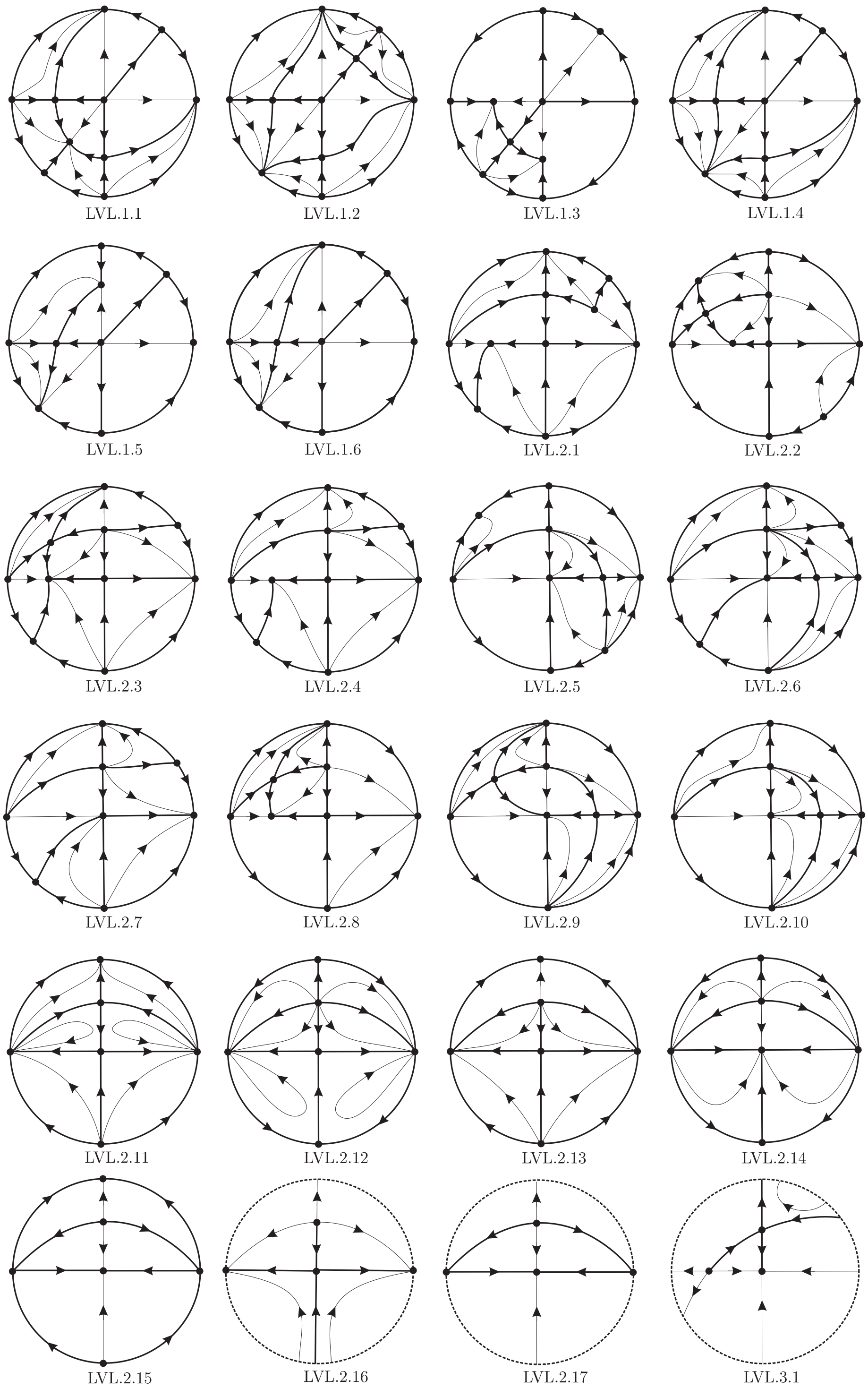

Figure 5 – Phase portraits of systems of type (LV) when they have a Darboux invariant.

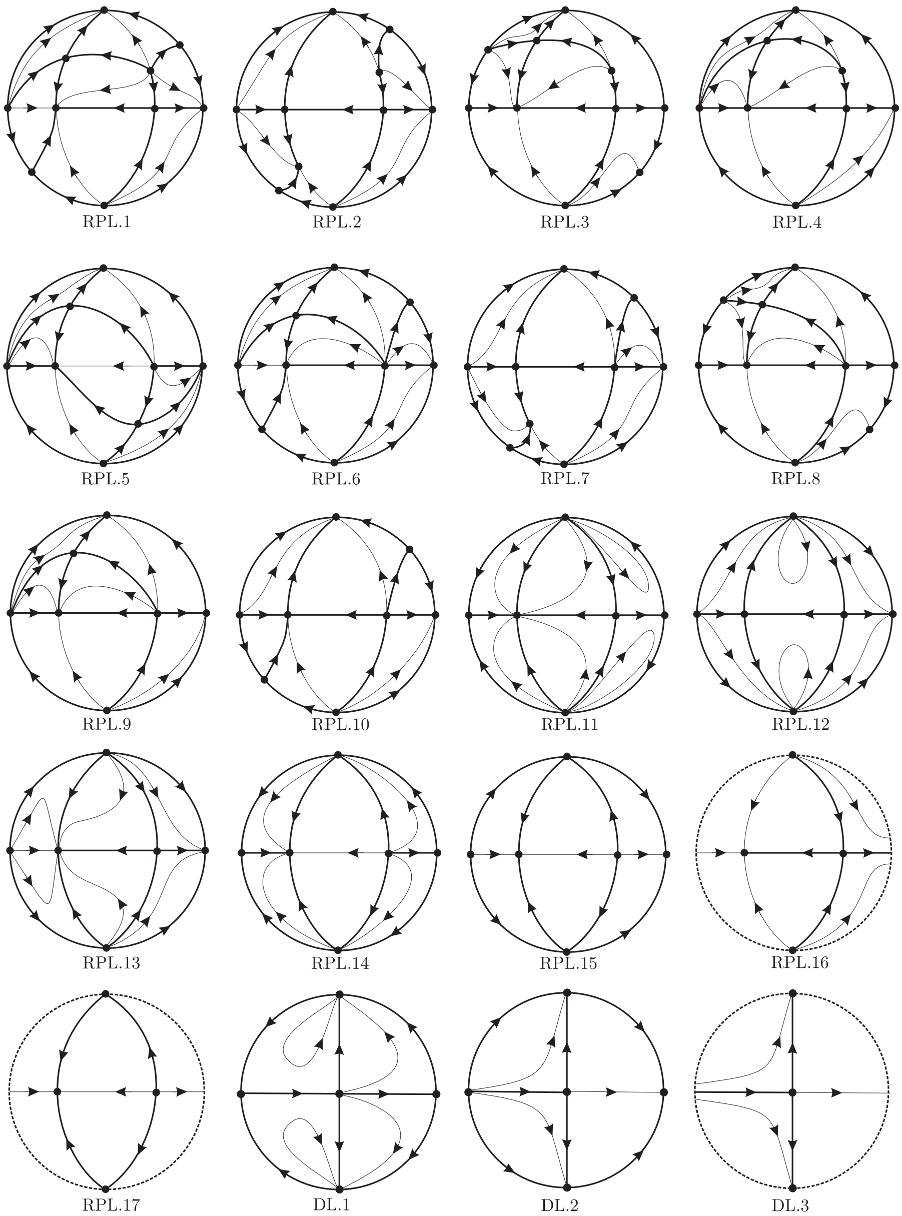

Figure 6 – Phase portraits of systems of type (RPL) and (DL) when they have a Darboux invariant.

Taking $\lambda_1 = 1/\alpha_2$ and replacing (2.34) in system $(E.2)$ we obtain the system

$$\dot{x} = \beta_2(y-1) + x(\beta_2 x + \alpha_2 y), \quad \dot{y} = (y-1)(\alpha_2 + \beta_2 x + \alpha_2 y), \tag{2.35}$$

which has the Darboux invariant

$$I_2(x,y,t) = e^{-t}(y-1)^{1/\alpha_2}(x^2+y^2-1)^{-\frac{1}{2\alpha_2}}.$$

In order to study the global phase portrait of system $(E.2)$ we start considering its finite singularities. Note that (2.35) has at most three finite singularities, namely $z_1 = (0,1)$, $z_2 = (-1/\beta_2, 1)$ and $z_3 = \left(-\frac{2\beta_2}{\beta_2^2+1}, \frac{\beta_2^2-1}{\beta_2^2+1}\right)$. The eigenvalues associated to z_1 are 2 and 1, if $\beta_2 \neq 0$, the eigenvalues associated to z_2 are -1 and 1 and the eigenvalues of z_3 are -1 and -2.

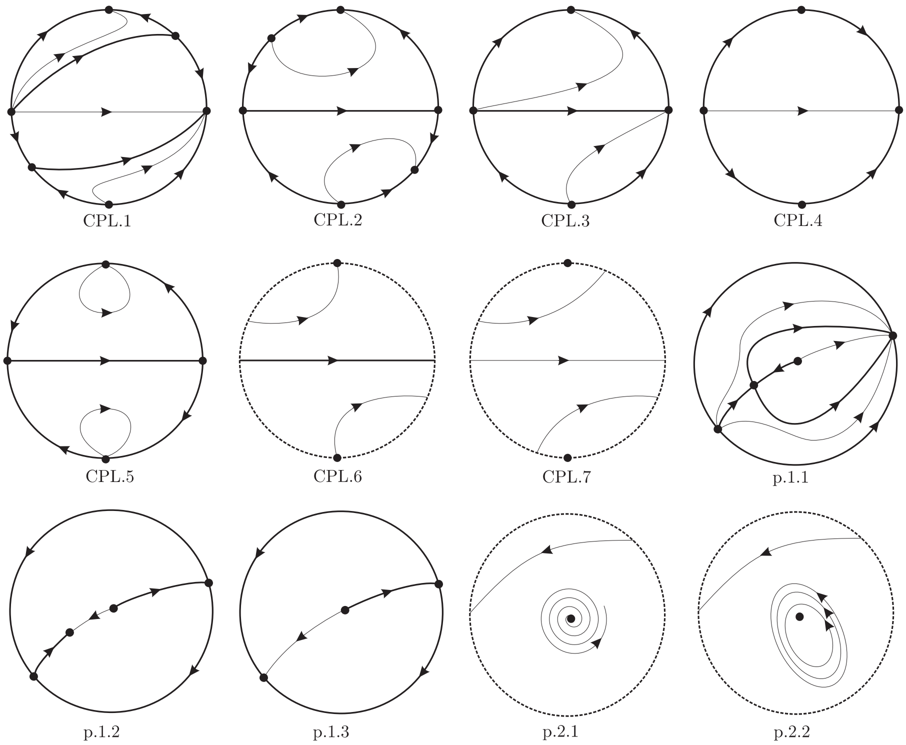

Figure 7 – Phase portraits of systems of type (CPL) and (p) when they have a Darboux invariant.

So for $\beta_2 \neq 0$ z_1, z_2 and z_3 are an unstable node, a saddle and a stable node, respectively. When $\beta_2 = 0$ we have only z_1 and z_3 as finite singularities.

In the local chart U_1 the compactified system is

$$\dot{u} = -v(\beta_2 + \beta_2 u^2 - \beta_2 uv + v), \quad \dot{v} = -v(\beta_2 + \beta_2 uv + u - \beta_2 v^2), \tag{2.36}$$

so $v = 0$ is a common factor, this means that $v = 0$ is a line of singular points. Eliminating the common factor v, system (2.36) has no singular points if $\beta_2 \neq 0$. Otherwise $u_1 = (0,0)$ is a singular point with eigenvalues -1 and 1 which implies that the origin is a hyperbolic saddle besides the line of singular points.

In the local chart U_2 the compactified system is written as

$$\dot{u} = v(\beta_2 + \beta_2 u^2 + uv - \beta_2 v), \quad \dot{v} = v(v-1)(\beta_2 u + v + 1).$$

Eliminating the common factor v the origin is not a singular point of the compactified system.

Note that if $\beta_2 = 0$ there are an additional invariant straight line given by $y + 1 = 0$. From the study of the finite and infinite behavior of system $(E.2)$ we conclude that such system has two non–equivalent phase portraits when $c = -1$: phase portrait EL.2.1, if $\beta_2 \neq 0$ and phase portrait EL.2.2, if $\beta_2 = 0$. See Figure 2. □

Proposition 2.5.4 (H)**.** Each real planar quadratic differential system with a hyperbola and a straight line having a Darboux invariant can be written, after an affine change of coordinates, as

(i) system $(H.2)$ with $\alpha_2 \neq 0$ and $c = -1$. Its Darboux invariant is

$$I_3(x,y,t) = e^{-\alpha_2 t}(x^2 - y^2 - 1)^{-1/2}(x-1).$$

ii) system $(H.3)$ with $A\alpha \neq 0$, $c = 0$ and $\beta = -\gamma$. Its Darboux invariant is

$$I_4(x,y,t) = e^{-A\alpha t}(x^2 - y^2 - 1)^{\gamma}(y-x)^A.$$

(iii) system $(H.3)$ with $\alpha \neq 0$ and $\beta = \gamma = 0$. Its Darboux invariant is

$$I_5(x,y,t) = e^{\alpha t}(y - x + c)^{-1}.$$

(iv) system $(H.4)$ with $\alpha \neq 0$ and $A = 2\beta$. Its Darboux invariant is

$$I_6(x,y,t) = e^{-\alpha t}(x^2 - y^2 - 1)^{-1/2}(y + ax - \sqrt{a^2 - 1}).$$

Moreover the are 12 non–equivalent phase portrait in the Poincaré disc of these systems. They are in Figure 2 HL.2.1–HL.2.3, HL.3.1–HL.3.9.

Proof. Fixing $f_1 = x^2 - y^2 - 1 = 0$, Proposition 2.1.6 says that system $(H.2)$ has a Darboux invariant if equation (2.9) holds for λ_1, λ_2 not both zero, where $s \in \mathbb{R} \setminus \{0\}$, and k_1, k_2 are cofactors of $f_1 = 0$ and $f_2 = x + c = 0$, respectively. Moreover $c = -1$ or $\alpha_2 = 0$ in system $(H.2)$. For $\alpha_2 = 0$ we have $k_1 = \gamma_1 y$ and $k_2 = \gamma_2 y$ and the equation $\lambda_1 k_1 + \lambda_2 k_2 + s = 0$ has no solution with $s \neq 0$. So in this case system $(H.2)$ has no Darboux invariant. If $\alpha \neq 0$ and $c = -1$ then $k_1 = 2\alpha_2 x + \gamma_1 y$ and $k_2 = \alpha_2 + \alpha_2 x + \gamma_2 y$ and (2.9) has a unique solution

$$s = -\alpha_2\lambda_2,\ \gamma_1 = 2\gamma_2,\ \lambda_1 = -\lambda_2/2.$$

The proof of (i) follows taking $\lambda_2 = 1$ and replacing $\gamma_1 = 2\gamma_2$ in system $(H.2)$, from that we have the system

$$\dot{x} = (x-1)(\alpha_2 + \alpha_2 x + \gamma_2 y), \quad \dot{y} = -\gamma_2(x^2 - y^2 - 1) + x(-\gamma_2 + \gamma_2 x + \alpha_2 y), \tag{2.37}$$

having the Darboux invariant

$$I_3(x,y,t) = e^{-\alpha_2 t}(x^2 - y^2 - 1)^{-1/2}(x-1).$$

To prove (ii) and (iii) we study system $(H.3)$ where we consider two cases: $c = 0$ and $\beta = -\gamma$. It is easy to see that if $c = 0$ $(H.3)$ has a Darboux invariant when $\alpha \neq 0$ and $\beta = -\gamma$. In this case we have the differential system

$$\dot{x} = (A/2)(x^2 - y^2 - 1) - y(\alpha - \gamma x + \gamma y), \quad \dot{y} = (A/2)(x^2 - y^2 - 1) - x(\alpha - \gamma x + \gamma y), \tag{2.38}$$

having the Darboux invariant

$$I_4(x,y,t) = e^{-A\alpha t}(x^2 - y^2 - 1)^{\gamma}(y-x)^A.$$

If $\beta = -\gamma$ system $(H.3)$ has a Darboux invariant only when $\gamma = 0$ and $\alpha \neq 0$. In this case the system is

$$\dot{x} = (A/2)(x^2 - y^2 - 1) - \alpha y(1 - cx - cy), \quad \dot{y} = (A/2)(x^2 - y^2 - 1) - \alpha x(1 - cy) + c\alpha(y^2 + 1), \tag{2.39}$$

and it has the Darboux invariant

$$I_5(x,y,t) = e^{\alpha t}(y - x + c)^{-1}.$$

The study of (iv) follows from system $(H.4)$ where the invariant line is $f_2 = y + ax - \sqrt{a^2 - 1} = 0$. In this case the unique solution of equation (2.9) is

$$s = -\alpha\lambda_2, A = 2\beta, \lambda_1 = -\lambda_2/2. \tag{2.40}$$

So taking $\lambda_2 = 1$ we obtain the Darboux invariant

$$I_6(x,y,t) = e^{-\alpha t}(x^2 - y^2 - 1)^{-1/2}(y + ax - \sqrt{a^2 - 1}).$$

We start the study of the phase portraits of system (2.37). Since $\alpha_2 \neq 0$ we can take $\alpha_2 = 1$ and the transformation $x = X, y = -Y$ takes the system with parameter γ_2 to the system with parameter $-\gamma_2$. So we may also assume $\gamma_2 \geq 0$.

Considering the finite singularities, if $\gamma_2 \notin \{0,1\}$ system (2.37) has three finite singularities, namely $z_1 = (0,1)$, $z_2 = (1, -1/\gamma_2)$ and $z_3 = \left(\frac{\gamma_2^2+1}{\gamma_2^2-1}, -\frac{2\gamma_2}{\gamma_2^2-1}\right)$. The eigenvalues associated to z_1 are 2 and 1, if $\beta_2 \neq 0$, the eigenvalues associated to z_2 are -1 and 1 and the eigenvalues of z_3 are -1 and -2. So for $\gamma_2 \notin \{0,1\}$ z_1, z_2 and z_3 are respectively, an unstable node, a saddle and a stable node. When $\beta_2 = 0$ we have only z_1 and z_3 as finite singularities.

In the local chart U_1 the compactified system is

$$\dot{u} = v(-\gamma_2 + \gamma_2 u^2 + uv + \gamma_2 v), \quad \dot{v} = v(v-1)(\gamma_2 u + v + 1), \tag{2.41}$$

so v is a common factor, this means that $v = 0$ is a line of singular points. Eliminating the common factor v, system (2.41) has no singular points if $\gamma_2 \neq 1$. Otherwise $u_1 = (-1,0)$ is a singular point with eigenvalues -2 and -1, which implies that u_1 is a hyperbolic stable node. Moreover if $\gamma_2 = 0$ there an additional invariant straight line given by $x + 1 = 0$.

In the local chart U_2 the compactified system is written as

$$\dot{u} = -v(\gamma_2 - \gamma_2 u^2 + \gamma_2 uv + v), \quad \dot{v} = -v(\gamma_2 + \gamma_2 v^2 - \gamma_2 uv + u).$$

So after eliminating the common factor v the origin is a singular point of the compactified system if and only if $\gamma_2 = 0$. In this case $(0,0)$ is a hyperbolic saddle.

It is easy to see that if $\gamma_2 \in (0,1)$ the singulatities z_1 and z_3 are in distinct branches of the hyperbola, and if $\gamma_2 \in (1,+\infty)$ they are in the same branch as shows Figure 8.

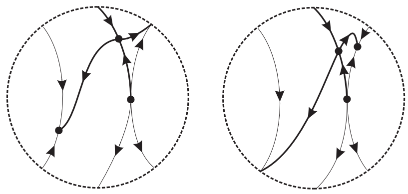

Figure 8 – Possible phase portraits of sytem (2.37) when $\gamma_2 \notin \{0,1\}$.

From Theorem 1.43 of (DUMORTIER; LLIBRE; ARTÉS, 2006) (Markus-Neumann-Peixoto Theorem) we conclude that these two phase portraits are topologically equivalent. By continuity and the study done previously we conclude that system of type $(H.2)$ having a Darboux invariant can have three non–equivalent phase portrait. The case $\gamma_2 \neq 0,1$ corresponds to HL.2.1 in Figure 2 and when $\gamma_2 = 1$ or $\gamma_2 = 0$ we have the phase portraits HL.2.2 and HL.2.3 of Figure 2, respectivelly.

Now we study the global phase portrait of system $(H.3)$. Remember that the parameters of $(H.3)$ must satisfies $c(\gamma+\beta) = 0$. We start considering $c = 0$, then the differential system is

$$\dot{x} = (A/2)(x^2 - y^2 - 1) - y(\alpha - \gamma x + \gamma y), \quad \dot{y} = (A/2)(x^2 - y^2 - 1) - x(\alpha - \gamma x + \gamma y), \quad (2.42)$$

that has $f_1 = x^2 - y^2 - 1 = 0$ and $f_2 = y - x = 0$ as invariant algebraic curves. Since $\alpha \neq 0$ we can take $\alpha = 1$ and the transformation $x = -X, y = -Y$ allows to assume $A > 0$.

If $\gamma \neq 0$ then $z_1 = (-A/2, -A/2)$ and $z_2 = \big((\gamma^2+1)/(2\gamma), (\gamma^2-1)/(2\gamma)\big)$ are the two finite singular points. If $\gamma = 0$ exists only one finite singular point.

The eigenvalues associated to z_1 are -1 and 1 so z_1 is a saddle. The eigenvalues associated to z_2 are A/γ and -1, so z_2 is a stable node if $\gamma < 0$, and a saddle if $\gamma > 0$. Moreover z_1 is on the straight line and z_2 is on the hyperbola.

In the local chart U_2 we have the system

$$\dot{u} = (1/2)(u-1)(Av^2 - (A+2\gamma)u^2 + 2uv + 2v + A + 2\gamma),$$

$$\dot{v} = (1/2)v(Av^2 - (A+2\gamma)u^2 + 2\gamma u + 2uv + A),$$

and the origin is a singular point only when $A + 2\gamma = 0$ but in this case the line $v = 0$ is filled up of singular points.

In the local chart U_1 we have system

$$\dot{u} = (1/2)(u-1)((A+2\gamma)u^2 + Av^2 + 2uv + 2v - A - 2\gamma),$$

$$\dot{v} = (1/2)v((A+2\gamma)u^2 + Av^2 + 2uv - 2\gamma u - A),$$

which has the infinity filled up by singularities when $A+2\gamma=0$, otherwise, there are two singularities $u_1=(-1,0)$ and $u_2=(1,0)$.

Assuming $A+2\gamma\neq 0$. The point u_1 has eigenvalues 2γ and $2(A+2\gamma)$, and u_2 is linearly zero because the Jacobian matrix of the linear part of the system evaluated in u_2 is null. To decide the local behavior of u_2 we must do *blow up*. From now on we fix $l_1=\gamma$, $l_2=A+2\gamma$.

After translate the singular point u_2 to the origin, making the change of coordinates $u=U, v=UW$ and rescaling the common factor U we get the differential system

$$\dot{U}=(1/2)U(AUW^2+(A+2\gamma)U+2UW+4W+2A+4\gamma), \quad \dot{W}=-W(W+\gamma).$$

Note that such system have two singularities when $l_1l_2\neq 0$, namely, $\overline{U_1}=(0,0)$ and $\overline{U_2}=(0,-\gamma)$; one singular point when $l_1=0$ and $l_2\neq 0$, namely $\overline{U_1}=\overline{U_2}$. The eigenvalues of $\overline{U_1}$ are $-\gamma$ and $A+2\gamma$, whereas the eigenvalues of $\overline{U_2}$ are A and γ. From the combination of the signs of l_1 and l_2, as described in Figure 9, we get the possible local behavior of $\overline{U_1}$ and $\overline{U_2}$.

$(1)\ l_1>0$: $\xrightarrow{(1.1)}\ l_2>0$; $\xrightarrow{(1.2)}\ l_2<0$

$(2)\ l_1<0 \xrightarrow{(2.1)} l_2>0$

$(3)\ l_1=0 \xrightarrow{(3.1)} l_2>0$

Figure 9 – The possible combination of signs of l_1 and l_2 describe the cases to be considered for system $(H.3)$ when $c=0$.

Applying the *blow down* we get all possible phase portraits for system $(H.3)$ when $c=0$. Note that each one is realizable, indeed, the phase portrait HL.3.2 corresponds to subcase (1.1) which is realizable with $A=4$ and $\gamma=-1$; HL.3.3 corresponds to subcase (1.2) which is realizable with $A=1$ and $\gamma=-1$. Notice that if $\gamma\neq 0$ there is a third invariant straight line, given by $f_3=\gamma(x-y)-1=0$ so HL.3.3 is the only possible phase portrait for subcase (1.2). The phase portraits HL.3.4 and HL.3.5 correspond, respectively, to subcases (2.1) and (3.1). The phase portrait HL.3.4 is realizable with $A=1$ and $\gamma=1$, and HL.3.5 is realizable with $A=1$ and $\gamma=0$.

It remains to consider the case $l_2=0$. With this condition the infinity is filled up of singular points. After eliminating the common factor v we have only one singular point at the local chart U_1. The eigenvalues associated to this point are 2 and 1, so this is a unstable node. By continuity the only possible phase portrait in this case is HL.3.1 of Figure 2, which is realizable with $A=2$ and $\gamma=-1$.

Now considering system $(H.3)$ with $\beta+\gamma=0$ we have seen above that the system has a Darboux invariant when $\beta=\gamma=0$ and $\alpha\neq 0$. Under these conditions the differential system is

$$\begin{aligned}\dot{x}&=(A/2)(x^2-y^2-1)-\alpha y(1-cx-cy),\\ \dot{y}&=(A/2)(x^2-y^2-1)+c\alpha(y^2+1)-\alpha x(1-cy).\end{aligned} \tag{2.43}$$

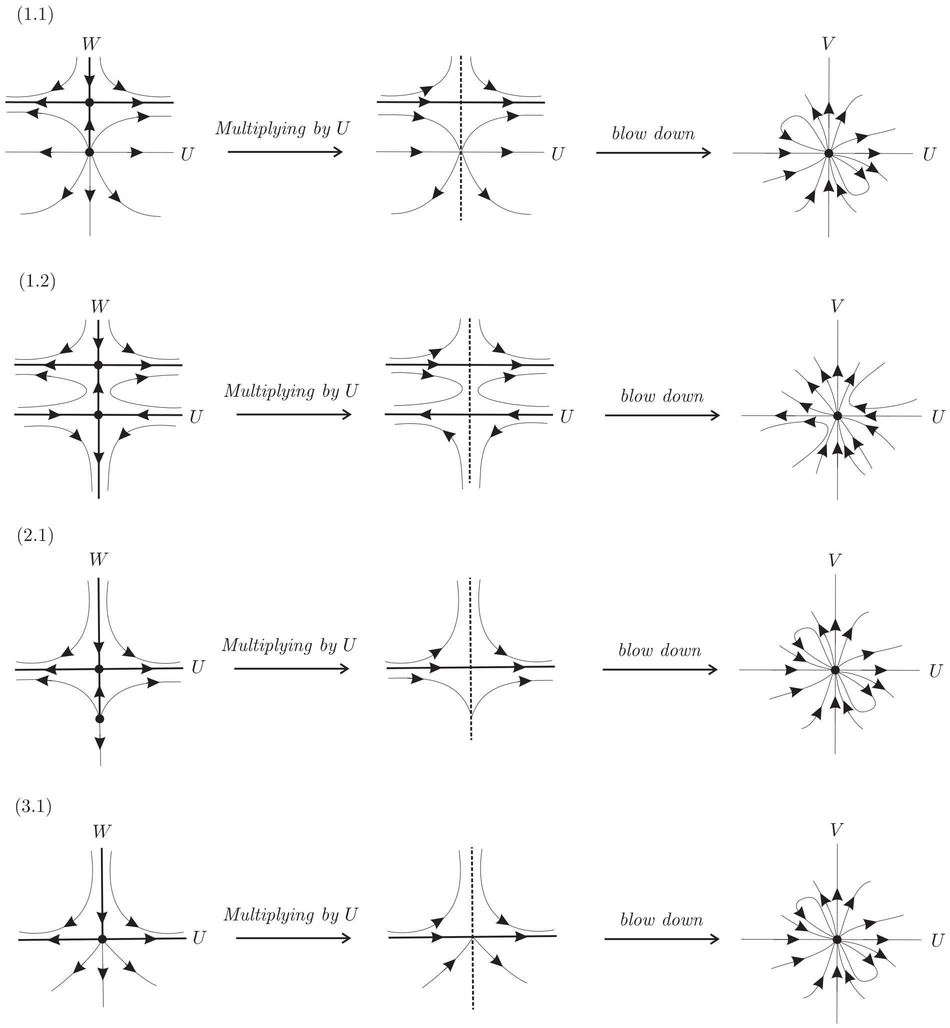

Figure 10 – On the left the local phase portrait after blow up. Here they are indexed according to the signs of l_1 and l_2. On the right the local behavior at origin after the *Blow down* for system $(H.3)$.

Such system has $f_1 = x^2 - y^2 - 1 = 0$ and $f_2 = y - x + c = 0$ as algebraic invariant curves. If $c = 0$ then we get system (2.42) when $\gamma = 0$, so we can take $c \neq 0$ here. Moreover, doing the transformation $x = -X$, $y = -Y$ in the algebraic cubic we can assume $c > 0$. Finally, since α is different from zero we can take $\alpha = 1$ in (2.43).

System (2.43) has two finite singular points, namely $z_1 = ((2c - A)/2, -A/2)$ and $z_2 = ((c^2+1)/(2c), (1-c^2)/(2c))$. Defining $l_1 = c^2 - Ac - 1$, $l_2 = A - c$ and $l_3 = A - 2c$, we have z_1 coalesces with z_2 if and only if $l_1 = 0$. Moreover the eigenvalues associated to z_1 are l_1 and 1, and the eigenvalues associated to z_2 are $-l_1$ and 1. So we conclude that z_1 is a unstable node and z_2 is a saddle if $l_1 > 0$; z_1 is a saddle and z_2, an unstable node, if $l_1 < 0$ and, if $l_1 = 0$, $z_1 = z_2$ is a saddle-node.

In the local chart U_1 the system becomes

$$\dot{u} = (1/2)((A-2c)u^3 - Au^2 + Auv^2 - Au - Av^2 + 2cu + 2cv^2 + 2u^2v - 2v + A),$$
$$\dot{v} = (1/2)v((A-2c)u^2 + Av^2 - 2cu + 2uv - A), \tag{2.44}$$

which has three singularities $u_1 = (-1,0)$ and $u_2 = (1,0)$ and $u_3 = (\frac{A}{A-2c}, 0)$, if $A \neq 2c$. Note that when $l_3 = 0$ the point u_3 does not exist and $u_1 = u_3$ when $l_2 = 0$. The eigenvalues associated to u_1 are $2l_2$ and 0, the point u_2 has both eigenvalues equal to $-2c$, and u_3 has eigenvalues 0 and $2cl_2/l_3$. It is not difficult to see that when $l_2 \neq 0$, u_1 and u_3 are saddle–nodes. In the local chart U_2 the origin $(0,0)$ is a singular point if and only if $l_3 = 0$.

Assuming $l_1 l_2 \neq 0$ and considering all possible combinations of the sign of l_1, l_2 and l_3 we observe that there are some impossible combinations, for instance when $l_2 < 0$ we have $l_3 < 0$. In Figure 11 we describe the possible combinations and introduce a label for each one.

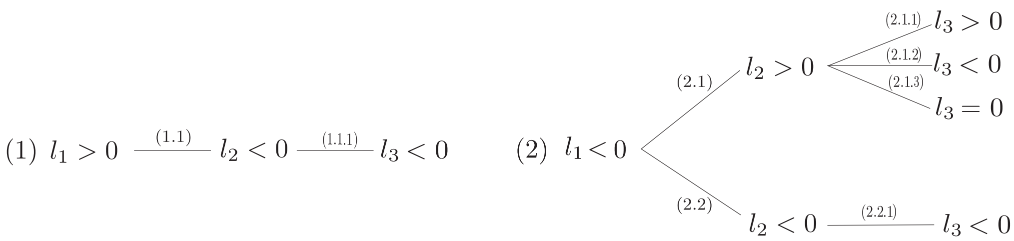

Figure 11 – The possible combinations of signs of l_1, l_2 and l_3 for system $(H.3)$ when $c \neq 0$.

The case $(2.2.1)$ presents a unique phase portrait, HL.3.6 of Figure 2 and it is realizable with $A = 1/2$ and $c = 1$.

In case $(2.1.1)$ we have three possibilities for the finite saddle separatrix ω-limit set: we can have a connection of separatrix as in HL.3.7; the separatrix can go to the stable node, generating a phase portrait equivalent to HL.3.6, or the separatrix can go to the parabolic sector of the saddle node u_3 which corresponds to HL.3.8. Moreover HL.3.8 is realizable with $A = 2$ and $c = 1/2$, and as we see above, HL.3.6 is realizable with $A = 1/2$ and $c = 1$. Since HL.3.6 and HL.3.8 are realizable then by continuity of the parameters we conclude that HL.3.7 is also realizable.

The analysis of case $(2.1.2)$ can be done as the case $(2.1.1)$ and it has the phase portraits equivalent to them.

The possible phase portraits of $(2.1.3)$ are also equivalent to the phase portraits of $(2.1.1)$. Also the case $(1.1.1)$ has a phase portrait equivalent to $(2.2.1)$.

When $l_2 = 0$ it follows that $l_1, l_3 < 0$ and in the local chart U_1 the singular point $u_1 = u_3$ is non–elementary. After translate this singular point to the origin, making the change of coordinates

$u = U$, $v = UW$ and rescaling the common factor U we get

$$\dot{U} = (U/2)(AUW^2 - AU + 2UW - 4W + 2A), \quad \dot{W} = W(W - A).$$

This system has two singularities $\overline{U_1} = (0,0)$ and $\overline{U_2} = (0,A)$ being both saddles. Figure 12 shows the *blow down*.

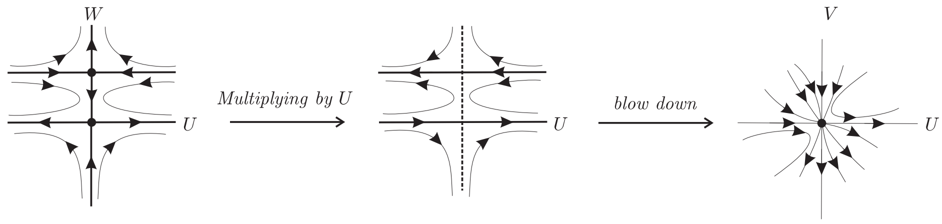

Figure 12 – The local phase portrait of system (2.44) when $l_2 = 0$. On the left the local phase portrait after blow up. On the right the local behavior at the origin after blow down.

To obtain the phase portrait for system (2.43) with $l_2 = 0$ we note that there is more two invariant straight lines, given by $f_3 = x + y = 0$ and $f_4 = Ax + Ay - 1$. The finite saddle z_1 is on $f_3 = 0$ and the finite node is on the intersection of $f_2 = 0$ and $f_4 = 0$ so by continuity there is only one phase portrait, which is topologically equivalent to HL.3.3.

Finally it remains to study the case $l_1 = 0$. Here $l_2 < 0$ and $l_3 < 0$ so the only possibility is the phase portrait HL.3.9 of Figure 2, which is realizable with $A = 0$ and $c = 1$.

To conclude the proof of Proposition 2.5.4 it remains to study the global phase portrait of system $(H.4)$ when $A = 2\beta$ and $\alpha \neq 0$. In this case we assume $\alpha = 1$, so $(H.4)$ is written as

$$\dot{x} = \beta x^2 + (a\beta - \sqrt{a^2-1})xy + (a - \sqrt{a^2-1}\beta)y - \beta,$$

$$\dot{y} = (a\beta - \sqrt{a^2-1})y^2 + \beta xy + (a - \sqrt{a^2-1}\beta)x + (a\beta - \sqrt{a^2-1}).$$

Denoting $\delta = a\beta - \sqrt{a^2-1}$ and $\eta = a - \sqrt{a^2-1}\beta$ there are at most three finite singularities $z_1 = (-\delta/\eta, \beta/\eta)$, $z_2 = ((\delta\eta - \beta)/(\beta^2 - \delta^2), (\delta\eta - \beta)/(\beta^2 - \delta^2))$ and $z_3 = ((\beta + \delta\eta)/(\beta^2 - \delta^2), -(\beta + \delta\eta)/(\beta^2 - \delta^2))$. We observe that such singular points never coalesce but if $\eta = 0$, z_1 does not exist and if $\beta^2 - \delta^2 = 0$ the same happens with z_2 or z_3. With respect to the localization of these points, z_3 is the intersection of the hyperbola and the straight line, z_1 is on the straight line and z_2 is on the hyperbola. Moreover is not difficult to check that z_1, z_2 and z_3 are hyperbolic points, being z_1 a saddle, z_2 a stable node and z_3 an unstable node.

Concerning to the behavior at infinity, in the local chart U_1 the compactified system is given by

$$\dot{u} = v(\eta - \eta u^2 + \beta uv + \delta v), \quad \dot{v} = -v(\beta - \beta v^2 + \eta uv + \delta u),$$

so v is a common factor what means that $v = 0$ is a line of singular points. Eliminating this common line it remains singularities if and only if $\eta = 0$ or $\beta^2 - \delta^2 = 0$. When $\eta = 0$ the point

$u_1 = (-a, 0)$ is a saddle. When $\delta = \beta$ the point $u_2 = (-1, 0)$ is a node with eigenvalues η and 2η. Finally if $\delta = -\beta$ then the point $u_3 = (1, 0)$ has eigenvalues $-\eta$ and -2η so it is a node.

In the local chart U_2 the system becomes

$$\dot{u} = -v(-\eta + \beta v + \eta u^2 + \delta uv), \quad \dot{v} = v(\delta + \beta u + \eta uv + \delta v^2).$$

So eliminating the common factor v the origin is not a singular point.

By the previous study and continuity of the solutions we conclude that there exist three possible phase portraits and they are topologically equivalent to the ones obtained from system $(H.2)$ and described in Figure 2. Indeed when $\eta, \beta^2 - \delta^2 \neq 0$ we have the phase portrait HL.2.1, when $\beta^2 - \delta^2 = 0$ we have HL.2.2, and the case $\eta = 0$ corresponds to phase portrait HL.2.3. $\square$

Before to study the systems of type (P), we present two lemmas that will help to show the realization or not of the phase portraits that follow.

Lemma 2.5.5. On any straight line which is not composed of orbits the total number of contact points is at most two for any quadratic system. If there are two such points p_1 and p_2, then the orbits intersecting the segment ∞p_1 cross in the same sense as the orbits intersecting $p_2\infty$, and the opposite sense to the path intersecting p_1p_2.

Lemma 2.5.6. The straight line connecting one finite singular point and a pair of infinite singular points in a quadratic system is either formed by trajectories or a line with exactly one contact point. If this contact point is the finite singular point, the flow goes in different directions on each half straight line.

The proof of Lemma A.1.2 can be founded in (COPPEL, 1966). Lemma 2.5.6 in the case that the pair of infinite singular points are saddles was proved in (SOTOMAYOR; PATERLINI, 1983). When such a pair are saddle-nodes, the proof appeared in (ARTES, 1990).

Proposition 2.5.7 (P)**.** Each real planar quadratic differential system with a parabola and a straight line having a Darboux invariant can be written, after an affine change of coordinates, as

(i) $(P.1)$ with $\alpha_1 - 2\alpha_2 \neq 0$ and Darboux invariant

$$I_7(x, y, t) = e^{(\alpha_1 - 2\alpha_2)t}(y - x^2)^{-1}x^2.$$

(ii) $(P.2)$ with $\alpha_2(\beta_1 - \beta_2) \neq 0$, $c = 0$ and Darboux invariant

$$I_8(x, y, t) = e^{2\alpha_2(\beta_1 - \beta_2)t}(y - x^2)^{\beta_2}y^{-\beta_1},$$

(iii) $(P.2)$ with $c\gamma_2 \neq 0$, $\beta_1 = \beta_2$, $\alpha_2 = 0$ and Darboux invariant

$$I_9(x, y, t) = e^{-2c\gamma_2 t}(y - x^2)(y + c)^{-1},$$

Moreover there are 41 non–equivalent phase portrait in the Poincaré disc for these systems. They are in Figures 3 and 4.

Proof. We fix the invariant parabola as $f_1 = y - x^2 = 0$. Here we describe in details the proof of the existence of a Darboux invariant for system $(P.2)$, the other cases are analogous. System $(P.2)$ is given by

$$\dot{x} = -\beta_1(y-x^2) + y(\beta_2+\gamma_2 x) + (\alpha_2+\gamma_2 c)x + c\beta_2, \quad \dot{y} = 2(y+c)(\alpha_2+\beta_2 x+\gamma_2 y),$$

where $c\,\alpha_2 = 0$. If $c = 0$ then the additional invariant line is written as $f_2 = y = 0$ and if $\alpha_2 = 0$, such line is $f_2 = y + c = 0$.

System $(P.2)$ has a Darboux invariant if there exist λ_1, λ_2 not all zero satisfying equation (2.9) with $s \in \mathbb{R}\setminus\{0\}$, and k_1, k_2 being the cofactors of $f_1 = 0$ and $f_2 = 0$, respectively. For $c = 0$, $k_1 = 2(\alpha_2+\beta_1 x+\gamma_2 y)$ and $k_2 = 2(\alpha_2+\beta_2 x+\gamma_2 y)$. Equation (2.9), with $s \neq 0$ has the solution

$$s = -2\alpha_2(\lambda_1+\lambda_2),\ \beta_2 = -\beta_1\lambda_1/\lambda_2,\ \gamma_2 = 0, \tag{2.45}$$

Taking $\lambda_1 = \beta_2$ and $\lambda_2 = -\beta_1$ the solution can be rewritten as

$$s = -2\alpha_2(\beta_2-\beta_1),\ \lambda_1 = \beta_2,\ \lambda_2 = -\beta_1,\ \gamma_2 = 0, \tag{2.46}$$

and the Darboux invariant is

$$I_8(x,y,t) = e^{2\alpha_2(\beta_1-\beta_2)t}(y-x^2)^{\beta_2}y^{-\beta_1}.$$

In this case we assume $\beta_2 - \beta_1 \neq 0$ otherwise system $(P.2)$ has a common factor. Moreover if $\alpha_2 = c = 0$ $(P.2)$ does not admit a Darboux invariant.

When $\alpha_2 = 0$ then $f_2 = y + c$ and the cofactors of $f_1 = 0$ and $f_2 = 0$ are, respectively, $k_1 = 2(c\gamma_2+\beta_1 x+\gamma_2 y)$ and $k_2 = 2(\beta_2 x+\gamma_2 y)$. In this case equation (2.9) has only one solution

$$s = -2c\gamma_2\lambda_1,\ \beta_2 = \beta_1,\ \lambda_2 = -\lambda_1.$$

So taking $\lambda_1 = 1$ we get the Darboux invariant

$$I_7(x,y,t) = e^{-2c\gamma_2 t}(y-x^2)(y+c)^{-1}.$$

From now on we study the possible global phase portraits for systems (P) when they have a Darboux invariant. We start studying system $(P.1)$. Remember that such system is given by

$$\dot{x} = x(\alpha_2+\beta_2 x+\gamma_2 y), \qquad \dot{y} = \alpha_1(y-x^2)+2\alpha_2 x^2+2y(\beta_2 x+\gamma_2 y).$$

We consider two cases: $\gamma_2 \neq 0$ and $\gamma_2 = 0$. If $\gamma_2 \neq 0$ we assume $\gamma_2 = 1$. In this last case system $(P.1)$ have at most four singular points, given by

$$z_1 = (0,0), \qquad z_2 = (0, -\alpha_1/2),$$

$$z_3 = \left(-(\beta_2+\sqrt{\beta_2^2-4\alpha_2})/2,\ (\beta_2^2-2\alpha_2+\beta_2\sqrt{\beta_2^2-4\alpha_2})/2\right),$$

$$z_4 = \left(-(\beta_2-\sqrt{\beta_2^2-4\alpha_2})/2,\ (\beta_2^2-2\alpha_2-\beta_2\sqrt{\beta_2^2-4\alpha_2})/2\right).$$

Observe that unless of the change $x = -X$, $y = Y$ we can assume $\beta_2 \geq 0$. Let $l_1 = \alpha_1$, $l_2 = \alpha_2$, $l_3 = \beta_2^2 - 4\alpha_2 - \beta_2\sqrt{\beta_2^2 - 4\alpha_2}$ and $l_4 = \alpha_1 - 2\alpha_2$ be. It follows from Proposition 2.5.7 (i) $l_4 \neq 0$. Moreover

- z_1 has eigenvalues l_1 and l_2;
- z_2 has eigenvalues $-l_1$ and $-l_4$;
- z_3 has eigenvalues l_4 and $(\beta_2^2 - 4\alpha_2 + \beta_2\sqrt{\beta_2^2 - 4\alpha_2})/2$;
- z_4 has eigenvalues l_3 and l_4,

so $l_1^2 + l_2^2 \neq 0$ and the topological type of the finite singular points can be studied using the Hartman-Grobman Theorem and Theorem 2.19 of (DUMORTIER; LLIBRE; ARTÉS, 2006).

With respect to the position of the finite singularities, z_1 is on the intersection of the parabola and the straight line, z_2 is on the straight line, and z_3, z_4 are on the parabola.

In the local chart U_1 system $(P.1)$ is written as

$$\dot{u} = u^2 + \beta_2 u + (\alpha_1 - \alpha_2)uv + 2\alpha_2 - \alpha_1, \qquad \dot{v} = -v(\alpha_2 v + u + \beta_2),$$

which has at most two singular points when $v = 0$, namely

$$u_1 = (-\beta_2 - \sqrt{\beta_2^2 + 4(\alpha_1 - 2\alpha_2)}/2, 0), \quad u_2 = (-\beta_2 + \sqrt{\beta_2^2 + 4(\alpha_1 - 2\alpha_2)}/2, 0).$$

The eigenvalues associated to u_1 are $-\sqrt{\beta_2^2 + 4l_4}$ and $-(\beta_2 - \sqrt{\beta_2^2 + 4l_4})/2$ while the eigenvalues associated to u_2 are $\sqrt{\beta_2^2 + 4l_4}$ and $-(\beta_2 + \sqrt{\beta_2^2 + 4l_4})/2$.

Since we are assuming $\beta_2 \geq 0$ it follows that when $\beta_2^2 + 4l_4 > 0$ the point u_2 is a saddle and it is not difficult to see that if $l_4 > 0$, then u_1 is a saddle, and if $l_4 < 0$, u_1 is a stable node. When $\beta_2^2 + 4l_4 = 0$ u_1 and u_2 coalesce and we conclude that this point is a saddle-node, using Theorem 2.19 (DUMORTIER; LLIBRE; ARTÉS, 2006). When $\beta_2^2 + 4l_4 < 0$ there is no infinite points in the local chart U_1.

In the local chart U_2 the origin $(0,0)$ is a stable node.

Observe that $l_1, l_2, l_3, l_4, \beta_2^2 - 4\alpha_2$ and $\beta_2^2 + 4l_4$ are bifurcation surfaces, i.e. where topological changes in the global phase portrait of (P.1) can happen. To draw all non–equivalent phase portraits of system (P.1) we split the study in three cases: $\beta_2^2 - 4\alpha_2 > 0$, $\beta_2^2 - 4\alpha_2 = 0$ and $\beta_2^2 - 4\alpha_2 < 0$.

Choosing a representative of each region defined by such surfaces we have a configuration of finite and infinite points. Considering the behavior of the separatrices of these systems we obtain all possible phase portraits when $\beta_2^2 - 4\alpha_2 > 0$, thus we obtain the 40 phase portraits described in Figures 13 and 14 and the phase portraits $41 - 50$ of Figure 18. We study all these cases bellow.

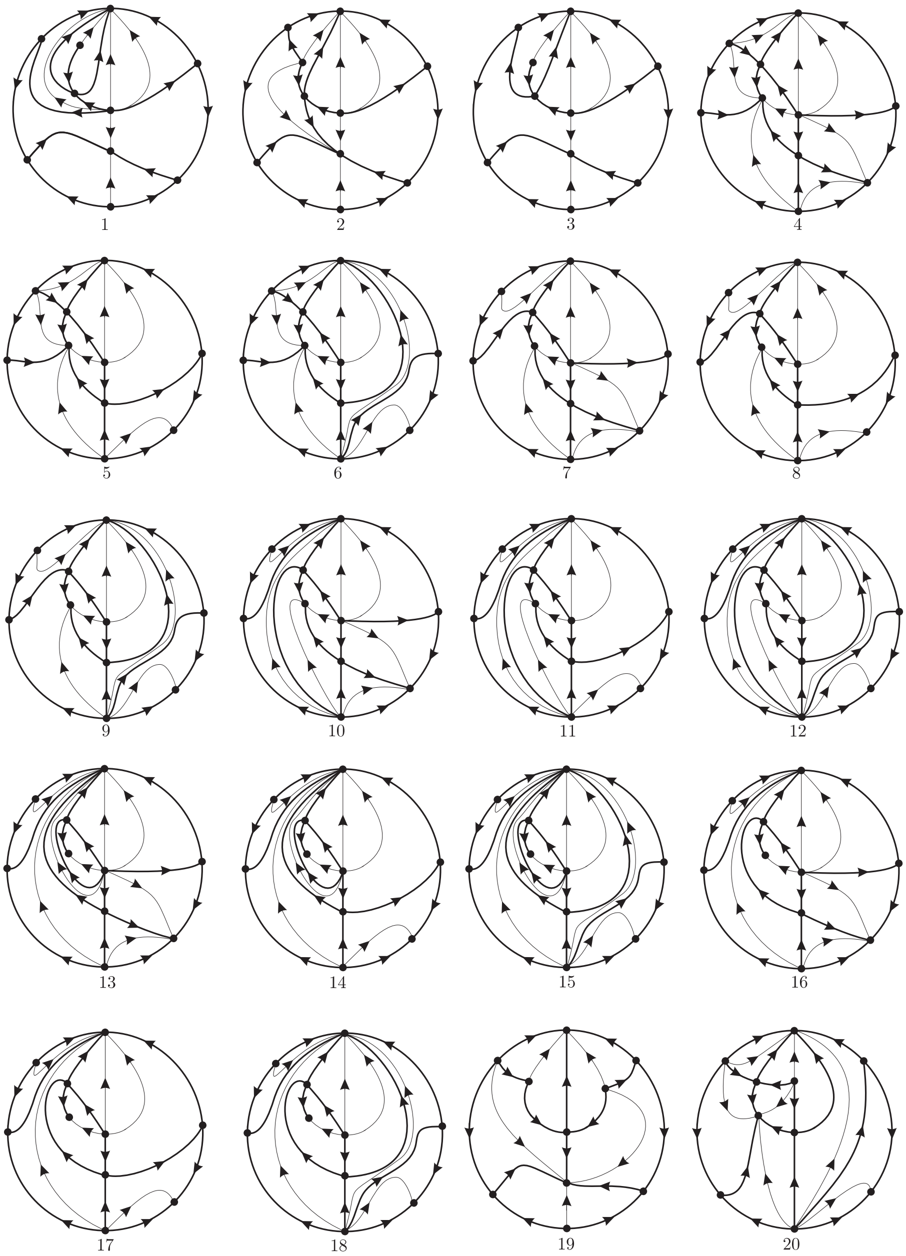

Figure 13 – Phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 > 0$.

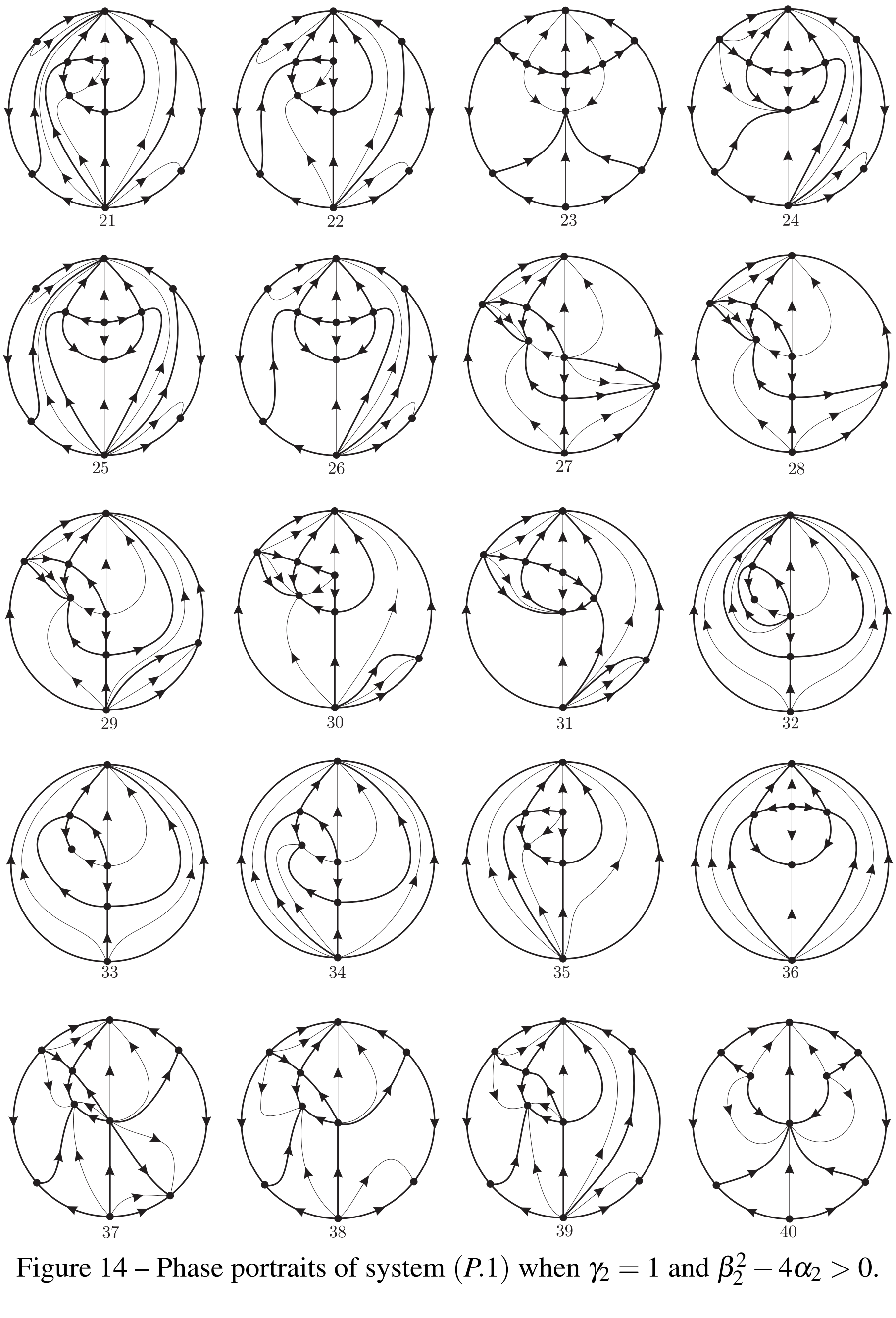

Figure 14 – Phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 > 0$.

Among the phase portraits $1-18$ of Figure 13, we claim that 1 and 3, as well as 7 to 18, are not realizable. Indeed these 18 phase portraits, $1-3$ present the possible combinations when the singular points in the local chart U_1 are both saddles. In the finite part we have z_1 and z_3 unstable nodes, z_2 is a stable node and z_4 is a saddle. So we have $l_1, l_2, l_4 > 0$ and $l_3 < 0$. In phase portrait 1 of Figure 13, consider the straight line joining the finite singular point z_3 to the infinity singular point u_1 as shows Figure 15. We can see that near the singular point z_3 but on opposite sides, the vector field has the same direction, which contradicts Lemma 2.5.6. So the phase portrait 1 of Figure 13 is not realizable. With the same argument the portrait 3 of Figure 13 is also not realizable. So phase portrait 2 of Figure 13 is the only realizable and corresponds to phase portrait PL.1.1 of Figure 3.

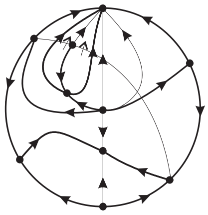

Figure 15 – The straight line joining the finite singular point z_3 to the infinity singular point u_1 in phase portrait 1 of Figure 13.

Considering the phase portraits $4-18$ of Figure 13 we shall prove that $7-18$ are not realizable. First consider the phase portrait 7 and the straight line joining the middle point between the infinity singular points u_1 and u_2 and the middle point between the finite singular points z_3 and z_4 as shows Figure 16. By Lemma A.1.2 this line should have at most two points of contact with the vector field, which does not occur. In Figure 16 we can see at least four contact points, represented by the smaller points that are not singularities of the system. This fact guarantees that the $\omega-$limit set of u_2 is the finite point z_4 on the parabola. So phase portraits $7-18$ are not realizable using similar arguments. So among the phase portraits $4-18$ only $4,5$ and 6 are realizable, which correspond, respectively to phase portraits PL.1.2, PL.1.3 and PL.1.4 of Figure 3. The values of the parameters that realize these systems can be found in Table 2.

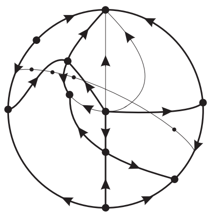

Figure 16 – The straight line joining the middle point between the infinity singular points u_1 and u_2 and the middle point between the finite singular points z_3 and z_4 in phase portrait 7 of Figure 13.

The phase portraits $19-20$ in Figure 13 and $21-26$ in Figure 14 are topologically equivalent to one of the phase portraits $1-18$ in Figure 13 so they can be realizable or not, depends on their configuration. In Table 1 we present the relation among the equivalent phase portraits of system $(P.1)$ when $c \neq 0$. In the case where they are topologically equivalent to a realizable phase portrait, we need not consider the study again. However if they are topologically equivalent to a phase portrait which was not realizable, we need to study it.

Considering the same straight line used to prove the non-realization of phase portraits $7-18$ of Figure 13 we apply Lemma A.1.2 to conclude that $21, 22, 25$ and 26 of Figure 14 are not realizable.

The phase portraits $27-31$ in Figure 14 present all the possibilities when there are four finite singular points and one infinite singular point on the local chart U_1. Phase portraits 27, 28 and 29 are realizable and correspond to phase portraits PL.1.5, PL.1.6 and PL.1.7 of Figure 3. The values of the parameters that realize these systems can be found in Table 2. Moreover 30 and 31 are topologically equivalent to one of these three phase portraits.

Phase portrait	Topologicaly equivalent to
19	2
20	6
21	12
22	9
23	2
24	6
25	12
26	9
30	29
31	29
35	34
36	34
60	50

Table 1 – Table of relations among all the possible phase portraits of system $(P.1)$ when $c \neq 0$.

Finally if there are four finite singular points and the local chart U_1 has no singular point we get the phase portraits $32-36$ in Figure 14. For phase portraits 32 and 33 of Figure 14 we consider the straight line $x = z_4^1$ where the finite singularity z_4 is $z_4 = (z_4^1, z_4^2)$, and apply Lemma 2.5.6 to see that they are not realizable (see Figure 17).

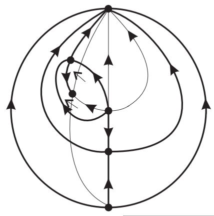

Figure 17 – The straight line $x = z_4^1 = -(\beta_2 - \sqrt{\beta_2^2 - 4\alpha_2})/2$ in phase portrait 32 of Figure 14.

Moreover the phase portraits 35 and 36 are topologically equivalent to the phase portrait 34 which is the only realizable phase portrait for this case and it is represented by PL.1.8 in Figure 3. The values of the parameters that realize this system can be found in Table 2.

For $\beta_2^2 - 4\alpha_2 > 0$ we consider the cases with three finite singular points. When $z_1 = z_2$ the origin is a saddle-node and there are ten possible phase portraits, namely $37-40$ in Figure 14 and $41-46$ in Figure 18. But since the nodal sector of the saddle node must have its orbits tangent to its separatrix, the phase portraits 37 and 38 in Figure 14 are not realizable. In other words the separatrices of the saddle-node z_1 must be on the invariant parabola. With the same argument the phase portraits $41, 42, 45$ and 46 of Figure 18 also are not realizable. So when $z_1 = z_2$ the realizable phase portraits are $39, 40$, 43 and 44 of Figure 18, corresponding to PL.1.9, PL.1.10, PL.1.11 and PL.1.12, in Figure 3, respectively. The values of the parameters that realize these systems can be found in Table 2.

When there are three finite singularities with $z_1 = z_4$ then by continuity we have the phase portraits $47, 48, 49$ and 50 of Figure 18. All these for phase portraits are realizable and correspond, to PL.1.13, PL.1.14, PL.1.15 and PL.1.16 in Figure 3, respectively. The values of the parameters that realize these systems can be found in Table 2

For $\beta_2^2 - 4\alpha_2 = 0$ there is another case with three finite singularities that correspond to the case $z_3 = z_4$. Here we can have ten phase portraits, given by $51-60$ in Figure 18. The phase portraits $51, 52$ and 55 are realizable and corresponds, respectively, to PL.1.17, PL.1.18 and PL.1.19 in Figure 3. The values of the parameters that realize these systems can be found in Table 2. The phase portraits 53 and 54 are not realizable. The ideia again is to use Lemma 2.5.6 with the straight line joining the origin of the local chart U_3 to the singular point u_2 of the local chart U_1. By Figure 19 and this lemma the phase portraits 53 and 54 are not realizable.

Considering the phase portraits 56 and 57 we will show that they are not realizable. Take the straight line passing through the origin of the local chart U_1 and the infinite singular point $u_1 = u_2$ (see Figure 20). The contact points on this straight line contradicts Lemma 2.5.6 so the phase portraits 56 and 57 are not realizable. About the phase portraits 58 and 59, considering the straight line passing through the points z_1 and z_3 we have Figure 21 that is a contradiction with

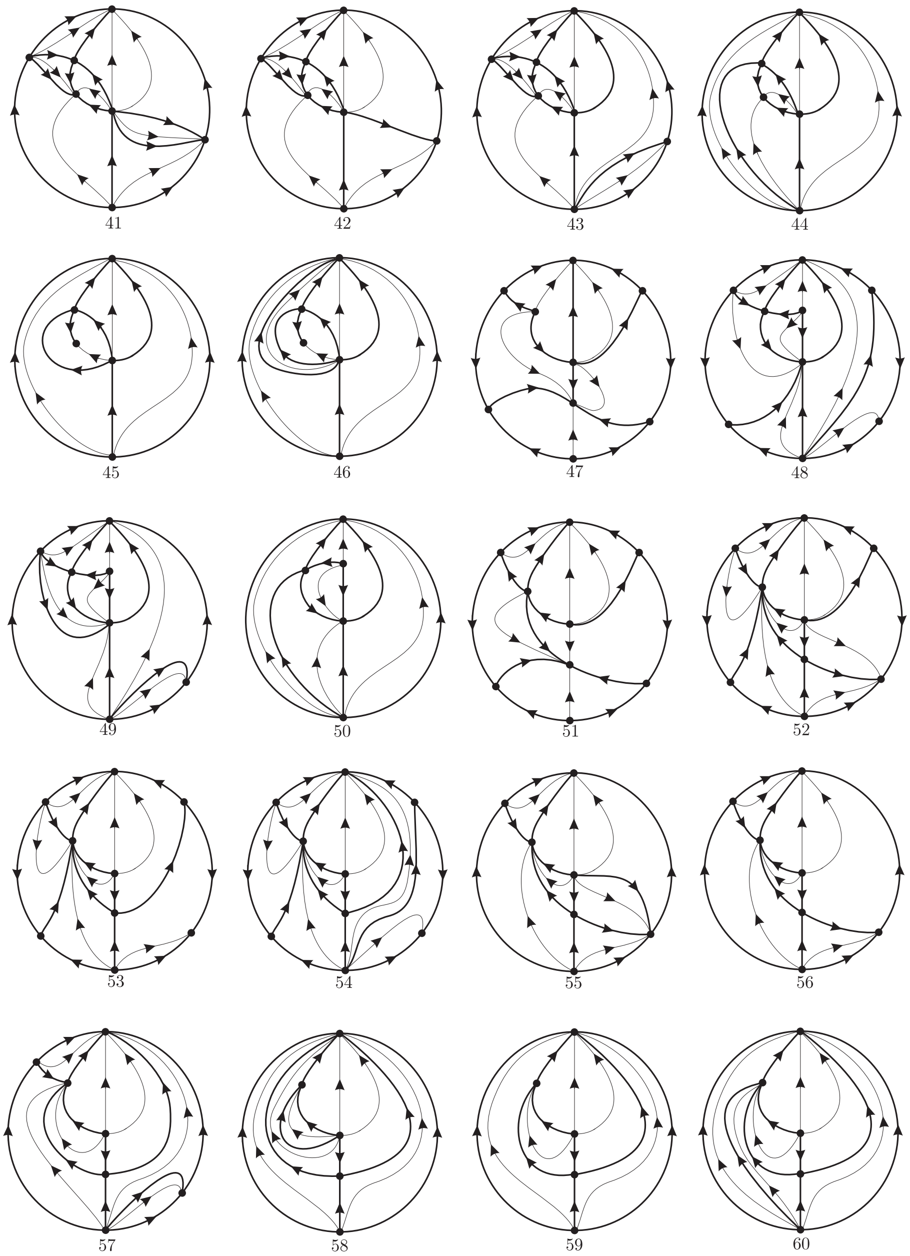

Figure 18 – Phase portraits 41 – 50 corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 > 0$;
Phase portraits 51 – 60 corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 = 0$.

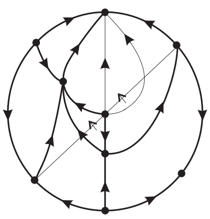

Figure 19 – The straight line connecting the origin of the local chart U_3 with the singular poin u_2 of the local chart U_1 in phase portrait 53 of Figure 18.

Lemma A.1.2. So they are not realizable. The phase portrait 60 is topologically equivalent to 50 of Figure 18.

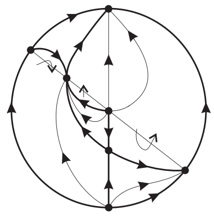

Figure 20 – The straight line connecting the origin of the local chart U_3 with the singular poin $u_1 = u_2$ of the local chart U_1 in phase portrait 56 of Figure 18.

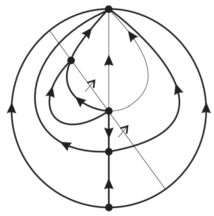

Figure 21 – The straight line passing through the points z_1 and z_3 in phase portrait 58 of Figure 18.

If $z_3 = z_4$ and $z_1 = z_2$ we have the phase portraits $61, 62$ and 63 of Figure 22. But using the straight line joining z_1 and z_3 as done in Figure 21 and applying Lemma A.1.2 we see that 61 and 62 are not realizable. The phase portrait 63 is realizable and corresponds to PL.1.20 in Figure 3. The values of the parameters that realize this system can be found in Table 2.

For $\beta_2^2 - 4\alpha_2 < 0$ the points z_3 and z_4 are complex. The possible phase portraits are described by $64 - 72$ of Figure 22. The phase portraits $64, 65, 68$ and 71 are realizable and corresponds, respectively, to PL.1.21, PL.1.22, PL.1.23 and PL.1.24 of Figure 3. The values of the parameters that realize these systems can be found in Table 2. To prove that the phase portraits $66, 67, 69$ and 70 are not realizable, it is enough to consider the straight line passing through the origin of the local chart U_3 and the infinity singularity $u_1 = u_2$ of the local chart U_1 (see Figure 23). This straight line generates a contradition with Lemma 2.5.6 so the phase portraits $66, 67, 69$ and 70 are not realizable.

To end the case $\gamma_2 = 1$ we consider the case where there is only one finite singular point. Using Theorem 2.19 of (DUMORTIER; LLIBRE; ARTÉS, 2006) we can see that the point is a saddle, which generates phase portrait 72 of Figure 22 which corresponds to phase portrait PL.1.25 of Figure 4. The values of the parameters that realize this system can be found in Table 2.

Now we consider the case $\gamma_2 = 0$. The system is

$$\dot{x} = x(\alpha_2 + \beta_2 x), \qquad \dot{y} = \alpha_1 (y - x^2) + 2x(\alpha_2 x + \beta_2 y). \tag{2.47}$$

When $\alpha_1 = 0$ such system has a common factor so assume $\alpha_1 = 1$. By the change $x = -X, y = Y$ it is enough to consider the case $\beta_2 \geq 0$.

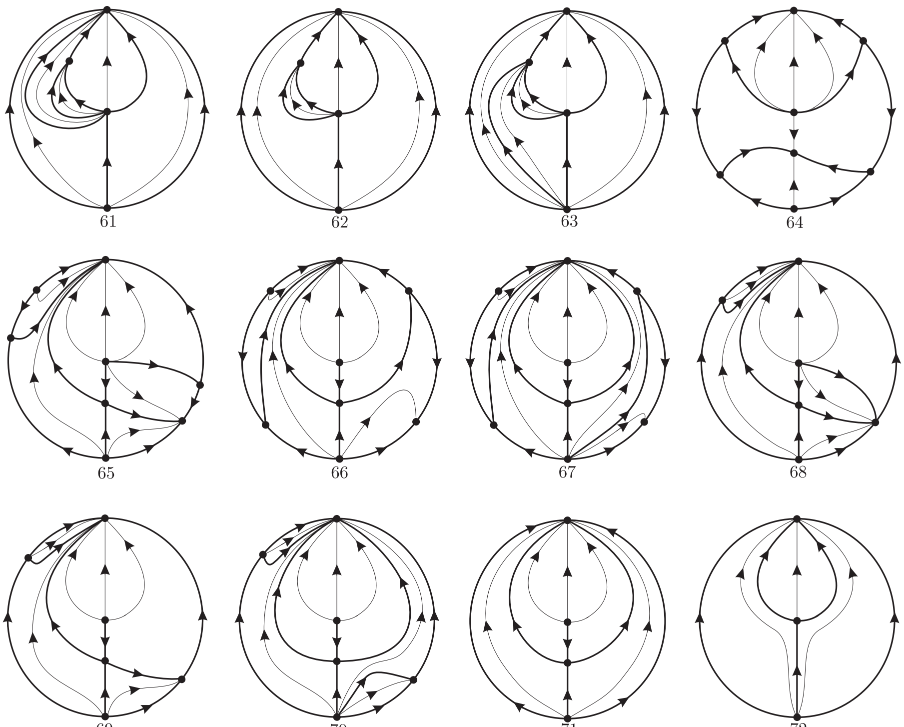

Figure 22 – Phase portraits 61 – 63 corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 = 0$;
Phase portraits 64 – 72 corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 < 0$.

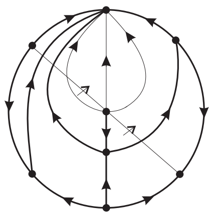

Figure 23 – The straight line connecting the origin of the local chart U_3 with the singular poin $u_1 = u_2$ in the local chart U_1 in phase portrait 66 of Figure 22.

Assuming $\beta_2 > 0$. In the finite part the points $z_1 = (0,0)$ and $z_2 = \left(-\alpha_2/\beta_2, (\alpha_2/\beta_2)^2\right)$ are the singular points and the system has an additional invariant straight line, given by $f_3 = x + \alpha_2/\beta_2 = 0$. Defining $l_1 = \alpha_2$ and $l_2 = 1 - 2\alpha_2$ the eigenvalues associated to z_1 are 1 and l_1, while the eigenvalues associated to z_2 are $-l_1$ and l_2. We assume $l_2 \neq 0$ (otherwise such system has a common factor and it is equivalent to a linear system).

In the local chart U_1 the unique singular point is $u_1 = (l_2/\beta_2, 0)$ and it is a saddle. In the local chart U_2 the compactified system is

$$\dot{u} = u((1-2\alpha_2)u^2 + (\alpha_2 - 1)v - \beta_2 u), \qquad \dot{v} = v((1-2\alpha_2)u^2 - 2\beta_2 u - v).$$

The origin $(0,0)$ is a linearly zero singularity. Doing the blow up $u = UV$, $v = V$ and rescaling by V we get the system

$$\dot{U} = U(\alpha_2 + \beta_2 U), \qquad \dot{V} = V((1 - 2\alpha_2 U^2 V) - 2\beta_2 U - 1).$$

When $V = 0$ the singularities are $\overline{u}_1 = (0,0)$ and $\overline{u}_2 = (-\alpha_2/\beta_2, 0)$. The eigenvalues associated to $\overline{u}_1$ are -1 e l_1 while the eigenvalues of $\overline{u}_2$ are $-l_1$ e $-l_2$. The blowing down process is described in Figure 24 (1)-(4) according to the signs of l_1 and l_2.

When $\beta_2 = 0$ the point z_1 is the unique finite singular point, being a saddle or an unstable node depending on the sign of l_1. In the local chart U_1 there is no singular point and the origin $(0,0)$ of U_2 is linearly zero. To study such point we apply *blow ups*, in Figure 24 is described the blowing down (5) and (6).

Summarizing the study done previously we get the local behaviour at origin of U_2:

1. $\beta_2 > 0$, $l_1 > 0$ and $l_2 > 0$: the origin of U_2 has two elliptic sectors;

2. $\beta_2 > 0, l_1 > 0$ and $l_2 < 0$: the origin of U_2 has two hyperbolic sectors;

3. $\beta_2 > 0, l_1 < 0$ and $l_2 > 0$: the origin of U_2 has two elliptic sectors;

4. $\beta_2 > 0, l_1 = 0$ and $l_2 > 0$: the origin of U_2 has two elliptic sectors.

5. $\beta_2 = 0, l_1 > 0$: the origin of U_2 has two hyperbolic sectors;

6. $\beta_2 = 0, l_1 < 0$: the origin of U_2 has two elliptic sectors;

By continuity and the above analysis we conclude that the case (3) is topologically equivalent to case (1) and the cases $(1),(2),(4),(5)$ and (6) correspond, respectively, to the phase portraits PL.1.26, PL.1.27, PL.1.28, PL.1.29 e PL.1.30 of Figure 4. Table 4 has the values of the parameters that realizes the phase portraits of system $(P.1)$

System $(P.2)$ with $c \neq 0$ has a Darboux invariant if $\gamma_2 \neq 0$, and it can be written as

$$\dot{x} = \beta_1 (x^2 + c) + \gamma_2 x(y + c), \qquad \dot{y} = 2(y + c)(\beta_1 x + \gamma_2 y).$$

Note that if $\beta_1 = 0$ such system has a common factor so we can assume $\beta_1 = 1$. Applying the change of coordinates $x = -X, y = Y$ and rescaling the time we can assume $\gamma_2 > 0$.

If $c < 0$ the system has three finite singular points $z_1 = (-1/\gamma_2, 1/\gamma_2^2)$, $z_2 = (-\sqrt{-c}, -c)$ and $z_3 = (\sqrt{-c}, -c)$. Otherwise, only z_1.

Defining $l_1 = c \neq 0$ and $l_2 = 1 + c\gamma_2^2$ the eigenvalues associated to z_1 are $2\gamma_2 l_1$ and l_2/γ_2, the eigenvalues associated to z_2 are $-2\sqrt{-c}$ and $-2(\gamma_2 c + \sqrt{-c})$; the eigenvalues associated to z_3 are $2\sqrt{-c}$ and $-2(\gamma_2 c - \sqrt{-c})$. So when $c < 0$ the point z_3 exists and it is an unstable node.

In the local chart U_1 we have two singular points $u_1 = (0,0)$ being a hyperbolic saddle and $u_2 = (-1/\gamma_2, 0)$ being a saddle-node. In the local chart U_2 the origin is a stable node.

When $l_2 = 0$ then $z_1 = z_3$ is a semi-hyperbolic node and the infinity part does not change. Note that z_1 is a saddle-node in this case. So by continuity and the reasoning above, if $c > 0$

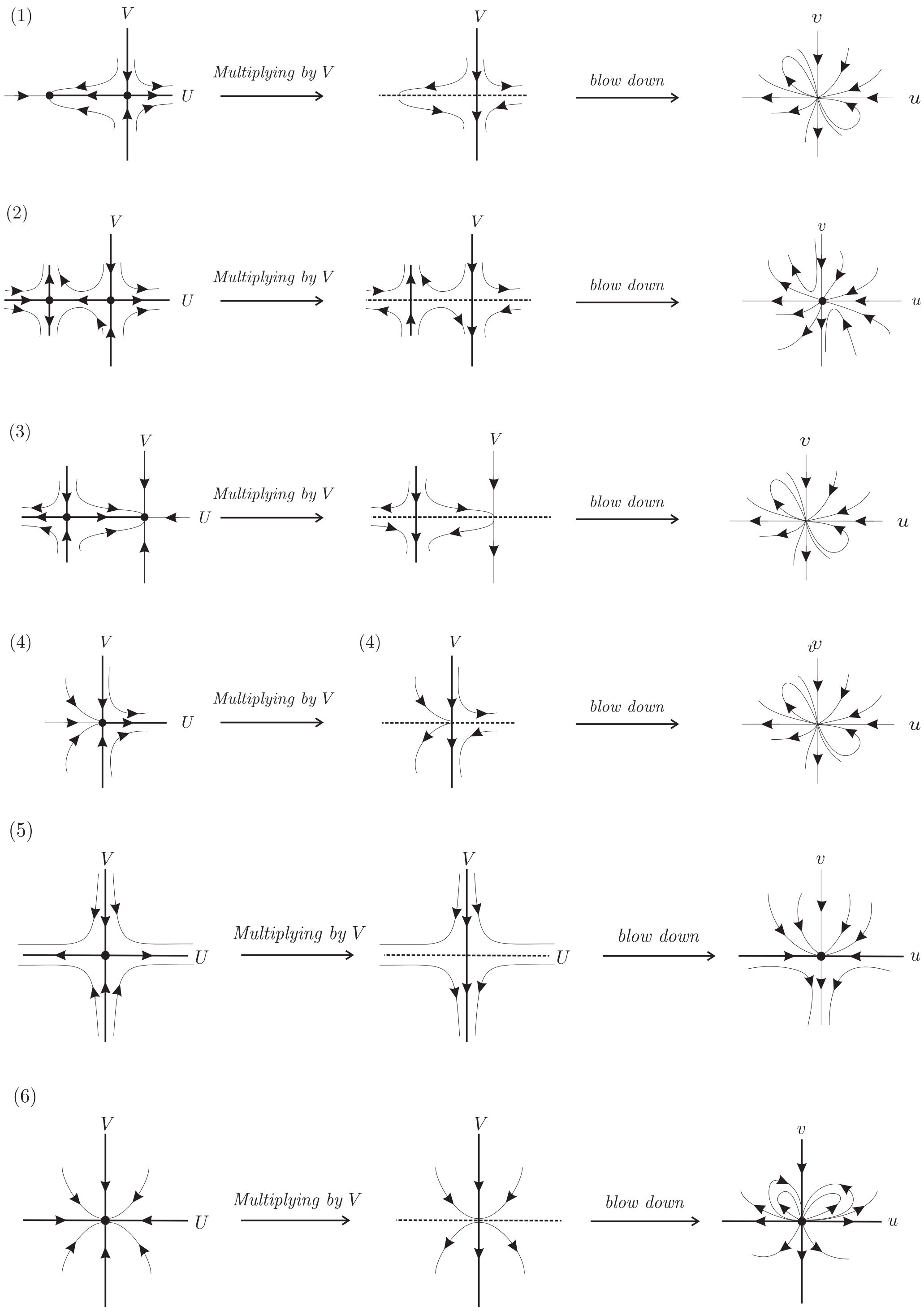

Figure 24 – *Blow down* of system $(P.1)$ when $\gamma_2 = 0$.

	γ_2	β_2	α_2	α_1
PL.1.1	1	1	1/8	1
PL.1.2	1	1	1/16	1/16
PL.1.3	by continuity			
PL.1.4	1	1	1/16	1/150
PL.1.5	1	1/2	3/64	1/32
PL.1.6	by continuity			
PL.1.7	1	1	-3/8	-1
PL.1.8	1	1	3/16	1/16
PL.1.9	1	1	1/16	0
PL.1.10	1	1	-1	0
PL.1.11	1	1	1/18	0
PL.1.12	1	1	3/16	0
PL.1.13	1	1	0	1
PL.1.14	1	1	0	-1/8
PL.1.15	1	1	0	-1/4
PL.1.16	1	1	0	-1
PL.1.17	1	1	1/4	1
PL.1.18	1	1	1/4	3/8
PL.1.19	1	1	1/4	1/4
PL.1.20	1	1	1/4	0
PL.1.21	1	1	2	5
PL.1.22	1	3	4	6
PL.1.23	1	1	9/8	2
PL.1.24	1	1	2	13/4
PL.1.25	1	1	2	0

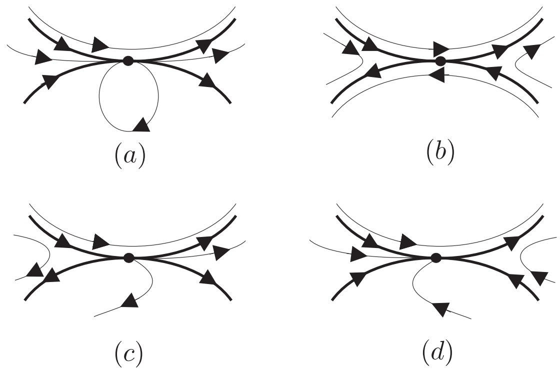

Figure 25 – Local phase portraits

we have phase portrait PL.2.1 of Figure 4 which is realizable with $c = \gamma_2 = 1$. When $c < 0$ and $l_2 \neq 0$ the system has two possible phase portraits, also described in Figure 4: PL.2.2 (realizable with $c = -1/2$ and $\gamma_2 = 1$) and PL.2.3 (realizable with $c = -2$ and $\gamma_2 = 1$).

Finally if $c < 0$ and $l_2 = 0$, we see that the line $y + c = 0$ is one of the separatix of the saddle-node. So the only possible phase picture is PL.2.4 (realizable with $c = -1$ and $\gamma_2 = 1$).

Now we study the global phase portraits of systems $(P.2)$ when $c = 0$ and they have a Darboux invariant. The differential system is

$$\dot{x} = -\beta_1(y - x^2) + \beta_2 y + \alpha_2 x, \qquad \dot{y} = 2y(\beta_2 x + \alpha_2).$$

Since $\alpha_2 \neq 0$ we take $\alpha_2 = 1$. Moreover doing the change of coordinates $x = -X, y = Y$ we can assume $\beta_2 \geq 0$. The system has at most three finite singular points, namely, $z_1 = (0,0)$ and $z_2 = (-1/\beta_1, 0)$ and $z_3 = (-1/\beta_2, 1/\beta_2^2)$. The point z_1 has eigenvalues 2 and 1, so it is an unstable node. On the other hand the topological type of z_2 and z_3 depends on the numbers $l_1 \doteq \beta_1$ and $l_2 \doteq \beta_1 - \beta_2 \neq 0$. Indeed the point z_2 has eigenvalues -1 and $2l_2/l_1$ and z_3 has the eigenvalues -1 and $-2l_2/l_1$.

In the local chart U_1 the system has $u_1 = (0,0)$ as a singularity with eigenvalues $-l_1$ and $-l_3$, where $l_3 \doteq \beta_1 - 2\beta_2$.

In the local chart U_2 the compactified system has the origin as a *nilpotent* singularity. This mean that the linear part of the system, evaluated in $(0,0)$, is not null but their eigenvalues are both equal to zero. To classify this type of singular point we use Theorem 3.5 of (DUMORTIER; LLIBRE; ARTÉS, 2006). This result use two functions, $F(u) = a_M u^M + o(u^M)$ and $G(u) = b_N u^N + o(u^N)$, defined from the differential system. In short the caracterization is done using a_M, b_N and the natural numbers M, N.

For the compactified system in the local chart U_2 these functions are

$$G(u) = -\frac{2(\beta_2 - 3\beta_2)}{l_2} u + \frac{5l_3}{l_2^2} u^2, \qquad F(u) = \frac{2\beta_2 l_3}{l_2^2} u^3 + \frac{2l_3^2}{l_2^3} u^4.$$

So when $l_3 > 0$ the origin $(0,0)$ is a saddle as in (b) of Figure 25. If $l_3 < 0$ the origins consists of one hyperbolic and one elliptic sector as in (a) of Figure 25. By continuity, when $l_1 > 0$ and

$l_3 > 0$ we have the phase portrait PL.2.5 of Figure 4 (realizable with $\beta_1 = 4$ and $\beta_2 = 1$). If $l_3 < 0$ we have the phase portraits PL.2.6 (realizable with $\beta_1 = 3/2$ and $\beta_2 = 1$) and PL.2.7 (realizable with $\beta_1 = 1/2$ and $\beta_2 = 1$) of Figure 4. Now if $l_1 < 0$ the only possibility is $l_3 < 0$ and we have the phase portrait PL.2.8 (realizable with $\beta_1 = -1$ and $\beta_2 = 1$) of Figure 4.

If $l_1 = 0$ the point z_2 goes to the infinity and collide with u_1 becoming a saddle-node. Moreover $l_1 = 0$ implies $l_3 < 0$, so the origin of U_2 has a hyperbolic and one elliptic sector. This case corresponds to phase portrait PL.2.9 of Figure 4, realizable with $\beta_1 = 0$ and $\beta_2 = 1$.

If $\beta_2 = 0$ the point z_3 goes to the infinity and collide with the origin of U_2 becoming $(0,0)$ a nilpotente saddle-node as (c) or (d) in Figure 25. Moreover the only possible phase portrait is given by PL.2.10 of Figure 4, realizable with $\beta_1 = 1$ and $\beta_2 = 0$).

Finally when $l_3 = 0$ then the infinity if filled of singular points, without special singularities and the corresponding phase portrait is PL.2.11 of Figure 4 (realizable with $\beta_1 = 2$ and $\beta_2 = 1$). □

Proposition 2.5.8 (LV)**.** Each real polynomial differential system having two real lines that intersect at a single point and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as

(i) $(LV.1)$ with $\alpha(q-\beta) \neq$ and Darboux invariant

$$I_{10}(x,y,t) = e^{\alpha(q-\beta)t}\, y^{\beta}\, x^{\beta-q+r}(y-x)^{-(\beta+r)},$$

(ii) $(LV.2)$ with $c = q = 0$, $p \neq 0$ and Darboux invariant

$$I_{11}(x,y,t) = e^{-pt} x y^{-r},$$

(iii) $(LV.2)$ with $c = -1$ and Darboux invariant

$$I_{12}(x,y,t) = e^{t}\, y\,(y-1)^{-1},$$

(iv) $(LV.3)$ with $\alpha = -(\beta+1)$, $c\beta \neq 0$ and Darboux invariant

$$I_{13}(x,y,t) = e^{-c\beta t}\, y\,(y+ax+c)^{-1}.$$

Moreover there are 27 non–equivalent phase portraits in the Poincaré disc. They are in Figure 5.

Proof of Proposition 2.5.8 (LV). Let $f_1 = x = 0$, $f_2 = y = 0$ be the two real straight lines intersecting in a point. Considering system $(LV.1)$ the third line is $f_3 = y - x$ and the cofactors associated to f_1, f_2 and f_3 are, respectivelly, $k_1 = \alpha + ry + \beta x$, $k_2 = \alpha + y(\beta - q + r) + qx$ and $k_3 = \alpha + y(\beta - q + r) + \beta x$. One solution for equation $\lambda_1 k_1 + \lambda_2 k_2 + s = 0$ is

$$\lambda_2 = \frac{\beta\lambda_1}{\beta - q + r}, \lambda_3 = -\frac{(\beta+r)\lambda_1}{\beta - q + r}, s = \frac{\alpha(q-\beta)\lambda_1}{\beta - q + r},$$

Taking $\lambda_1 = \beta - q + r$ we obtain the Darboux invariant

$$I_{10}(x,y,t) = e^{\alpha(q-\beta)t}\, y^{\beta}\, x^{\beta-q+r}(y-x)^{-(\beta+r)}.$$

Now we analize system $(LV.2)$ that has $f_3 = y + c$ as the third invariant straigh line(remember that $c = 0$ or $c = -1$). Here the cofactors are $k_1 = p + qx + ry$, $k_2 = y + c$ and $k_3 = y$. If $c = 0$ then equation (2.9) has only one the solution

$$q = 0,\, \lambda_3 = -r\lambda_1 - \lambda_2,\, s = -p\lambda_1.$$

Taking $\lambda_1 = 1$ we get the Darboux invariant

$$I_{11}(x,y,t) = e^{-pt}xy^{-r}.$$

Otherwise if $c = -1$ then the more general solution is

$$\lambda_1 = 0,\, \lambda_3 = -\lambda_2,\, s = \lambda_2.$$

Taking $\lambda_2 = 1$ we obtain the Darboux invariant

$$I_{12}(x,y,t) = e^{t}y(y-1)^{-1}.$$

The last case to be considered is system $(LV.3)$ that has $f_3 = y + ax + c = 0$ as the third straight line. The cofactors are $k_1 = -\alpha(y + ax + c) - y$, $k_2 = \beta(y + ax + c) + ax$ and $k_3 = \beta y - a\alpha x$. Solving equation (2.9) we get the solution

$$\alpha = -(\beta+1),\, \lambda_2 = -\lambda_1 - \lambda_2,\, s = -c(\lambda_1 + \beta(\lambda_1 + \lambda_2)).$$

Taking $\lambda_1 = 0$ and $\lambda_2 = 1$ then we obtain the Darboux invariant

$$I_{13}(x,y,t) = e^{-c\beta t}\, y\,(y + ax + c)^{-1}.$$

We begin the study of the global phase portraits with systems $(LV.1)$ when they have a Darboux invariant. Remember that if system $(LV.1)$ has a Darboux invariant then $\beta - q \neq 0$ and $\alpha \neq 0$ so we can take $\alpha = 1$ getting

$$\dot{x} = x(1 + \beta x + ry), \qquad \dot{y} = y(1 + qx + (\beta - q + r)y). \tag{2.48}$$

Define $l_1 = (\beta - q)/(\beta - q + r)$, $l_2 = (\beta - q)/\beta$ and $l_3 = (\beta - q)/(\beta + r)$. The finite part presents at most four singularities

- $z_1 = (0,0)$ with eigenvalues both equal to 1;
- $z_2 = (0, -1/(\beta - q + r))$ with eigenvalues -1 and l_1;
- $z_3 = (-1/\beta, 0)$ with eigenvalues -1 and l_2;

- $z_4 = (-1/(\beta+r), -1/(\beta+r))$ with eigenvalues -1 and $-l_3$.

.

In the local chart U_1 the compactified system has two singular points, being $u_1 = (0,0)$ with eigenvalues $-\beta$ and $-(\beta-q)$ and $u_2 = (1,0)$ with eigenvalues $\beta-q$ and $-(\beta+r)$. Moreover in the local chart U_2 the origin $(0,0)$ is a singular point with eigenvalues $-(\beta-q)$ and $-(\beta-q+r)$. Thus when one of the finite singularities goes to infinity, it collides with u_1, u_2, or the origin of the local chart U_2.

When l_1, l_2 and l_3 are non-zero, the combinations between their signs generate the possible phase portraits of system (2.48). There are exactally three possible phase portraits, all of them described in Figure 5: LVL.1.1, realizable for $\beta = 1, q = r = 0$; LVL.1.2, realizable for $\beta = 1, q = r = -2$; LVL.1.3, realizable for $\beta = 1, q = -r = 3/4$.

Now we consider the case $\beta = -r \neq 0$. Here only the point z_4 goes to the infinity and collides with u_2 making it a semi hyperbolic saddle-node. There are two possible phase portraits, given by LVL.1.4 of Figure 5 (realizable with $\beta = 1, q = r = -1$) and LVL.1.5 of Figure 5 (realizable with $\beta = 2, q = 1, r = -2$). The cases where z_2 or z_3 goes to the infinity generate phase portraits equivalent to the previous ones.

Finally when two finite singular points go to the infinity (for example when $\beta = -r$ and $q = 0$), then there is only one phase portrait, given by LVL.1.6 of Figure 5. This last phase portrait is realizable for $\beta = 1, q = 0$ and $r = -1$.

Now consider the systems $(LV.2)$ when they have a Darboux invariant we split in two cases. First we consider the case $c = -1$, when the system is given by

$$\dot{x} = x(p + qx + ry), \qquad \dot{y} = y(y-1).$$

If $q \neq 0$ unless of the change $x = X/q$ we can assume $q = 1$. Considering $q = 1$ and defining $l_1 = p$, $l_2 = -(p+r)$ and $l_3 = r-1$ the system has at most four finite singular points, namely

- $z_1 = (0,0)$ with eigenvalues -1 and l_1;
- $z_2 = (0,1)$ with eigenvalues 1 and $-l_2$;
- $z_3 = (-p,0)$ with eigenvalues -1 and $-l_1$;
- $z_4 = (-p-r,1)$ with eigenvalues 1 and l_2.

In the local chart U_2 the origin $(0,0)$ is a singularity with eigenvalues -1 and l_3. In the local chart U_1 the sytem has two singularities if $l_3 \neq 0$: $u_1 = (0,0)$ being a hyperbolic unstable node and $u_2 = (1/l_3, 0)$ with eigenvalues 1 and $1/l_3$. Hence if $l_3 = 0$ the point u_2 collides with the origin of U_2 making it a semi-hyperbolic singularity of type saddle node. By continuity and using all the possible combinations of the signs of l_1, l_2 and l_3 when $q = 1$ and $l_3 \neq 0$ we obtain the phase portraits LVL.2.1- LVL.2.7 of Figure 5. When $l_3 = 0$, i.e., $r = 1$ has three possible phase portraits: LVL.2.8, LVL.2.9 and LVL.2.10 of Figure 5. The values of the parameters that realize these systems can be found in Table 3. Now it remains to study the case $q = 0$. Note that since the system cannot have commom factors it follows that l_1 and l_2 are different from zero. When $q = 0$ both the finite part and the analyzes in the local chart U_2 remain almost the same. The only difference in the finite part is that the singularities z_3 and z_4 go to infinity. However in the local chart U_1 the compactified system is

$$\dot{u} = -u((r-1)u + (p+1)v), \qquad \dot{v} = -v(pv + ru).$$

So the origin is a linearly zero singular point if $l_3 \neq 0$ and we apply the *blow up* doing the change of coordinates $u = U, v = UW$. The new system is

$$\dot{U} = -U^2((p+1)W + r - 1), \qquad \dot{W} = UW(W-1).$$

After eliminating the common factor U it remains two singular points on $U = 0$: $\overline{u_1} = (0,0)$ with eigenvalues -1 and $-l_3$, and $\overline{u_2} = (0,1)$ with eigenvalues 1 and l_2. Hence they are hyperbolic points and doing the *blow down* the origin of U_2 has (for $l_3 \neq 0$)

- two elliptic sectors if $\overline{u_1}$ is a saddle and $\overline{u_2}$ is a unstable node. This case corresponds to phase portrait LVL.2.11 of Figure 5;
- two elliptic sectors if $\overline{u_1}$ is a stable node and $\overline{u_2}$ is a saddle. This case corresponds to phase portrait LVL.2.12 of Figure 5;
- two parabolic sectors if $\overline{u_1}$ and $\overline{u_2}$ are both saddles and there is a saddle and a node as singular finite points. This case corresponds to phase portrait LVL.2.13 of Figure 5;
- two parabolic sectors if $\overline{u_1}$ and $\overline{u_2}$ are both saddles and there are two nodes as singular finite points. This case corresponds to phase portrait LVL.2.14 of Figure 5;
- six parabolic sectors if $\overline{u_1}$ and $\overline{u_2}$ are both saddles and there are two nodes as singular finite points. This case corresponds to phase portrait LVL.2.15 of Figure 5.

The last possibility when $c = -1$ is $q = 0$ and $l_3 = 0$. But when this happens the system has the infinity line $v = 0$ filled up of singular points. After eliminating the common factor v, in the local chart U_1 the point $u_1 = (0,0)$ is a singular point, with eigenvalues $-l_1$ and l_2. In the local chart U_2, After eliminating the common factor v, the origin is a singularity. By continuity the possible phase portraits are LVL.16 and LVL.2.17 of Figure 5. In Table 3 we put the values of the parameters that realizes each one of the phase portraits described in Figure 5.

Finally when $c = 0$ we get the differential system

$$\dot{x} = x^2, \qquad \dot{y} = y(p + rx), \tag{2.49}$$

	q	r	p
LVL.2.1	1	-1	1/2
LVL.2.2	1	2	1
LVL.2.3	1	-1	2
LVL.2.4	1	-1	1
LVL.2.5	1	2	-2
LVL.2.6	1	1/2	-1/2
LVL.2.7	1	0	0
LVL.2.8	1	1	1
LVL.2.9	1	1	-1/2
LVL.2.10	1	1	-1
LVL.2.11	0	-2	1
LVL.2.12	0	2	1
LVL.2.13	0	0	1
LVL.2.14	0	2	-1
LVL.2.15	0	3/4	-1/4
LVL.2.16	0	1	1
LVL.2.17	0	1	-1/2

Table 3 – Table of values for the parameters of system $(LV.2)$ when $c = -1$.

with $p \neq 0$. So we can take $p = 1$ and the system becomes a particular case of system (DL) of Theorem 2.5.1. The global phase portraits of this system will be done in the proof of Proposition 2.5.10 and the correponding phase portraits of system (2.49) are described by DL.1, DL.2 and DL.3 of Figure 6.

To end the proof of Proposition 2.5.8 we study the glogal phase portraits of systems $(LV.3)$. When $(LV.3)$ has a Darboux invariant the parameter α must be equal to $-(\beta+1)$ so the differential system is

$$\dot{x} = x(ax + \beta(y + ax + c) + c), \qquad \dot{y} = y(ax + \beta(y + ax + c)).$$

In the finite part there are three singular points, namely $z_1 = (0,0)$, $z_2 = (0,-c)$ and $z_3 = (-c/a, 0)$ (remember that $ac \neq 0$). Defining $l_1 = c\beta \neq 0$ and $l_2 = c(\beta+1) \neq 0$, then the eigenvalues of the z_1 are l_1 and l_2; the eigenvalues of z_2 are c and $-l_1$, and the eigenvalues associated to z_3 are $-c$ and $-l_2$.

In the local chart U_1 the compactified system becomes

$$\dot{u} = -cuv, \qquad \dot{v} = -v(cv + \beta(u + cv + a) + a).$$

Hence the line $v = 0$ is filled of singular points after eliminating the common factor v there are no singular points. The same happens in the local chart U_2. So by continuity the only possible phase portrait is LVL.3.1 of Figure 5, which is realizable for $\beta = 1$ and $a = c = -1$. $\square$

Proposition 2.5.9 (RPL)**.** Each real planar quadratic differential system with two parallel real straight lines and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as system (RPL) and it has the Darboux invariant

$$I_{14}(x,y,t) = e^{2t}(x+1)(x-1)^{-1}.$$

Moreover there are 17 non–equivalent phase portraits in the Poincaré disc for this system. They are described by RPL.1–RPL.17 in Figure 6.

Proof. Let $f_1 = x + 1 = 0$, $f_2 = x - 1 = 0$ and $f_3 = y = 0$ be the three invariant straight lines. The cofactors of f_1, f_2 and f_3 are, respectivelly, $k_1 = x - 1, k_2 = x + 1, k_3 = \alpha + \beta x + \gamma y$. With these cofactors equation (2.9) with $s \in \mathbb{R} \setminus \{0\}$ has two solutions, namely

$$s_1 = \{\gamma = 0,\, s = 2\lambda_1 + (\beta - \alpha)\lambda_3,\, \lambda_2 = -(\lambda_1 + \beta\lambda_3)\}$$

$$s_2 = \{s = 2\lambda_1,\, \lambda_2 = -\lambda_1,\, \lambda_3 = 0\}.$$

Since the second solution s_2 is more general we conclude that every quadratic system that has two real parallel straight lines and a third real straight line as invariant straight lines also has a Darboux invariant. Taking $\lambda_1 = 1$ we get the invariant

$$I_{14}(x,y,t) = e^{2t}(x+1)(x-1)^{-1}.$$

To draw the possible global phase portraits, remember that the system is

$$\dot{x} = x^2 - 1, \qquad \dot{y} = y(\alpha + \beta x + \gamma y).$$

When $\gamma \neq 0$ we can take $\gamma = 1$ (indeed, just do the change $x = X, y = Y/\gamma$). So the system can present at most four finite singularities, namely, $z_1 = (-1,0)$, $z_2 = (-1, \beta - \alpha)$, $z_3 = (1,0)$ and $z_4 = (1, -\beta - \alpha)$. Define $l_1 = \alpha - \beta$ and $l_2 = \alpha + \beta$. The eigenvalues associated to z_1 are -2 and l_1 while the eigenvalues associated to z_2 are -2 and $-l_1$. Moreover $z_1 = z_2$ when $l_1 = 0$. Analogously the eigenvalues of z_3 are 2 and l_2, while the eigenvalues associated to z_4 are 2 and $-l_2$, with $z_3 = z_4$ when $l_2 = 0$. So in the finite part the system can have two, three or four singulaties, depending on the values of l_1 and l_2.

In the local chart U_1 the compactified system has at most two singularities on the infinity line: $u_1 = (0,0)$ and $u_2 = (1-\beta, 0)$. Defining $l_3 = \beta - 1$ we see that $u_1 = u_2$ when $l_3 = 0$ and the topological type of these singularities depends on the sign of l_3. Indeed the eigenvalues associated to u_1 are -1 and l_3 while the associated to u_2 are -1 and $-l_3$.

In the local chart U_2 we just need to check if the origin $(0,0)$ is a singularity, which is true. It is a node, with the two eigenvalues equal to -1.

So considering $\gamma \neq 0$ and combining all the possibilities of the signs of l_1, l_2 and l_3 we obtain the phase portraits RPL.1–RPL.10 of Figure 6. In Table 3 we put the values of the parameters that realizes each one of the phase portraits described in Figure 6.

If $\gamma = 0$ then z_2 and z_4 goes to the infinity and the compactified system in the local chart U_2 becomes

$$\dot{u} = (1-\beta)u^2 - \alpha uv - v^2, \qquad \dot{v} = -v(\beta u + \alpha v).$$

Note that when $l_3 = 0 (\beta = 1)$ the line $v = 0$ is filled up of singular points, and when $l_3 \neq 0$ the origin $(0,0)$ is a linearly zero singularity. Considering this case first and applying the *blow up* $u = U, v = UW$ and dividing by U we get the system

$$\dot{U} = -U(\beta + W^2 + \alpha W - 1), \qquad \dot{W} = W(W-1)(W+1). \tag{2.50}$$

When $U = 0$ the singularities of (2.50) are $\overline{u_1} = (0,-1)$ with eigenvalues 2 and l_1, $\overline{u_2} = (0,0)$ with eigenvalues -1 and $-l_3$, and $\overline{u_3} = (0,1)$ with eigenvalues 2 and $-l_2$.

After blow-down we get the local phase portraits of the origin of U_2 which depend on the signs of l_1, l_2 and l_3. Doing all the combinations the origin of U_2 consists of:

- two elliptic sectors and parabolic sectors, see phase portraits RPL.11 and RPL.12 of Figure 6;
- two hyperbolic sectors and parabolic sectors, see phase portraits RPL.13 and RPL.14 of Figure 6;
- six hyperbolic sectors, see phase portrait RPL.15 of Figure 6.

	α	β	γ
RPL.1	-5/4	1/4	1
RPL.2	0	-1	1
RPL.3	-3	2	1
RPL.4	-2	1	1
RPL.5	0	1	1
RPL.6	-1/2	1/2	1
RPL.7	1/2	-1/2	1
RPL.8	-2	2	1
RPL.9	-1	1	1
RPL.10	0	0	1
RPL.11	-3	2	0
RPL.12	0	-1	0
RPL.13	-1	0	0
RPL.14	-1	2	0
RPL.15	-1/4	3/4	0
RPL.16	-2	1	0
RPL.17	0	1	0

Table 4 – Table of values for the parameters of system (RPL).

Finally if we consider $\beta = 1$ and after eliminating the common factor v the origin of the local chart U_2 is either a hyperbolic node or a hyperbolic saddle, described respectively by the phase portraits RPL.16 and RPL.17 of Figure 6. The Table 4 has the values of the parameters that realizes the phase portraits of Figure 6. □

Proposition 2.5.10 (DL)**.** Each real planar quadratic differential system with a double real straight line and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as system (DL), with $\gamma = 0$ and $\alpha \neq 0$, and the Darboux invariant is

$$I_{15}(x,y,t) = e^{-\alpha t} y x^{-\beta}.$$

Moreover there are 3 non–equivalent phase portraits in the Poincaré disc for this systems. They are described by DL.1–DL.3 in Figure 6.

Proof. Let $f_1 = x = 0$ be the double real invariant straight line. By the proof of Proposition 2.3.3 we know that the second invariant straight line is $f_2 = y = 0$. The cofactors of f_1 and f_2 are, respectively, $k_1 = x, k_2 = \alpha + \beta x + \gamma y$. Equation (2.9) with $s \in \mathbb{R} \setminus \{0\}$ has only one solution $\gamma = 0$, $s = -\alpha\lambda_2$, $\lambda_1 = -\beta\lambda_2$.

Taking $\lambda_2 = 1$ and using this solution we get

$$\dot{x} = x^2, \qquad \dot{y} = y(\alpha + \beta x),$$

with Darboux invariant $I_{15}(x,y,t) = e^{-\alpha t} y x^{-\beta}$.

In order to study the global phase portraits of systems (DL), since $\alpha \neq 0$ we can take $\alpha = 1$. The origin of the system is the only finite singularity, which is a saddle-node. For the infinity singularities we assume first that $\beta - 1 \neq 0$. In the local chart U_1 the origin is a saddle if $\beta - 1 > 0$, and a stable node if $\beta - 1 < 0$. In the chart U_2 the system becomes

$$\dot{u} = -u((\beta - 1)u + v), \qquad \dot{v} - v(\beta u + v),$$

and the origin is a linearly zero singularity. Applying the *blow up* $u = U$, $v = UW$ we get the system

$$\dot{U} = -U^2(\beta - 1 + W), \qquad \dot{W} = -UW,$$

which after eliminating the common factor U has the origin as only singular point. If $\beta - 1 > 0$ the origin is a hyperbolic stable node and if $\beta - 1 < 0$ the origin is a saddle.

After *blow down* we get the local phase portraits of the origin of U_2 which depend on β. When $\beta - 1 > 0$ the origin has two elliptic sectors and parabolic sectors, see phase portrait DL.1 of Figure 6. If $\beta - 1 < 0$ then there are two hyperbolic sectors and parabolic ones, see phase portrait DL.2 of Figure 6.

When $\beta = 1$ the infinity is filled up of singular points and the origin in the local chart U_2 is a hyperbolic stable node. The phase portrait of this case can be found and it is described by DL.3 of Figure 6. □

Proposition 2.5.11 (CPL)**.** Each real planar quadratic differential system with two parallel complex straight line and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as system (CPL). A Darboux invariant is given by

$$I_{16}(x,y,t) = e^t e^{arctan(1/x)}$$

Moreover there are 7 non–equivalent phase portraits in the Poincaré disc for this system. They are described by CPL.1–CPL.7 in Figure 7.

Proof. Let $f_1 = x+i = 0$, $f_2 = x-i = 0$ be the two complex parallel straight lines. By the proof of Proposition 2.3.3 we know that the third invariant straight line is $f_3 = y = 0$. The cofactors of f_1, f_2 and f_3 are, respectivelly, $k_1 = x-i, k_2 = x+i, k_3 = \alpha+\beta x+\gamma y$. The equation (2.9) with $s \in \mathbb{R}\setminus\{0\}$ has two solutions, namely

$$s_1 = \{\gamma = 0, s = i(2\lambda_1 + (\beta + i\alpha)\lambda_3), \lambda_2 = -\beta\lambda_3 - \lambda_1\}$$

$$s_2 = \{s = 2i\lambda_1, \lambda_2 = -\lambda_1, \lambda_3 = 0\}.$$

Using s_2 (which is more general) we conclude that all systems with two parallel complex straight lines and a real straight line as invariants curves have a Darboux invariant. Moreover taking $\lambda_1 = -i/2$ we get

$$I_{16}(x,y,t) = e^t(x-i)^{i/2}(x+i)^{-i/2}.$$

Using the polar form of the complex numbers it follows that $(x-i)^{i/2}(x+i)^{-i/2} = e^{\arctan(1/x)}$ so the Darboux invariant is $I_{16}(x,y,t) = e^{\arctan(1/x)+t}$.

In (GASULL; LI-REN; LLIBRE, 1986) the authors already study the quadratic systems with $f = x^2+1 = 0$ as invariant curve, given by

$$\dot{x} = x^2+1, \qquad \dot{y} = Q(x,y),$$

with Q an arbitrary polynomial of degree 2. In this paper we have $Q(x,y) = y(\alpha+\beta x+\gamma y)$. So the system studied here is a subcase of systems (VI) of the article (GASULL; LI-REN; LLIBRE, 1986). In that article the study of those systems is divided in six cases and since we have the invariant straigh line $y = 0$ there are seven possible phase portraits. The case $(VI.1)$ provides the phase portraits 1 and 2 of (GASULL; LI-REN; LLIBRE, 1986)(Fig. 1), i.e. the phase portraits CPL.1 and CPL.2 of Figure 7; the case $(VI.2)$ gives the phase portrait 6 of (GASULL; LI-REN; LLIBRE, 1986)(Fig. 1), i.e. the phase portrait CPL.3 of Figure 7; the case $(VI.4)$ generates the phase portraits 16 and 17 of (GASULL; LI-REN; LLIBRE, 1986)(Fig. 1), i.e. the phase portraits CPL.4 and CPL.5 of Figure 7; the case $(VI.5)$ gives the phase portrait 20 of (GASULL; LI-REN; LLIBRE, 1986)(Fig. 1), i.e. the phase portrait CPL.6 of Figure 7. Finally the case $(VI.6)$ provides the phase portrait 21 of (GASULL; LI-REN; LLIBRE, 1986)(Fig. 1), i.e. the phase portrait CPL.7 of Figure 7. □

Proposition 2.5.12 (p)**.** Each real planar quadratic differential system with two complex straight lines that intersects in a real point and a third straight line having a Darboux can be written, after an affine change of coordinates, as

(i) $(p.1)$ with $\alpha_3(\beta-2\beta_3) \neq 0$ and Darboux invariant

$$I_{17}(x,y,t) = e^{\alpha_3(\beta-2\beta_3)t}\, e^{-2\gamma_3 \arctan(y/x)}\,(x^2+y^2)^{\beta_3}\, y^{-\beta}.$$

(ii) $(p.2)$ with $c \neq 0$, $\alpha = -1$ and Darboux invariant

$$I_{18}(x,y,t) = e^{-\arctan(y/x)-ct}$$

Moreover there are 5 non–equivalent phase portraits in the Poincaré disc for this system. They are described by p.1.1–p.1.3 and p.2.1, p.2.2 in Figure 7.

Proof. Let $f_1 = x+iy = 0$ and $f_2 = x-iy = 0$ be the two complex straight lines that intersect at a real point. We have two systems, $(p.1)$, with $f_3 = y$, and $(p.2)$ with $f_3 = y+ax+c$. We shall do the calculations for $(p.1)$, and for system $(p.2)$ the computations are analogous.

Consider system $(p.1)$ the cofactors of f_1, f_2 and f_3 are, respecively,

$$\begin{aligned} k_1 &= (1/2)(\beta x + 2\gamma_3 y + 2\alpha_3 - i(\beta - 2\beta_3)y), \\ k_2 &= (1/2)(\beta x + 2\gamma_3 y + 2\alpha_3 + i(\beta - 2\beta_3)y), \\ k_3 &= \alpha_3 + \beta_3 x + \gamma_3 y. \end{aligned} \tag{2.51}$$

Solving equation (2.9) the most general solution is

$$\lambda_1 = \beta_3 + i\gamma_3, \quad \lambda_2 = \beta_3 - i\gamma_3, \quad \lambda_3 = -\beta, \quad s = \alpha_3(\beta - 2\beta_3).$$

Hence assuming $\alpha_3(\beta - 2\beta_3) \neq 0$ system $(p.1)$ of Theorem 2.5.1 has the Darboux invariant

$$I_{17}(x,y,t) = e^{\alpha_3(\beta-2\beta_3)t} y^{-\beta}\,(x-iy)^{\beta_3 - i\gamma_3}(x+iy)^{\beta_3+i\gamma_3}. \tag{2.52}$$

Using the polar form of the complex numbers it follows that $(x-i)^{i/2}(x-iy)^{\beta_3-i\gamma_3}(x+iy)^{\beta_3+i\gamma_3} = e^{-2\gamma_3 \arctan(y/x)}\,(x^2+y^2)^{\beta_3}$ and we get the Darboux invariant

$$I_{17}(x,y,t) = e^{\alpha_3(\beta-2\beta_3)t}\, e^{-2\gamma_3 \arctan(y/x)}\,(x^2+y^2)^{\beta_3}\, y^{-\beta}$$

For system $(p.2)$ the third invariant straight line is $f_3 = y+ax+c$ with $c \neq 0$. In this case the system has a Darboux invariant if and only if $\alpha = -1$, and with the same reasoning applied above we get the invariant

$$I_{18}(x,y,t) = e^{-\arctan(y/x)-ct}.$$

We start the study of the global phase portraits with systems $(p.1)$. Since $\alpha_3 \neq 0$ we can take $\alpha_3 = 1$. Systems $(p.1)$ have at most two finite singularities, namely $z_1 = (0,0)$ and $z_2 = (-2/\beta, 0)$. When $\beta = 0$ the point z_2 goes to infinity. The point z_1 is an unstable node and the eigenvalues associated to z_2 are -1 and $(\beta - 2\beta_3)/\beta$. So the point z_2 is either a stable node or a saddle.

In the local chart U_2 the origin is not a singularity for the compactfied system. In the local chart U_1 the system compactified has only one infinity singularity $u_1 = (0,0)$ with eigenvalues $-\beta/2$ and $-(\beta - 2\beta_3)/2$.

Then if $\beta(\beta - 2\beta_3) > 0$, z_2 is a saddle and u_1 is a stable node and the only phase portrait is p.1.1 of Figure 7, realizable for $\beta = 1$, $\gamma_3 = 1$ and $\beta_3 = -1/2$. If $\beta(\beta - 2\beta_3) < 0$, z_2 is a stable node and u_1 is a saddle and the corresponding phase portrait of this case is p.1.2 of Figure 7, realizable for $\beta = 1$, $\gamma_3 = 1$ and $\beta_3 = 3/2$. Finally if $\beta = 0$ then z_2 goes to the infinity and u_2 becomes a semi hyperbolic saddle-node generating the phase portrait p.1.3 of Figure 7, which is realizable for $\beta = 0$, $\gamma_3 = 1$ and $\beta_3 = 2$.

In order to study the global phase portraits of systems $(p.2)$ we start with the infinity singular points. In the local chart U_1 system $(p.2)$ becomes

$$\dot{u} = -cv(u^2+1), \qquad \dot{v} = -v(a\beta + au + cuv + \beta cv + \beta u - 1).$$

So the line $v = 0$ is filled up of singular points. The same happens in the local chart U_2. In the finite part the point $(0,0)$ is the only singularity, with complex eigenvalues. So the origin can be a node or a center. Both cases are described, respectively, by the phase portraits p.2.1 ,realizable with $a = \beta = 1$ and $c = 2$, and p.2.2, realizable with $a = 1$, $\beta = 0$ and $c = 2$, of Figure 7. □

To end this first chapter we study the systems that do not a Darboux invariant.

Theorem 2.5.13. Systems of type $(CE), (E.1), (H.1), (H.5), (P.3)$ do not admit Darboux invariants of the form $e^{-st} f(x,y)$.

Proof. First we consider systems of type (CE), i.e, the ones which has an invariant cubic of the form $f = f_1 f_2 = 0$ where $f_1 = x^2 + y^2 + 1$ and $f_2 = ax + by + c$. By Theorem 2.5.1 these systems can be written as

$$\dot{x} = -(x^2+y^2+1) - 2\alpha_1 y(y+ax+c), \qquad \dot{y} = a(x^2+y^2+1) + 2\alpha_1 x(y+ax+c),$$

with $f_1 = x^2+y^2+1$ and $f_2 = y+ax+c$. The cofactors of f_1 and f_2 are $k_1(x,y) = 2(ay+x)$ and $k_2(x,y) = -2\alpha_1(ay-x)$, respectively. So the cofactors have no constant terms, i.e, $k_1(0,0) = k_2(0,0) = 0$. The consequence of this is that equation (2.9) has no solution considering $s \neq 0$. Hence these systems do not have a Darboux invariant of the form $e^{st} f_1^{\lambda_1} f_2^{\lambda_2}$.

The proofs for other systems are very similar. In fact it suffices to observe that the cofactors of the invariant curves never have a constant term. □

CHAPTER

3

AVERAGING THEORY FOR NONSMOOTH SYSTEMS

The purpose of this chapter is to present applications of the classic averaging theory of first order to nonsmooth differential systems. Averaging theory is a very useful technique to investigate the existence of periodic orbits in differential systems. We start this chapter introducing the nonsmooth differential systems and the classic averaging theory of first order.

3.1 Piecewise smooth differential systems

Definition 3.1.1. A piecewise smooth vector field defined on an open bounded set $U \subset \mathbb{R}^n$ is a function $F : U \to \mathbb{R}^n$ which is continuous except on a set Σ of measure 0, called the set of discontinuity of the vector field F.

Here we assume that $U \setminus \Sigma$ is a finite collection of disjoint open sets U_i, $i = 1, 2, \ldots, m$, such that the restriction $F_i = F\big|_{U_i}$ is continuously extendable to the compact set $\overline{U_i}$. The local trajectory of F at a point $p \in U_i$ is given by the usual notion. However the local trajectory of F at a point $p \in \Sigma$ needs to be given with some care. In (FILIPPOV, 1988), taking advantage of the theory of differential inclusion (see (AUBIN; CELLINA, 1984)), Filippov established some conventions for what would be a local trajectory at points of discontinuity where the set Σ is locally a codimension one embedded submanifold of $\mathbb{R}^n$. For a such point $p \in \Sigma$, we consider a sufficiently small neighborhood U_p of p such that Σ splits $U_p \setminus \Sigma$ in two disjoint open sets U_p^+ and U_p^- and denote $F^{\pm}(p) = F\big|_{U_p^{\pm}}(p)$. In short, if the vectors $F^{\pm}(p)$ point at the same direction then the local trajectory of F at p is given as the concatenation of the local trajectories of $F^{\pm}$ at p. In this case we say that the trajectory crosses the set of discontinuity and that p is a crossing point. If the vectors $F^{\pm}(p)$ point in opposite directions then the local trajectory of F at p slides on Σ. In this case we say that p is a sliding point. For more details on the Filippov conventions see (FILIPPOV, 1988; GUARDIA; SEARA; TEIXEIRA, 2011).

The first objective is to estabilish conditions for the existence of crossing limit cycles for a class of discontinuous piecewise smooth vector fields, that is limit cycles which only cross the set of discontinuity Σ.

Remark 3.1.2. If Σ is locally described as $h^{-1}(0)$, being $h : U \to \mathbb{R}$ a smooth function and 0 a regular value, then $\langle \nabla h(p), F^{+}(p)\rangle\langle \nabla h(p), F^{-}(p)\rangle > 0$ is the condition in order that p is a crossing point. For nonautonomous system the same definition can be applied considering the extended phase space where the system becomes autonomous by taking the time as a new space variable with constant velocity equal 1.

3.2 The averaging theory of first order

The averaging theory is one of the best tools to provide sufficient conditions for the existence of isolated periodic solutions of a differential system. In chapter 4 we shall present a background on this theory and the new results obtained. But in this chapter we deal with the averaging theory of first order.

The thecnique is applied in systems on the form

$$\dot{x}(t) = \sum_{i=0}^{k} \varepsilon^{i} F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{3.1}$$

The form presented above is related to an expansion that is made around $\varepsilon = 0$ and when $k = 1$ we call first order averaging. So considering $k = 1$ and a continuous piecewise differential system we have the result(Theorem B of (LLIBRE; NOVAES; TEIXEIRA, 2014b))

Theorem 3.2.1. Consider the following system

$$\dot{x}(t) = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{3.2}$$

where $D \subset \mathbb{R}^n$ is an open subset, ε is a small parameter, the functions $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 0,1$ and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are T–periodic in the variable t, and for each $t \in \mathbb{R}$ the functions $F_0(t,.) \in C^1$, $F_1(t,.) \in C^0$, $D_x F_0$ and $R \in C^0$ are locally Lipschitz in the second variable. We denote by $x(t,z,\varepsilon)$ the solution of system (3.2) such that $x(0,z,\varepsilon) = z$. Assume that there exists an open and bounded subset V of D with its closure $\overline{V} \subset D$ such that for each $z \in \overline{V}$, the solution $x(t,z,0)$ is T- periodic. We denote by $M_z(t)$ the fundamental matrix solution of the variational equation hgh

$$\dot{x}(t) = D_x F_0(t, x(t,z,0)),$$

associated to the periodic solution $x(t,z,0)$ such that $M_z(0)$ is the identity.

If $a \in V$ is a zero of the map $f : \overline{V} \to \mathbb{R}^n$ defined by

$$f(z) = \int_0^T M_z^{-1}(t) F_1(t, x(t,z)) dt \tag{3.3}$$

and $\det(D_z f(a)) \neq 0$, then for $\varepsilon > 0$ sufficiently small, system (3.2) has a T–periodic solution $x(t, a_\varepsilon, \varepsilon)$ such that $a_\varepsilon \to a$ as $\varepsilon \to 0$. Moreover the linear stability type of the periodic solution $x(t, a_\varepsilon, \varepsilon)$ is given by the eigenvalues of the matrix $D_z f(a)$.

Now considering the averaging of first order ($k = 1$) and a special case of discontinuous piecewise differential system there is also a result, similar to Theorem 3.2.1. Let $D \subset \mathbb{R}^n$ be an open subset and $h : \mathbb{R} \times D \to \mathbb{R}$ a C^1 function having 0 as regular value. Consider $F^1, F^2 : \mathbb{R} \times D \to \mathbb{R}^n$ continuous functions and $\Sigma = h^{-1}(0)$ and the Filippov's system

$$\dot{x}(t) = F(t,x) = \begin{cases} F^1(t,x) & \text{if } (t,x) \in \Sigma^+, \\ F^2(t,x) & \text{if } (t,x) \in \Sigma^-, \end{cases} \tag{3.4}$$

where $\Sigma^+ = \{(t,x) \in \mathbb{R} \times D : h(t,x) > 0\}$ and $\Sigma^- = \{(t,x) \in \mathbb{R} \times D : h(t,x) < 0\}$.

Consider the differential system associated to system (3.4)

$$\dot{x}(t) = F(t,x) = \chi_+(t,x)F^1(t,x) + \chi_-(t,x)F^2(t,x), \tag{3.5}$$

where χ_+, χ_- are the characteristic functions defined as

$$\chi_+(t,x) = \begin{cases} 1 \text{ if } h(t,x) > 0, \\ 0 \text{ if } h(t,x) < 0. \end{cases}$$

and

$$\chi_-(t,x) = \begin{cases} 0 \text{ if } h(t,x) > 0, \\ 1 \text{ if } h(t,x) < 0. \end{cases}$$

Systems (3.4) and (3.5) do not coincide in Σ, but applying the Filippov's convention for the solutions of systems (3.4) and (3.5) (see (FILIPPOV, 1988)) passing through a point $(t,x) \in \Sigma$ we have that these solutions do not depend on the value of $F(t,x)$, so the solutions are the same.

Let $\mathscr{P}$ be the space formed by the periodic solutions of (3.5). If $\dim \mathscr{P} = \dim D = d$ then the following result follows directly from Theorem B of (LLIBRE JAUME NOVAES, 2015).

Theorem 3.2.2. Consider the differential system

$$\dot{x}(t) = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon). \tag{3.6}$$

where

$$F_i(t,x) = \quad \chi_+ F_i^1(t,x) + \chi_- F_i^2(t,x), \text{ for } i = 0,1, \text{ and}$$

$$R(t,x) = \quad \chi_+ R^1(t,x) + \chi_- R^2(t,x),$$

with $F_i^1 \in C^1$, for $i = 0,1$ and R^1, R^2 are continuous functions which are Lipschitz in the second variable, and all these functions are T-periodic functions in the variable $t \in \mathbb{R}$.

For $z \in D$ and $\varepsilon > 0$ sufficiently small denote by $x(t,z,\varepsilon)$ the solution of system (3.6) such that $x(0,z,\varepsilon) = z$.

Define the averaged function

$$f(z) = \int_0^T M(s,z)^{-1} F_1(s, x(s,z,0))ds \tag{3.7}$$

where $x(s,z,0)$ is a periodic solution of (3.6) with $\varepsilon = 0$ such that $x(0,z,0) = z$, and $M(s,z)$ is the fundamental matrix of the variational system $\dot{y} = D_x F_0(t, x(t,z,0))y$ associated to the unperturbed system evaluated on the periodic solution $x(s,z,0)$ such that $M(0,z) = Id$. Moreover we assume the following hypotheses.

(H_-) There exists an open bounded subset $C \subset D$ such that, for ε sufficiently small, every orbit starting in C reaches the set of discontinuity only at its crossing region.

(H_+) For $a \in C$ with $f(a) = 0$ there exists a neighborhood $U \subset C$ of a such that $f(z) \neq 0$, for all $z \in \bar{U} \backslash \{a\}$ and $\det(D_z f(a)) \neq 0$.

Then for $\varepsilon > 0$ sufficiently small there exists a T–periodic solution $x(t, a_\varepsilon, \varepsilon)$ of (3.6) such that $a_\varepsilon \to a$ as $\varepsilon \to 0$. Moreover the linear stability of the periodic solution $x(t,\varepsilon)$ is given by the eigenvalues of the matrix $D_z f(a)$.

Remark 3.2.3. If $f(x)$ is a C^1 function such that $f(a) = 0$ and $f'(a) \neq 0$ there exist a neighborhood $U \subset C$ of a such that $f(z) \neq 0$ for all $z \in \bar{U} \backslash \{a\}$ and $d_B(f, U, 0) \neq 0$. For more details about the Brouwer degree see for instance (BROWDER, 1983).

Using the results above we put two applications. The first one is published in Computational & Applied Mathematics (see (LLIBRE; OLIVEIRA; RODRIGUES, 2018)) and the second is a preprint (see (LLIBRE; OLIVEIRA; RODRIGUES, 2018)).

3.2.1 The Michelson differential system

The Michelson differential system is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= c^2 - y - \frac{x^2}{2}, \end{aligned} \tag{3.8}$$

with $(x,y,z) \in \mathbb{R}^3$ and the parameter $c \geq 0$. The dot denotes derivative with respect to an independent variable t, usually called the time. This system is due to Michelson (MICHELSON, 1986) for studying the traveling solutions of the Kuramoto-Sivashinsky equation.

This system has been largely investigated from the dynamical point of view. Michelson in (MICHELSON, 1986) proved that if $c > 0$ is sufficiently large, then system (3.8) has a unique bounded solution which is a transversal heteroclinic orbit connecting the two finite singularities $(-\sqrt{2}c, 0, 0)$ and $(\sqrt{2}c, 0, 0)$. When c decreases there will appear a cocoon bifurcation (see (KOKUBU; WILCZAK; ZGLICZYŃSKI, 2007; LAU, 1992; MICHELSON, 1986)). In (LLIBRE; ZHANG, 2011) there is an analytical proof of the existence of a zero-Hopf bifurcation for system (3.8).

In (CARMONA; FERNÁNDEZ-SÁNCHEZ; TERUEL, 2008; CARMONA *et al.*, 2010) the authors consider a continuous piecewise linear version of Michelson differential system changing the non linear function x^2 in (3.8) by the piecewise linear function $|x|$. For such system they proved that some dynamical aspects of the Michelson system remains as the existence of a reversible T–point heteroclinic cycle.

Doing the change of variable $(x, y, z, c) \to (2\varepsilon X, 2\varepsilon Y, 2\varepsilon Z, 2\varepsilon d)$ with $d \geq 0$ and $\varepsilon > 0$ sufficiently small to the Michelson differential system (3.8), followed by the change of the function $X^2 \to |X|$, and denoting again X, Y, Z by x, y, z, we obtain the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon(2d^2 - |x|), \end{aligned} \tag{3.9}$$

that we call the Michelson continuous piecewise linear differential system . We note that this system is reversible because it is invariant under the change of variables $(x, y, z, t) \mapsto (-x, y, -z, -t)$.

If in the continuous Michelson differential system (3.8) we change the continuous function $|x|$ by the discontinuous one $|x| + \operatorname{sign}(x)$, where

$$\operatorname{sign} x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

we obtain the Michelson discontinuous piecewise linear differential system given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon(2d^2 - |x| - \operatorname{sign} x). \end{aligned} \tag{3.10}$$

The first objective is studying analytically the periodic solutions of the Michelson continuous and discontinuous piecewise linear differential systems. The result is presented bellow.

Theorem 3.2.4. For all $d > 0$ and $\varepsilon = \varepsilon(d) > 0$ sufficiently small the Michelson continuous piecewise linear differential system (3.9) has a periodic solution of the form

$$x(t) = -\pi d^2 + \mathscr{O}(\varepsilon), \quad y(t) = \pi d^2 \sin t + \mathscr{O}(\varepsilon), \quad z(t) = \pi d^2 \cos t + \mathscr{O}(\varepsilon).$$

Moreover this periodic solution is linearly stable.

Proof. Doing to the Michelson continuous piecewise linear differential system (3.9) the change to cylindrical coordinates $x = x$, $y = r\sin\theta$ and $z = r\cos\theta$, the system becomes

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon(2d^2 - |x|)\cos\theta, \\ \dot{\theta} &= 1 - \frac{\varepsilon}{r}(2d^2 - |x|)\sin\theta. \end{aligned} \tag{3.11}$$

Taking θ as the new independent variable we can write the previous differential system as

$$\begin{aligned} \frac{dx}{d\theta} &= x' = r\sin\theta + \varepsilon(2d^2 - |x|)\sin^2\theta + \mathscr{O}(\varepsilon^2), \\ \frac{dr}{d\theta} &= r' = \varepsilon(2d^2 - |x|)\cos\theta + \mathscr{O}(\varepsilon^2). \end{aligned} \tag{3.12}$$

The unperturbed system is

$$\begin{aligned} x' &= r\sin\theta, \\ r' &= 0. \end{aligned} \tag{3.13}$$

For each (x_0, r_0) the solution $(x(\theta, x_0, r_0), r(\theta, x_0, r_0))$ such that $(x(0, x_0, r_0), r(0, x_0, r_0)) = (x_0, r_0)$ is

$$(x(\theta, x_0, r_0), r(\theta, x_0, r_0)) = (x_0 + r_0(1 - \cos\theta), r_0),$$

which is 2π–periodic for all $(x_0, r_0) \neq (0,0)$. When $r_0 = 0$ we have a straight line of equilibrium points. Now note that the function $F_0(\theta, (x, r)) = (r\sin\theta, 0)$ is C^∞ and in particular C^1, and that the function

$$F_1(\theta, (x, r)) = (2d^2 - |x|)(\sin^2\theta, \cos\theta)$$

is C^0, and both are Lipschitz. So the differential system (3.11) satisfies the assumptions of Theorem 3.2.1. Then, by Theorem 3.2.1, we need to calculate the averaged function

$$f(x_0, r_0) = \int_0^{2\pi} M(\theta)^{-1} F_1(\theta, x(\theta, (x_0, r_0)))d\theta,$$

where

$$M(\theta) = \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}$$

is the fundamental matrix of the variational differential system associated to system (3.13) evaluated on the periodic solution $(x_0+r_0(1-\cos\theta), r_0)$ such that $M(0)$ is the identity matrix. Therefore we have

$$
\begin{aligned}
f(x_0,r_0) &= \int_0^{2\pi} (2d^2 - |x_0+r_0(1-\cos\theta)|) \begin{pmatrix} 1 & \cos\theta-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix} d\theta \\
&= \int_0^{2\pi} (2d^2 - |x_0+r_0(1-\cos\theta)|) \begin{pmatrix} 1-\cos\theta \\ \cos\theta \end{pmatrix} d\theta \\
&= \int_0^{2\pi} g(\theta)d\theta.
\end{aligned}
$$

Note that $g(\theta)=g(-\theta)$ and $g(\theta)$ is 2π-periodic. So

$$\int_0^{2\pi} g(\theta)d\theta = \int_{-\pi}^{\pi} g(\theta)d\theta = 2\int_0^{\pi} g(\theta)d\theta.$$

To calculate this integral we need to study the zeros of the function $G(\theta)=x_0+r_0(1-\cos\theta)$.

As $G(\theta)=0$ if and only if $\theta=\pm\arccos\left(\dfrac{x_0+r_0}{r_0}\right)$ and the function $\arccos(x)$ takes real values when $x\in[-1,1]$ we have to consider the following three cases.

Case 1: $x_0 \le -2r_0$ or equivalently $\dfrac{x_0+r_0}{r_0} \le -1$. Then $r_0+x_0-r_0\cos\theta \le 0$ in $[0,\pi]$.

Case 2: $x_0 \in (-2r_0, 0)$ or equivalently $\left|\dfrac{x_0+r_0}{r_0}\right| < 1$. Then

(i) $r_0+x_0-r_0\cos\theta < 0$ if $\theta \in \left(0, \arccos\left(\dfrac{r_0+x_0}{r_0}\right)\right)$,

(ii) $r_0+x_0-r_0\cos\theta > 0$ if $\theta \in \left(\arccos\left(\dfrac{r_0+x_0}{r_0}\right), \pi\right)$.

Case 3: $x_0 \ge 0$ or equivalently $\dfrac{x_0+r_0}{r_0} \ge 1$. Then $r_0+x_0-r_0\cos\theta \ge 0$ in $[0,\pi]$.

In the computation of the integral $\displaystyle\int_0^{\pi} g(\theta)d\theta$ we distinguish the previous three cases.

Case 1. In this case the averaged function is

$$
\begin{aligned}
f(x_0,r_0) &= 2\int_0^{2\pi} (2d^2 - |x_0+r_0(1-\cos\theta)|) \begin{pmatrix} 1-\cos\theta \\ \cos\theta \end{pmatrix} d\theta \\
&= 2\pi(4d^2+3r_0+2x_0, -r_0),
\end{aligned}
$$

whose unique zero is $(x_0, r_0) = (-2d^2, 0)$. Since this initial condition corresponds to an equilibrium point of the unperturbed system (3.13), the averaging theory in this case does not provide periodic solutions.

Case 3. Analogously to Case 1 we have

$$
\begin{aligned}
f(x_0, r_0) &= 2\int_0^{2\pi} (2d^2 - |x_0 + r_0(1-\cos\theta)|) \begin{pmatrix} 1-\cos\theta \\ \\ \cos\theta \end{pmatrix} d\theta \\
&= 2\pi(4d^2 - 3r_0 - 2x_0, r_0),
\end{aligned}
$$

whose unique zero is $(x_0, r_0) = (2d^2, 0)$. The conclusion follows as in Case 1.

Case 2. Here

$$
\begin{aligned}
f(x_0, r_0) =& 2\int_0^{\arccos\left(\frac{r_0+x_0}{r_0}\right)} (2d^2 + (x_0 + r_0(1-\cos\theta))) \begin{pmatrix} 1-\cos\theta \\ \\ \cos\theta \end{pmatrix} d\theta \\
&+ 2\int_{\arccos\left(\frac{r_0+x_0}{r_0}\right)}^{\pi} (2d^2 - (x_0 + r_0(1-\cos\theta))) \begin{pmatrix} 1-\cos\theta \\ \\ \cos\theta \end{pmatrix} d\theta \\
=& \begin{pmatrix} f_1(x_0, r_0) \\ \\ f_2(x_0, r_0) \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(x_0, r_0) = &\quad 4\pi d^2 - \frac{\sqrt{-x_0(2r_0+x_0)}}{r_0}(6r_0 + 2x_0) - 2(3r_0 + 2x_0)\arcsin\left(\frac{r_0+x_0}{r_0}\right), \\
f_2(x_0, r_0) = &\quad 2(r_0 + x_0)\frac{\sqrt{-x_0(2r_0+x_0)}}{r_0} + 2r_0 \arcsin\left(\frac{r_0+x_0}{r_0}\right).
\end{aligned}
$$

In order to solve the system $f_1(x_0, r_0) = f_2(x_0, r_0) = 0$ we do the change of variables $x_0 \to X$ where $x_0 = -r_0 - Xr_0$ with $-1 < X < 1$, recall that $\left|\frac{x_0 + r_0}{r_0}\right| < 1$. Then the system becomes

$$
\begin{aligned}
&2\pi d^2 + \big((X-2)\sqrt{1-X^2} + (1-2X)\arcsin X\big)r_0 = 0, \\
&r_0\left(X\sqrt{1-X^2} + \arcsin X\right) = 0.
\end{aligned}
$$

Since r_0 must be positive, from the second equation it follows that $X = 0$, and from the first that $r_0 = \pi d^2$. So we have the solution $(x_0, r_0) = (-\pi d^2, \pi d^2)$.

The Jacobian of the map (f_1, f_2) evaluated at $(x_0, r_0) = (-\pi d^2, \pi d^2)$ is 4. It follows from Theorem 3.2.1 and for any $d > 0$ and $\varepsilon = \varepsilon(d) > 0$ sufficiently small that system (3.12) has a periodic solution $\varphi(\theta, \varepsilon) = (x(\theta, \varepsilon), r(\theta, \varepsilon)) = (-d^2\pi + \mathscr{O}(\varepsilon), d^2\pi + \mathscr{O}(\varepsilon))$. Moreover the eigenvalues of the Jacobian matrix of the map (f_1, f_2) at the solution $(-d^2\pi, d^2\pi)$ are $\pm 2i$, so the periodic solution is linearly stable.

Now we must identify the periodic solution of system (3.9) which corresponds to the periodic solution found. Going back to system (3.11) with the independent variable t we obtain the periodic solution

$$(x(t,\varepsilon), r(t,\varepsilon), \theta(t,\varepsilon)) = (-d^2\pi, d^2\pi, t(mod\, 2\pi)) + \mathscr{O}(\varepsilon).$$

Finally coming back to system (3.9) we find the periodic solution

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (-d^2\pi\,,\, d^2\pi \sin t\,,\, d^2\pi \cos t) + \mathscr{O}(\varepsilon).$$

This concludes the proof of Theorem 3.2.4. □

Remark 3.2.5.

1. A periodic solution is asymptotically stable if all the eigenvalues corresponding to the fixed point of the Poincaré map associated to this solution have negative real part, then this periodic solution is locally asymptotically stable. If one of the eigenvalues has positive real part the periodic solution is unstable. If all the eigenvalues have zero real parts, then we say that the periodic solution is linearly stable, in this case the linear stability does not provide any information on the kind of stability that the periodic solution has when we take into account the nonlinear terms.

2. The stability of the periodic solutions of system (3.2) when it is applied to the Michelson continuous piecewise linear differential system can be obtained from the stability of a differential system associated to it. In fact given the continuous system (3.8) consider a band of amplitude $\varepsilon > 0$ around the plane $x = 0$ and a differentiable extension of the continuous system (3.8) to this band. Studying the limit of this extended differentiable system when $\varepsilon \to 0$ we conclude that the linear stability of system (A.2) is given by the eigenvalues of $D_z\mathscr{F}(a)$.

Remark 3.2.6. The periodic orbit obtained in Theorem 3.2.4 is not reversible because their dominant terms $(-d^2\pi\,,\, d^2\pi \sin t\,,\, d^2\pi \cos t)$ are not invariant under the change of variables $(x, y, z, t) \mapsto (-x, y, -z, -t)$. So this periodic orbit has no relation with the periodic orbit of the noose bifurcation studied by the Michelson continuous piecewise linear differential system (1.2) of (CARMONA *et al.*, 2015) which is reversible. We also note that our Michelson continuous piecewise linear differential system (3.9) and system (1.2) of (CARMONA *et al.*, 2015) do not coincide.

With respect to the number of periodic solutions of system (3.10) we have the following result

Theorem 3.2.7. For $\varepsilon > 0$ sufficiently small the Michelson discontinuous piecewise linear differential system (3.10) satisfies the following statements.

(a) If $(-1+2d^2)\pi < 0$ then system (3.10) has two periodic solutions $(x(t,\varepsilon), r(t,\varepsilon), \theta(t,\varepsilon))$ of the form

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (x_0, r_0 \sin t, r_0 \cos t) + \mathscr{O}(\varepsilon), \tag{3.14}$$

where

$$r_0 = \frac{2\sqrt{1-a^2}}{a\sqrt{1-a^2}+\arcsin a}, \qquad x_0 = -r_0(1+a),$$

and a takes the value of the two unique zeros of the function

$$g(a) = \frac{2a^2-2+\pi a\sqrt{1-a^2}d^2+\arcsin a\left(\pi d^2+\arcsin a - a\sqrt{1-a^2}\right)}{a\sqrt{1-a^2}+\arcsin a},$$

in the interval $(-1,1)$.

(b) If $(-1+2d^2)\pi > 0$, then system (3.10) has a periodic solution of the form (3.14) given by the unique zero of the function $g(a)$ in the interval $(-1,1)$.

Proof. Doing the change to cylindrical coordinates $x = x$, $y = r\sin\theta$ and $z = r\cos\theta$ the Michelson discontinuous piecewise linear differential system becomes

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon(2d^2 - |x| - sign x)\cos\theta, \\ \dot{\theta} &= 1 - \frac{\varepsilon}{r}(2d^2 - |x| - sign x)\sin\theta. \end{aligned} \tag{3.15}$$

Now taking as new independent variable the angle θ we get the system

$$\begin{aligned} x' &= r\sin\theta + \varepsilon(2d^2 - |x| - sign x)\sin^2\theta + \mathscr{O}(\varepsilon^2), \\ r' &= \varepsilon(2d^2 - |x| - sign x)\cos\theta + \mathscr{O}(\varepsilon^2), \end{aligned} \tag{3.16}$$

where the prime denotes the derivative with respect to θ. This differential system satisfies the assumptions of Theorem 3.2.2, so we shall apply it for finding some of its periodic solutions. Using the notation of Theorem 3.2.2 we have that

$$F_1(\theta, x, r) = (2d^2 - |x| - sign x)\begin{pmatrix} \sin^2\theta \\ \\ \cos\theta \end{pmatrix}.$$

As in Theorem 3.2.4 the unperturbed system is given by (3.13) and the fundamental matrix is $M(\theta)=\begin{pmatrix}1 & 1-\cos\theta\\ 0 & 1\end{pmatrix}$. Then by Theorem 3.2.2 we need to calculate

$$f(x_0,r_0)=\int_0^{2\pi}M(\theta)^{-1}F_1(\theta,x(\theta,(x_0,r_0)),r(\theta,(x_0,r_0)))d\theta=\int_0^{2\pi}g(\theta)d\theta.$$

where $g(\theta)=\left(2d^2-|x_0+r_0(1-\cos\theta)|-sign(x_0+r_0(1-\cos\theta))\right)\begin{pmatrix}1-\cos\theta\\ \cos\theta\end{pmatrix}$.

Since $g(\theta)$ is 2π–periodic and $g(\theta)=g(-\theta)$ then

$$\int_0^{2\pi}g(\theta)d\theta=2\int_0^{\pi}g(\theta)d\theta.$$

As in the study of the continuous differential system, we separate the calculation of the averaged function corresponding to the discontinuous system (3.16) in the same three cases that appear in the proof of Theorem 3.2.4.

Case 1. In this subcase $r_0+x_0-r_0\cos\theta<0$ in $[0,\pi]$. Then the averaged function is

$$f(x_0,r_0)=\pi\left(2+4d^2+3r_0+2x_0,r_0\right).$$

This function has no zeros with $r_0>0$, so the averaging theory in this case does not detect any periodic solution.

Case 2. Now we have

$$f(x_0,r_0)=\pi(-2+4d^2-3r_0-2x_0,r_0).$$

The conclusion follows as in Case 1.

Case 3. Here $r_0+x_0-r_0\cos\theta<0$ when $\theta\in\left(0,\arccos\left(\dfrac{r_0+x_0}{r_0}\right)\right)$, and $r_0+x_0-r_0\cos\theta>0$, when $\theta\in\left(\arccos\left(\dfrac{r_0+x_0}{r_0}\right),\pi\right)$, so

$$f(x_0,r_0)=\begin{pmatrix}f_1(x_0,r_0)\\ f_2(x_0,r_0)\end{pmatrix},$$

with

$$
\begin{aligned}
f_1(x_0,r_0) = &\sqrt{-\frac{x_0(2r_0+x_0)}{r_0^2}}(3r_0+x_0+2)+\frac{\pi}{2}\left(4d^2-3r_0-2x_0+2\right)\\
&+(3r_0+2x_0+2)\arccos\left(\frac{r_0+x_0}{r_0}\right),\\
f_2(x_0,r_0) = &\frac{\pi r_0}{2}-\arccos\left(\frac{r_0+x_0}{r_0}\right)r_0+(r_0+x_0+2)\sqrt{-\frac{x_0(2r_0+x_0)}{r_0^2}}.
\end{aligned}
$$

Again for solving the system $f_1(x_0,r_0)=f_2(x_0,r_0)=0$ we do the change of variables $x_0 \to X$ where

$$x_0 = -r_0 - Xr_0, \tag{3.17}$$

with $-1 < X < 1$. Then the system becomes

$$2\pi d^2+\sqrt{1-X^2}((X-2)r_0-2)+(2-2Xr_0+r_0)\arcsin X = 0,$$

$$\sqrt{1-X^2}(2-Xr_0)-r_0\arcsin X = 0.$$

From the second equation we get

$$r_0 = \frac{2\sqrt{1-X^2}}{X\sqrt{1-X^2}+\arcsin X}. \tag{3.18}$$

Substituting r_0 in the first equation we obtain $g(X)=0$, where $g(X)$ is

$$\frac{2X^2-2+\pi X\sqrt{1-X^2}d^2+\arcsin X\left(\pi d^2+\arcsin X-X\sqrt{1-X^2}\right)}{X\sqrt{1-X^2}+\arcsin X}.$$

It is easy to compute that

$$\lim_{X\searrow -1} g(X) = (-1+2d^2)\pi,$$

$$\lim_{X\nearrow 0} g(X) = +\infty,$$

$$\lim_{X\searrow 0} g(X) = -\infty,$$

$$\lim_{X\nearrow 1} g(X) = (1+2d^2)\pi.$$

So, by continuity of the function $g(X)$ in the intervals $(-\infty,0)$ and $(0,\infty)$, it follows that $g(X)$ has one zero in the interval $(-\infty,0)$ if $(-1+2d^2)\pi<0$, and that $g(X)$ always has one zero in the interval $(0,\infty)$. Moreover, since the derivative $g'(X)>0$ in those two intervals, such zeros are the unique zeros of the function $g(X)$.

Since for each zero of $g(X)$ we have a unique zero (x_0,r_0) of system $f_1(x_0,r_0)=f_2(x_0,r_0)=0$ (see (3.18) and (3.17)), we obtain two solutions of the system $f_1(x_0,r_0)=f_2(x_0,r_0)=0$ if $(-1+2d^2)\pi\le 0$, and only one if $(-1+2d^2)\pi>0$. Computing the Jacobian of the map $(f_1(x_0,r_0),f_2(x_0,r_0))$ at (3.18) and (3.17) we get

$$\frac{X^4-5X^2+6X\sqrt{1-X^2}\arcsin X+\left(4X^2-1\right)(\arcsin X)^2+4}{1-X^2}\ge 4,$$

if $X\in(-1,1)$. Therefore, by Theorem 3.2.2 we obtain two periodic solutions of the differential system (3.16) if $(-1+2d^2)\pi<0$, and one periodic solution if $(-1+2d^2)\pi>0$. Such periodic solutions are of the form

$$(x(\theta,\varepsilon),r(\theta,\varepsilon))=(x_0+\mathscr{O}(\varepsilon),r_0+\mathscr{O}(\varepsilon)),$$

where x_0 and r_0 are given by (3.18) and (3.17) when X is a zero of $g(X)$.

Going back to system (3.15) with the independent variable t we obtain the periodic solution

$$(x(t,\varepsilon),r(t,\varepsilon),\theta(t,\varepsilon))=(x_0,r_0,t(mod\,2\pi))+\mathscr{O}(\varepsilon).$$

Finally coming back to system (3.9) we find the periodic solution

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))=(x_0,r_0\sin t,r_0\cos t)+\mathscr{O}(\varepsilon).$$

This concludes the proof of Theorem 3.2.7 □

3.2.2 Limit cycles of control piecewise linear differential systems

In control theory are relevant the continuous piecewise linear differential systems of the form

$$\dot{x}=Ax+\varphi(x_1)b, \tag{3.19}$$

with A a $m\times m$ matrix, $x,b\in\mathbb{R}^m$, $\varphi:\mathbb{R}\to\mathbb{R}$ is the continuous piecewise linear function

$$\varphi(x_1)=\begin{cases}-1 & \text{if } x_1\in(-\infty,-1),\\ x_1 & \text{if } x_1\in[-1,1],\\ 1 & \text{if } x_1\in(1,\infty),\end{cases} \tag{3.20}$$

where $x=(x_1,\dots,x_m)^T$, and the dot denotes the derivative with respect to the independent variable t, the time.

Also in control theory are important the discontinuous piecewise linear differential systems of the form (3.19) where instead of the function φ we have the discontinuous piecewise linear function

$$\psi(x_1)=\begin{cases}-1 & \text{if } x_1\in(-\infty,0),\\ 1 & \text{if } x_1\in(0,\infty).\end{cases} \tag{3.21}$$

For more details on these continuous and discontinuous piecewise linear differential systems see for instance the books (AIZERMAN, 1963) and (BARNETT; CAMERON, 1985).

The goal of this subsection is to study analytically the existence of limit cycles for a class of continuous and a class of discontinuous piecewise linear differential of the form (3.19).

More precisely, first we consider the class of continuous piecewise linear differential systems

$$\dot{x} = A_0 x + \varepsilon\big(Ax + \varphi(x_1)b\big), \tag{3.22}$$

with $|\varepsilon| \neq 0$ a sufficiently small real parameter, where A_0 is the $2n \times 2n$ matrix having on its principal diagonal the following 2×2 matrices

$$\begin{pmatrix} 0 & -(2k-1) \\ 2k-1 & 0 \end{pmatrix} \qquad \text{for } k = 1, \dots, n,$$

and zeros in the complement, A is an arbitrary $2n \times 2n$ matrix and $b \in \mathbb{R}^{2n} \setminus \{0\}$. Note that for $\varepsilon = 0$ system (3.22) becomes

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \quad \dots \quad , \dot{x}_{2n-1} = -(2n-1)x_{2n}, \quad \dot{x}_{2n} = (2n-1)x_{2n-1}. \tag{3.23}$$

Moreover, the origin of (3.23) is a *global isochronous center* in $\mathbb{R}^{2n}$, i.e. all its orbits different from the origin are periodic with period 2π.

In a similar way we consider the discontinuous piecewise linear differential systems

$$\dot{x} = A_0 x + \varepsilon\big(Ax + \psi(x_1)b\big). \tag{3.24}$$

The main results on the limit cycles of the continuous and discontinuous piecewise linear differential systems (3.22) are described below.

Theorem 3.2.8. For $|\varepsilon| > 0$ sufficiently small and if the conditions for applying the averaging theory of first order hold, then at most one limit cycle γ_ε of the continuous piecewise linear differential system (3.22) bifurcates from the periodic orbits of system (3.23), i.e. γ_ε tends to a periodic solution of system (3.23) when $\varepsilon \to 0$. Moreover there are systems (3.22) with $|\varepsilon| > 0$ sufficiently small having a such limit cycle.

The main tool for proving Theorem 3.2.8 is the averaging theory of first order for continuous differential systems presented in Theorem 3.2.1. In order to use this theorem we need to write the differential system (3.22) in the normal form (3.1), and for obtaining this we need to some changes of variables.

Lemma 3.2.9. Doing the change of variables $(x_1,x_2,\dots,x_{2n}) \mapsto (\theta,r,\theta_1,r_1,\dots,\theta_{n-1},r_{n-1})$ defined by

$$
\begin{aligned}
x_1 &= r\cos\theta,\\
x_2 &= r\sin\theta,\\
x_{2j-1} &= r_{j-1}\cos((2j-1)\theta+\theta_{j-1}),\\
x_{2j} &= r_{j-1}\sin((2j-1)\theta+\theta_{j-1}),
\end{aligned}
$$

for $j=2,\dots,n$ system (3.22) is transformed into the system

$$
\begin{aligned}
\frac{dr}{d\theta} &= \ \varepsilon H_1(\theta,r,\theta_1,r_1,\dots,\theta_{n-1},r_{n-1})+\mathscr{O}(\varepsilon^2),\\
\frac{dr_{j-1}}{d\theta} &= \ \varepsilon H_{2(j-1)}(\theta,r,\theta_1,r_1,\dots,\theta_{n-1},r_{n-1})+\mathscr{O}(\varepsilon^2),\\
\frac{d\theta_{j-1}}{d\theta} &= \ \varepsilon H_{2j-1}(\theta,r,\theta_1,r_1,\dots,\theta_{n-1},r_{n-1})+\mathscr{O}(\varepsilon^2),
\end{aligned}
\tag{3.25}
$$

where

$$
H_1 = \sum_{l=1}^{n} r_{l-1}\Big(F_{1,l}\cos\theta+F_{2,l}\sin\theta\Big)+\varphi(r\cos\theta)(b_1\cos\theta+b_2\sin\theta),
$$

and for $j=2,3,\dots,n$ we have

$$
\begin{aligned}
H_{2(j-1)} = \ & \sum_{l=1}^{n} r_{l-1}\Big(F_{2j-1,l}\cos((2j-1)\theta+\theta_{j-1})+F_{2j,l}\sin((2j-1)\theta+\theta_{j-1})\Big)\\
&+\varphi(r\cos\theta)\big[b_{2j-1}\cos((2j-1)\theta+\theta_{j-1})+b_{2j}\sin((2j-1)\theta+\theta_{j-1})\big],
\end{aligned}
$$

$$
\begin{aligned}
H_{2j-1} = \ & \sum_{l=1}^{n} \frac{r_{l-1}}{r_{j-1}}\Big(F_{2j,l}\cos((2j-1)\theta+\theta_{j-1})-F_{2j-1,l}\sin((2j-1)\theta+\theta_{j-1})\Big)\\
&+(2j-1)\sum_{l=1}^{n}\frac{r_{l-1}}{r}\Big(F_{1,l}\sin\theta-F_{2,l}\cos\theta\Big)\\
&+\varphi(r\cos\theta)\left(\frac{b_{2j}}{r_{j-1}}\cos((2j-1)\theta+\theta_{j-1})-\frac{b_{2j-1}}{r_{j-1}}\sin((2j-1)\theta+\theta_{j-1})\right)\\
&-(2j-1)\varphi(r\cos\theta)\left(\frac{b_2}{r}\cos\theta-\frac{b_1}{r}\sin\theta\right),
\end{aligned}
$$

with

$$
F_{i,l}=F_{i,l}(r,\theta,\theta_{l-1})=a_{i(2l-1)}\cos((2l-1)\theta+\theta_{l-1})+a_{i(2l)}\sin((2l-1)\theta+\theta_{l-1}).
$$

We take ε_0 sufficiently small, m arbitrarily large and

$$
D_m=\left\{(r,\theta_1,r_1,\dots,\theta_{n-1},r_{n-1})\in\left(\frac{1}{m},m\right)\times\left[\mathbb{S}^1\times\left(\frac{1}{m},m\right)\right]^{n-1}\right\}.
$$

Then the vector field of system (3.25) is well defined and continuous on $\mathbb{S}^1 \times D_m \times (-\varepsilon_0, \varepsilon_0)$. Moreover the system is 2π-periodic with respect to variable θ and locally Lipschitz with respect to variables $(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})$.

Proof. In the variables $(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})$ the differential system (3.22) becomes

$$\begin{aligned}
\dot{\theta} = &\ 1 + \frac{\varepsilon}{r}\left[\sum_{l=1}^{n} r_{l-1}\left(F_{2,l}\cos\theta - F_{1,l}\sin\theta\right) + \varphi(r\cos\theta)(b_2\cos\theta - b_1\sin\theta)\right],\\
\dot{r} = &\ \varepsilon H_1(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}),\\
\dot{r}_{j-1} = &\ \varepsilon H_{2(j-1)}(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}),\\
\dot{\theta}_{j-1} = &\ \varepsilon H_{2j-1}(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}),
\end{aligned}$$

for $j = 2, 3, \dots, n$. Note that for $\varepsilon = 0$, $\dot{\theta}(t) > 0$ and hence for $|\varepsilon| \neq 0$ sufficiently small this property remains valid for each t when $(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) \in \mathbb{S}^1 \times D_m$. Now we take θ as the new independent variable. The right-hand side of the new system is well defined and continuous in $\mathbb{S}^1 \times D_m \times (-\varepsilon_0, \varepsilon_0)$ and it is 2π-periodic with respect to the new variable θ and locally Lipschitz with respect to $(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})$. Now system (3.25) can be obtained doing a Taylor series expansion in the parameter ε around $\varepsilon = 0$. $\square$

The next step is to find the corresponding average function (3.3) of system (3.25) that we denoted by $f = (f_1, f_2, \dots, f_{2(n-1)}, f_{2n-1}) : D_m \to \mathbb{R}^{n-1}$ and it is defined by

$$\begin{aligned}
f_1 = &\ f_1(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) = \int_0^{2\pi} H_1(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})d\theta,\\
f_{2(j-1)} = &\ f_{2(j-1)}(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) = \int_0^{2\pi} H_{2(j-1)}(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})d\theta,\\
f_{2j-1} = &\ f_{2j-1}(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) = \int_0^{2\pi} H_{2j-1}(r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})d\theta,
\end{aligned}$$

for $j = 1, 2, \dots, n$. To calculate these integrals we will use the following equalities

$$\int_0^{2\pi} \cos((2j-1)\theta + \theta_{j-1})\sin((2l-1)\theta + \theta_{l-1})d\theta = 0 \qquad \text{for all integers } l, j > 1,$$

$$\int_0^{2\pi} \cos((2j-1)\theta + \theta_{j-1})\cos((2l-1)\theta + \theta_{l-1})d\theta = \begin{cases} \pi & \text{if } l = j,\\ 0 & \text{if } l \neq j,\end{cases}$$

$$\int_0^{2\pi} \sin((2j-1)\theta + \theta_{j-1})\sin((2l-1)\theta + \theta_{l-1})d\theta = \begin{cases} \pi & \text{if } l = j,\\ 0 & \text{if } l \neq j,\end{cases}$$

and the next lemma.

For $r > 0$ and $j = 1, 2, \dots, n$ we denote

$$
\begin{aligned}
I_j(r) = &\int_0^{2\pi} \varphi(r\cos\theta)\cos((2j-1)\theta)d\theta,\\
J_j(r) = &\int_0^{2\pi} \varphi(r\cos\theta)\sin((2j-1)\theta)d\theta,
\end{aligned}
$$

where φ is the piecewise linear function (3.20).

Lemma 3.2.10. The integrals I_j and $J_j(r)$ satisfy

$$
\begin{aligned}
I_j(r) = &\begin{cases} \pi r & \text{if } j=1 \text{ and } 0<r\le 1,\\ 0 & \text{if } j>1 \text{ and } 0<r\le 1,\\ K(r) & \text{if } j=1 \text{ and } r>1,\\ L_j(r) & \text{if } j>1 \text{ and } r>1; \end{cases}\\
J_j(r) = &\ 0 \qquad \text{for all } j=1,2,\dots,n \text{ and } r>0.
\end{aligned}
$$

where

$$
\begin{aligned}
L_j(r) = &\ \frac{2}{j(2j-1)^2}\Big((2j-1)\sqrt{r^2-1}\cos((2j-1)\arctan\sqrt{r^2-1})\\
&-\sin((2j-1)\arctan\sqrt{r^2-1})\Big),\\
K(r) = &\ \pi r + \frac{2}{r}\sqrt{r^2-1} - 2r\arctan(\sqrt{r^2-1}).
\end{aligned}
$$

Proof. We consider two cases: $0 < r \le 1$ and $r > 1$.

Case 1: $0 < r \le 1$ In this case $|r\cos\theta| \le 1$ and hence $\varphi(r\cos\theta) = r\cos\theta$ for all $\theta \in [0, 2\pi]$. Then if $j = 1$

$$\int_0^{2\pi} \varphi(r\cos\theta)\cos\theta d\theta = r\int_0^{2\pi}\cos^2\theta d\theta = \pi r,$$

and

$$\int_0^{2\pi} \varphi(r\cos\theta)\sin\theta d\theta = r\int_0^{2\pi}\cos\theta\sin\theta d\theta = 0.$$

And if $j > 1$ then

$$
\begin{aligned}
&\int_0^{2\pi} \varphi(r\cos\theta)\cos((2j-1)\theta)d\theta = r\int_0^{2\pi}\cos\theta\cos((2j-1)\theta)d\theta = 0,\\
&\int_0^{2\pi} \varphi(r\cos\theta)\sin((2j-1)\theta)d\theta = r\int_0^{2\pi}\cos\theta\sin((2j-1)\theta)d\theta = 0.
\end{aligned}
$$

Case 2: $r > 1$ In this case choose $\theta_c \in (0, \pi/2)$ such that $\cos\theta_c = 1/r$. If $j = 1$ we have

$$
\begin{aligned}
I_1(r) = &\int_0^{\theta_c} \cos\theta d\theta + r\int_{\theta_c}^{\pi-\theta_c} \cos^2\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} \cos\theta d\theta + r\int_{\pi+\theta_c}^{2\pi-\theta_c} \cos^2\theta d\theta + \int_{2\pi-\theta_c}^{2\pi} \cos\theta d\theta \\
= &\ \pi r + \frac{2}{r}\sqrt{r^2-1} - 2r\arctan(\sqrt{r^2-1}).
\end{aligned}
$$

The same reasoning can be applied to see that $J_1(r) = 0$. If $j > 1$ then

$$
\begin{aligned}
I_j(r) = &\int_0^{\theta_c} \cos((2j-1)\theta)d\theta + r\int_{\theta_c}^{\pi-\theta_c} \cos\theta\cos((2j-1)\theta)d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} \cos((2j-1)\theta)d\theta \\
&+ r\int_{\pi+\theta_c}^{2\pi-\theta_c} \cos\theta\cos((2j-1)\theta)d\theta + \int_{2\pi-\theta_c}^{2\pi} \cos((2j-1)\theta)d\theta \\
= &\ \frac{2}{j(2j-1)^2}\left((2j-1)\sqrt{r^2-1}\cos((2j-1)\arctan\sqrt{r^2-1}) - \sin((2j-1)\arctan\sqrt{r^2-1})\right),
\end{aligned}
$$

and $J_j(r) = 0$. □

With the results presented previously we are able to prove Theorem 3.2.8. Since we can choose m sufficiently large to find the zeroes of the average function f in D_m it is sufficient to look for them in $(0,\infty) \times [\mathbb{S}^1 \times (0,\infty)]^{n-1}$. To calculate the expression of the average function we consider again two cases.

Case 1: $0 < r \leq 1$. In this case the system whose zeros can provide limit cycles of system (3.22) is

$$
\begin{aligned}
f_1 = &\ (a_{11} + a_{22} + b_1)\pi r, \\
f_2 = &\ (a_{33} + a_{44})\pi r_1, \\
f_3 = &\ (a_{43} - a_{34} + 3(a_{12} - a_{21} - b_2))\pi, \\
&\ \vdots \\
f_{2(n-1)} = &\ (a_{(2n-1)(2n-1)} + a_{(2n)(2n)}))\pi r_{n-1}, \\
f_{2n-1} = &\ (a_{(2n)(2n-1)} - a_{(2n-1)(2n)} + (2n-1)(a_{12} - a_{21} - b_2)\pi.
\end{aligned}
\tag{3.26}
$$

Note that the variables $\theta_1, \theta_2, \ldots, \theta_{n-1}$ does not appear explicitly into system (3.26). Hence, if this system has zeros, it has a continuum of zeros. Therefore the assumption $\det(D_z f(a)) \neq 0$ of the averaging theory, presented in Theorem 3.2.1, is not satisfied and this theorem does not provide any information about the limit cycles of system (3.25).

Case 2: $r>1$. Now the system whose zeros can provide limit cycles of system (3.25) is

$$
\begin{aligned}
f_1 = &\quad (a_{11}+a_{22})\pi r + b_1 K(r),\\
f_2 = &\quad (a_{33}+a_{44})\pi r_1 + (b_3\cos\theta_1 + b_4\sin\theta_2)L_2(r),\\
f_3 = &\quad (a_{43}-a_{34}+3(a_{12}-a_{21}))\pi - \frac{3b_2 r_1 K(r) - r(b_4\cos\theta_1 - b_3\sin\theta_1)L_2(r)}{r r_1},\\
&\quad \vdots \\
f_{2(n-1)} = &\quad (a_{(2n-1)(2n-1)}+a_{(2n)(2n)})\pi r_{n-1} + (b_{2n-1}\cos\theta_{n-1} + b_{2n}\sin\theta_{n-1})L_n(r),\\
f_{2n-1} = &\quad (a_{(2n)(2n-1)} - a_{(2n-1)(2n)} + (2n-1)(a_{12}-a_{21}))\pi -\\
&\quad \frac{(2n-1)b_2 r_{n-1}K(r) - r(b_{2n}\cos\theta_{n-1} - b_{2n-1}\sin\theta_{n-1})L_n(r)}{r r_{n-1}},
\end{aligned}
\tag{3.27}
$$

For each $j\in\{2,3,\dots,n\}$ we will study the zeros of the system

$$
\begin{aligned}
f_1 = &\quad (a_{11}+a_{22})\pi r + b_1 K(r),\\
f_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)}+a_{(2j)(2j)})\pi r_{j-1} + (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1})L_j(r),\\
f_{2j-1} = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + (2j-1)(a_{12}-a_{21}))\pi -\\
&\quad \frac{(2j-1)b_2 r_{j-1}K(r) - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1})L_j(r)}{r r_{j-1}},
\end{aligned}
$$

Note that the function $K:(1,\infty)\to(\pi,4)$ is a diffeomorphism. Indeed note that K is twice differentiable with

$$K'(r) = \pi - 2\frac{\sqrt{r^2-1}}{r^2} - 2\arctan\sqrt{r^2-1},$$

and

$$K''(r) = -\frac{4}{r^3\sqrt{r^2-1}} < 0$$

which implies that K' is a strictly decreasing function. Moreover $\lim_{r\to\infty}K'(r)=0$ what means that $K'(r)$ has a horizontal asymptote given by the axis r and then $K'(r)\geq 0$. Suppose that there exists an $r_0\in(1,\infty)$ such that $K'(r_0)=0$. Then for all $r>r_0$ we have $K'(r)<K'(r_0)=0$, contradiction. Therefore it follows that $K'(r)\neq 0$ for all $r\in(1,\infty)$ and the Inverse Function Theorem guarantees that K is a local diffeomorphism and since that K is a injective function we obtain the global diffeomorphism, ending the proof of this claim.

First we note that in order that the equation $f_1 = 0$ has solutions with $r > 1$ it is necessary that $b_1(a_{11} + a_{22}) < 0$. Moreover $K''(r) < 0$ implies that the graph of K is convex. In the plane of the graph of $K(r)$ the graph of $(a_{11} + a_{22})\pi r$ is a straight line passing through the origin and then both graphs can intersect at most in two points.

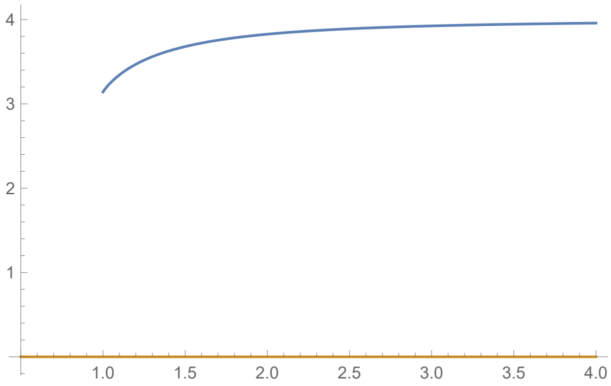

Figure 26 – The graphic of the function $K(r)$.

But if some straight line intercept the graph of $K(r)$ in two points then it cannot pass through the origin, as we can see in Figure 26. Then the equation $f_1 = 0$ has at most one solution if $r > 1$, and since that $K(r)$ is a diffeomorphism we can choose the coefficients a_{11}, a_{22} and b_1 so that this solution exists. We denote this solution by r_0 and we substitute it into the equations $f_{2(j-1)} = 0$ and $f_{2j-1} = 0$. Defining

$$\begin{aligned} A_j = &\ (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi, \qquad B_j = b_{2j-1}L_j(r_0), \qquad C_j = b_{2j}L_j(r_0), \\ D_j = &\ (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + (2j-1)(a_{12} - a_{21}))\pi - \frac{1}{r_0}(2j-1)b_2K(r_0), \\ u_j = &\ \cos\theta_{j-1}, \qquad v_j = \sin\theta_{j-1}, \end{aligned}$$

the system $f_{2(j-1)} = f_{2j-1} = 0$ is equivalent to the system

$$\begin{aligned} A_j r_{j-1} + B_j u_j + C_j v_j = &\ 0, \\ D_j r_{j-1} + C_j u_j - B_j v_j = &\ 0, \\ u_j^2 + v_j^2 - 1 = &\ 0. \end{aligned}$$

Using the two first equations we obtain

$$u_j = -\frac{(A_jB_j + C_jD_j)r_{j-1}}{B_j^2 + C_j^2}, \qquad v_j = \frac{(B_jD_j - A_jC_j)r_{j-1}}{B_j^2 + C_j^2}.$$

Substituting these two expressions in the third equation we get

$$(A_j^2+D_j^2)r_{j-1}^2-B_j^2-C_j^2=0.$$

Therefore at most there is one solution $r_{j-1}>0$, which provide a unique u_j and v_j. Since we fixed an arbitrarily j to solve this system, the same reasoning can be applied to each pair of equations $f_{2(j-1)}=0$ and $f_{2j-1}=0$, concluding that system (3.27) has at most one solution. Moreover taking conveniently the parameters of the initial system (3.22) this solution exists and its Jacobian is not zero. Hence at most one limit cycle can bifurcate from the periodic orbits of the center of system (3.23) when we perturbe it as in system (3.22), and there are systems for which a such limit cycles exist. This completes the proof of Theorem 3.2.8.

Replacing the function φ by ψ we get the discontinuous differential system (3.24) and the next result.

Theorem 3.2.11. For $|\varepsilon|>0$ sufficiently small and if the conditions for applying the averaging theory of first order hold, then at most one limit cycle γ_ε of the discontinuous piecewise linear differential system (3.24) bifurcates from the periodic orbits of system (3.23). Moreover there are systems (3.24) with $|\varepsilon|>0$ sufficiently small having a such limit cycle.

Proof. According to Theorem 3.2.2 the same kind of arguments used for proving Theorem 3.2.8 can be applied to the discontinuous system (3.24), obtaining that the average function $f=(f_1,f_2,\ldots,f_{2(n-1)},f_{2n-1}):D_m\to\mathbb{R}^{n-1}$ defined in (3.7) is

$$\begin{aligned}
f_1 = &\quad (a_{11}+a_{22})\pi r+b_1\widetilde{I}_1,\\
f_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)}+a_{(2j)(2j)})\pi r_{j-1}+(b_{2j-1}\cos\theta_{j-1}+b_{2j}\sin\theta_{j-1})\widetilde{I}_j,\\
f_{2j-1} = &\quad (a_{(2j)(2j-1)}-a_{(2j-1)(2j)}+(2j-1)(a_{12}-a_{21}))\pi-\\
&\quad \frac{(2j-1)b_2r_{j-1}\widetilde{I}_1-r(b_{2j}\cos\theta_{j-1}-b_{2j-1}\sin\theta_{j-1})\widetilde{I}_j}{rr_{j-1}},
\end{aligned} \tag{3.28}$$

for $j=2,3,\ldots,n$, where

$$\widetilde{I}_j=\begin{cases}-\dfrac{4}{(2j-1)} & \text{if } j \text{ is even},\\ \dfrac{4}{(2j-1)} & \text{if } j \text{ is odd}.\end{cases}$$

In fact if we define

$$\widetilde{I}_j=\int_0^{2\pi}\psi(r\cos\theta)\cos((2j-1)\theta)d\theta,\qquad \widetilde{J}_j=\int_0^{2\pi}\psi(r\cos\theta)\sin((2j-1)\theta)d\theta,$$

where ψ is the piecewise linear function given by (3.21). Then we have that

$$
\begin{aligned}
\widetilde{I}_j = & \int_0^{2\pi} \psi(r\cos\theta)\cos((2j-1)\theta)d\theta \\
= & \int_0^{\pi/2} \cos((2j-1)\theta)d\theta - \int_{\pi/2}^{3\pi/2} \cos((2j-1)\theta)d\theta + \int_{3\pi/2}^{2\pi} \cos((2j-1)\theta)d\theta \\
= & -\frac{4}{(2j-1)}\cos(j\pi),
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{J}_j = & \int_0^{2\pi} \psi(r\cos\theta)\sin((2j-1)\theta)d\theta \\
= & \int_0^{\pi/2} \sin((2j-1)\theta)d\theta - \int_{\pi/2}^{3\pi/2} \sin((2j-1)\theta)d\theta + \int_{3\pi/2}^{2\pi} \sin((2j-1)\theta)d\theta \\
= & -\frac{4}{(2j-1)}\sin(2j\pi)\cos(j\pi) = 0.
\end{aligned}
$$

Note that $\widetilde{I}_j$ is a constant real number different from zero, and hence f_1 is a straight line, and then system (3.28) has at most one positive zero. Moreover if we choose conveniently the coefficients b_1 a_{11} and a_{22} we can find a simple positive zero of system (3.28). This completes the proof of Theorem 3.2.11. □

Now we present an explicit example of a continuous piecewise linear differential system (3.22) in $\mathbb{R}^4$, and repeating for it the proof of Theorem 3.2.8 we will show it has one limit cycle.

Example 3.2.12. Consider the following differential system

$$
\dot{x} = A_0 x + \varepsilon(Ax + \varphi(x_1)b), \tag{3.29}
$$

where

$$
A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & \dfrac{18\pi-\sqrt{3}}{9\pi} \end{pmatrix}, b = \begin{pmatrix} -\dfrac{24\pi}{3\sqrt{3}+2\pi} \\ 1 \\ \dfrac{9(3-2\sqrt{3}\pi)}{2} \\ -1 \end{pmatrix}.
$$

Doing the change or variables $x_1 = r\cos\theta, x_2 = r\sin\theta, x_3 = r_1\cos(3\theta+\theta_1), x_4 = r_1\sin(3\theta+\theta_1)$ and taking θ as the new independent variable we obtain the system

$$
\begin{aligned}
r'(\theta) = \;& \frac{dr}{d\theta} = \varepsilon H_1(\theta, r, \theta_1, r_1) + \mathscr{O}(\varepsilon^2),\\
r_1'(\theta) = \;& \frac{dr_1}{d\theta} = \varepsilon H_2(\theta, r, \theta_1, r_1) + \mathscr{O}(\varepsilon^2),\\
\theta_1'(\theta) = \;& \frac{d\theta_1}{d\theta} = \varepsilon H_3(\theta, r, \theta_1, r_1) + \mathscr{O}(\varepsilon^2),\\
H_1(\theta, r, \theta_1, r_1) = \;& 2r + \varphi(r\cos\theta)\sin\theta + \cos\theta\left(r\sin\theta - \frac{24\pi\varphi(r\cos\theta)}{3\sqrt{3}+2\pi}\right),\\
H_2(\theta, r, \theta_1, r_1) = \;& -\frac{1}{18\pi}\Bigg(18\pi\varphi(r\cos\theta)\sin(3\theta+\theta_1) + 9\pi r_1\sin(2(3\theta+\theta_1))\\
& +\sqrt{3}r_1 + 81\pi(2\sqrt{3}\pi-3)\varphi(r\cos\theta)\cos(3\theta+\theta_1) -\\
& (\sqrt{3}-36\pi)r_1\cos(2(3\theta+\theta_1))\Bigg),\\
H_3(\theta, r, \theta_1, r_1) = \;& \sin^2(3\theta+\theta_1) + 2\sin(2(3\theta+\theta_1)) - \frac{\sin(2(3\theta+\theta_1))}{6\sqrt{3}\pi} + 3\sin^2\theta\\
& -\frac{72\pi\varphi(r\cos\theta)\sin\theta}{3\sqrt{3}r+2\pi r} - \frac{\varphi(r\cos\theta)\cos(3\theta+\theta_1)}{r_1} - \frac{3\varphi(r\cos\theta)\cos\theta}{r}\\
& +\frac{9\pi\sqrt{3}\varphi(r\cos\theta)\sin(3\theta+\theta_1)}{r_1} - \frac{27\varphi(r\cos\theta)\sin(3\theta+\theta_1)}{2r_1}.
\end{aligned}
\tag{3.30}
$$

After some computations the average function $f = (f_1, f_2, f_3)$ is

$$
\begin{aligned}
f_1(r, \theta_1, r_1) = \;& 4\pi r - \frac{24\pi}{3\sqrt{3}+2\pi}\left(\pi r + \frac{2\sqrt{r^2-1}}{r} - 2r\arctan(\sqrt{r^2-1})\right),\\
f_2(r, \theta_1, r_1) = \;& \frac{\sqrt{3}}{3}\sin\theta_1 + \frac{3}{2}(2\sqrt{3}\pi-3)\sqrt{3}\cos\theta_1 - \frac{\sqrt{3}}{9}r_1,\\
f_3(r, \theta_1, r_1) = \;& \frac{9\sqrt{3}\sin\theta_1}{2r_1} - \frac{9\pi\sin\theta_1}{r_1} + \frac{\sqrt{3}\cos\theta_1}{3r_1} - \frac{3}{2}\left(\sqrt{3}+\frac{2\pi}{3}\right) + 4\pi.
\end{aligned}
$$

In order to solve the system $f_1 = f_2 = f_3 = 0$ we can use the same reasoning applied in the proof of Theorem 3.2.8 obtaining that $(r^*, \theta_1^*, r_1^*) = (2, \pi/2, 3)$ is a zero of the average function. Moreover if $J = J(r, \theta_1, r_1)$ is the Jacobian matrix of h, then $\det J(2, \pi/2, 3) \neq 0$ which implies that we have a simple zero. By Theorem 3.2.2 system (3.30) and consequently system (3.29) has one limit cycle for $|\varepsilon| > 0$ sufficiently small.

If instead of the matrix A_0 we consider the matrix A_1 where A_1 is the $2n \times 2n$ matrix having on its principal diagonal the following 2×2 matrices

$$\begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \qquad \text{for } k = 1, \ldots, n,$$

and zeros in the complement, then the averaging theory of first order does not provide any information about the limit cycles of the systems. Indeed we have the following results.

Proposition 3.2.13. Assume that the conditions for applying the averaging theory of first order hold. Then this theory does not provide any information about the limit cycles of the continuous piecewise linear differential system

$$\dot{x} = A_1 x + \varepsilon \big(Ax + \varphi(x_1) b \big). \tag{3.31}$$

Proof. Doing the change of coordinates

$$\begin{aligned} &x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \\ &x_{2j-1} = r_{j-1}\cos(j\theta + \theta_{j-1}), \qquad x_{2j} = r_{j-1}\sin(j\theta + \theta_{j-1}) \quad j \in \{2, 3, \ldots, n\}, \end{aligned}$$

for $j = 2, 3, \ldots, n$, to the continuous piecewise linear differential system (3.31), and working as in the proof of Theorem 3.2.8 we obtain that the average function $f = (f_1, f_2, \ldots, f_{2n-1})$ now is given by

$$\begin{aligned} f_1 = {} & (a_{11} + a_{22})\pi r + b_1 I_1(r), \\ f_{2(j-1)} = {} & (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi r_{j-1} + (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1}) I_j(r), \\ f_{2j-1} = {} & (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + j(a_{12} - a_{21}))\pi - \\ & \frac{j b_2 r_{j-1} I_1(r) - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1}) I_j(r)}{r r_{j-1}}, \end{aligned} \tag{3.32}$$

where

$$I_j(r) = \int_0^{2\pi} \varphi(r\cos\theta)\cos(j\theta)\, d\theta.$$

Using exactly the same arguments than in the proof of Lemma 3.2.10 is possible to prove that

$$I_j(r) = \begin{cases} \pi r & \text{if } j = 1 \text{ and } 0 < r \le 1, \\ 0 & \text{if } j \text{ is even and } 0 < r \le 1, \\ L_j(r) & \text{if } j \text{ is odd and } r > 1, \end{cases}$$

where

$$L_j(r) = \frac{4}{j(j^2-1)} \Big(j\sqrt{r^2-1}\cos(j\arctan(\sqrt{r^2-1})) - \sin(j\arctan(\sqrt{r^2-1})) \Big).$$

The simple zeros of system (3.32) provide the existence of limit cycles for system (3.31) but since $I_j(r) = 0$ if j is even and $r > 1$, the variables θ_{j-1}, for $j = 2, 4, 6, ...$ do not appear in the system $f_1 = f_2 = \ldots = f_{2n-1} = 0$, so either this system has no zeros, or if it has zeros, then it has a continuum of zeros, and consequently the averaging theory cannot say anything about the limit cycles of system (3.31). The same occurs for the case $0 < r \leq 1$. So we conclude that, using the averaging theory of first order, we can say nothing about the number of the limit cycles of system (3.31). □

Proposition 3.2.14. Assume that the conditions for applying the averaging theory of first order hold. Then this theory does not provide any information about the limit cycles of the discontinuous piecewise linear differential system

$$\dot{x} = A_1 x + \varepsilon\big(Ax + \psi(x_1)b\big). \tag{3.33}$$

Proof. Now if we consider the discontinuous piecewise linear differential system (3.33), then its average function $f = (f_1, f_2, \ldots, f_{2n-1})$ is

$$\begin{aligned}
f_1 = &\quad (a_{11} + a_{22})\pi r + b_1 \widetilde{I}_1, \\
f_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi r_{j-1} + (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1})\widetilde{I}_j, \\
f_{2j-1} = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + j(a_{12} - a_{21}))\pi - \\
&\quad \frac{jb_2 r_{j-1}\widetilde{I}_1 - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1})\widetilde{I}_j}{rr_{j-1}},
\end{aligned} \tag{3.34}$$

where

$$\widetilde{I}_j = \int_0^{2\pi} \psi(r\cos\theta)\cos(j\theta)d\theta.$$

Again we have that

$$\widetilde{I}_j = \int_0^{2\pi} \psi(r\cos\theta)\cos(j\theta)d\theta = \begin{cases} 0 & \text{if } j \text{ is even}, \\ \pm\dfrac{4}{(2j-1)} & \text{if } j \text{ is odd}, \end{cases}$$

and we can see that again either no zeros of the function f, or a continuum of zeros, concluding that the averaging theory of first order given by Theorem 3.2.2 does not say anything about the limit cycles of system (3.33). □

Remark 3.2.15. Note the difference between the matrices A_0 and A_1, in the matrix A_0 the non-zero entries are only the odd numbers $1, 3, \ldots, 2n-1$, while in the matrix A_1 the non-zero entries are the numbers $1, 2, \ldots, n$. This difference provides that the continuous and discontinuous piecewise linear differential systems (3.22) and (3.24) can have limit cycles detected by the averaging theory, while for the continuous and discontinuous piecewise linear differential systems (3.31) and (3.33) the averaging theory cannot detect limit cycles.

CHAPTER

4

AN EXTENSION OF THE AVERAGING THEORY FOR NONSMOOTH SYSTEMS

The main goal of this chapter is present an extension of the high order averaging to nonsmooth differential systems defined in $\mathbb{R}^n$, $n \geq 2$. The first sections of this chapter are dedicated to described the previous results about high order averaging and the class of systems where the theory shall be developed. After that the averaged functions for nonsmooth differential systems are defined and the main result is presented. It is worth to mentioned that the averaged functions defined in this chapter can be implemented in algebraic manipulators as Mathematica and Maple. To finish this chapter, the proof of the main result is given as well applications.

4.1 Background on the averaging theory

Let D be an open bounded subset of $\mathbb{R}_+$ and denote $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$. Consider C^{k+1} functions $F_i : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 0, 1, 2, \ldots, k$, and $R : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$. Note that $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$ means that the above functions are 2π-periodic in the variable θ. Now consider the following differential equation

$$r'(\theta) = \sum_{i=0}^{k} \varepsilon^i F_i(\theta, r) + \varepsilon^{k+1} R(\theta, r, \varepsilon), \tag{4.1}$$

and assume that the solution $\varphi(\theta, \rho)$ of the unperturbed system $r'(\theta) = F_0(\theta, r)$, such that $\varphi(0, \rho) = \rho$, is 2π-periodic for every $\rho \in D$. Here the prime denotes the derivative in the variable θ.

A central question in the study of system (4.1) is to understand which periodic orbits of the unperturbed system $r'(\theta) = F_0(\theta, r)$ persists for $|\varepsilon| \neq 0$ sufficiently small. In others words to provide sufficient conditions for the persistence of isolated periodic solutions. The averaging theory is one of the best tools to track this problem. Summarizing, it consists in defining a collection of functions $f_i : D \to \mathbb{R}$, for $i = 1, 2, \ldots, k$, called averaged functions, such that their simple zeros provide the existence of isolated periodic solutions of the differential equation (4.1). In (LLIBRE; NOVAES; TEIXEIRA, 2014b; LLIBRE; NOVAES; TEIXEIRA, 2014a) it was proved that these averaged functions are

$$f_i(\rho) = \frac{y_i(2\pi,\rho)}{i!}, \tag{4.2}$$

where $y_i : \mathbb{R}\times D \to \mathbb{R}$ for $i = 1,2,\dots,k$, are defined recurrently by the following integral equations

$$\begin{aligned}
y_1(\theta,\rho) = & \int_0^\theta \Big(F_1(\phi,\varphi(\phi,\rho)) + \partial F_0(\phi,\varphi(\phi,\rho))y_1(\phi,\rho)\Big)d\phi,\\
y_i(\theta,\rho) = & \; i!\int_0^\theta \Big(F_i(\phi,\varphi(\phi,\rho)) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\\
& \cdot\partial^L F_{i-l}(\phi,\varphi(\phi,\rho))\prod_{j=1}^{l} y_j(\phi,\rho)^{b_j}\Big)d\phi, \text{ for } i = 2,\dots,k.
\end{aligned} \tag{4.3}$$

Here $\partial^L G(\phi,\rho)$ denotes the derivative order L of a function G with respect to the variable ρ, and S_l is the set of all l-tuples of non-negative integers $(b_1,b_2,\dots,b_l)$ satisfying $b_1+2b_2+\cdots+lb_l = l$, and $L = b_1+b_2+\cdots+b_l$.

When we consider the above problem in the world of discontinuous piecewise smooth differential systems it is not always true that the higher averaged functions (4.2) allow to study the persistence of isolated periodic solutions. In (LLIBRE; NOVAES; TEIXEIRA, 2015; LLIBRE; MEREU; NOVAES, 2015) this problem was considered for general Filippov systems when $F_0(\theta,r)\equiv 0$ and it was proved that the averaged function of first order can provide information on the existence of crossing isolated periodic solutions. Furthermore the authors have found conditions on those systems in order to assure that the averaged function of second order also provides information on the existence of crossing isolated periodic solutions. When $F_0(\theta,r)\not\equiv 0$ but the solutions of the unperturbed system $\dot r = F_0(\theta,r)$ are 2π-periodic the authors in (LLIBRE JAUME NOVAES, 2015) have found conditions on those systems in order to assure that the averaged function of first order provides information on the existence of crossing isolated periodic solutions.

4.2 Standard form

In what follows we introduce a class of discontinuous nonautonomous piecewise smooth differential equations for which the averaged functions (4.2) at any order provide informations on the existence of isolated periodic solutions.

Let $n>1$ be a positive integer, $\alpha_0 = 0$, $\alpha_n = 2\pi$ and $\alpha = (\alpha_1,\dots,\alpha_{n-1})\in\mathbb{T}^{n-1}$ a $(n-1)$-tuple of angles such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n = 2\pi$. For $i = 0,1,\dots,k$ and $j = 1,2,\dots,n$, let $F_i^j : \mathbb{S}^1\times D\to\mathbb{R}$ and $R^j : \mathbb{S}^1\times D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}$ be C^{k+1} functions, where

D is an open bounded interval of $\mathbb{R}_+$ and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$. Denote

$$\begin{aligned} F_i(\theta,r) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) F_i^j(\theta,r),\ i=0,1,\dots,k, \quad \text{and} \\ R(\theta,r,\varepsilon) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) R^j(\theta,r,\varepsilon), \end{aligned} \tag{4.4}$$

where $\chi_A(\theta)$ denotes the characteristic function of an interval A:

$$\chi_A(\theta) = \begin{cases} 1 & \text{if } \theta \in A, \\ 0 & \text{if } \theta \notin A. \end{cases}$$

We note that $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$ means that the above functions are 2π-periodic in the variable θ.

The main result of this chapter concern about the existence of isolated periodic solutions of the following discontinuous nonautonomous 2π-periodic piecewise smooth differential equation

$$r'(\theta) = \sum_{i=0}^{k} \varepsilon^i F_i(\theta,r) + \varepsilon^{k+1} R(\theta,r,\varepsilon). \tag{4.5}$$

In this case the set of discontinuity is given by $\Sigma = (\{\theta = 0 \equiv 2\pi\} \cup \{\theta = \alpha_1\} \cup \cdots \cup \{\theta = \alpha_{n-1}\}) \cap \mathbb{S}^1 \times D$. In short, we shall provide sufficient conditions in order to show that, for $|\varepsilon| \neq 0$ sufficiently small, the averaged functions (4.2) at any order can be used to ensure the existence of crossing limit cycles. It is worth to mention that the L^{th} derivative of the discontinuous function F_i with respect to the second variable, $\partial^L F_i(\theta,r)$, which appears in the averaged functions (4.2), is given by

$$\partial^L F_i(\theta,r) = \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) \partial^L F_i^j(\theta,r),\ i=0,1,\dots,k.$$

Denote by $\varphi(\theta,\rho)$ the solution of the system $r'(\theta) = F_0(\theta,r)$ such that $\varphi(0,\rho) = \rho$. From now on this last system will be called unperturbed system. We assume the following hypothesis:

(H1) For each $\rho \in D$ the solution $\varphi(\theta,\rho)$ is defined for every $\theta \in \mathbb{S}^1$, it reaches Σ only at crossing points, and it is 2π-periodic.

4.3 The averaged functions

In this section we develop a recurrence to compute the averaged function (4.2) in the particular case of the discontinuous differential equation (4.5). So consider the functions

$z_i^j : (\alpha_{j-1}, \alpha_j] \times D \to \mathbb{R}$ defined recurrently for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$, as:

$$
\begin{aligned}
z_1^1(\theta,\rho) = & \int_0^\theta \Big(F_1^1(\phi,\varphi(\phi,\rho)) + \partial F_0^1(\phi,\varphi(\phi,\rho)) z_1^1(\phi,\rho) \Big) d\phi, \\
z_i^1(\theta,\rho) = & \; i! \int_0^\theta \Big(F_i^1(\phi,\varphi(\phi,\rho)) \\
& + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2!2!^{b_2} \cdots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^1(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m^1(\phi,\rho)^{b_m} \Big) d\phi, \qquad (4.6) \\
z_i^j(\theta,\rho) = & \; z_i^{j-1}(\alpha_{j-1},\rho) + i! \int_{\alpha_{j-1}}^\theta \Big(F_i^j(\phi,\varphi(\phi,\rho)) \\
& + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2!2!^{b_2} \cdots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^j(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m^j(\phi,\rho)^{b_m} \Big) d\phi.
\end{aligned}
$$

Thus we have the next result.

Proposition 4.3.1. For $i = 1, 2, \ldots, k$, the averaged function (4.2) of order i, is

$$
f_i(\rho) = \frac{z_i^n(2\pi,\rho)}{i!}. \qquad (4.7)
$$

Proof. For each $i = 1, 2, \cdots, k$, define

$$
z_i(\theta,\rho) = \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) z_i^j(\theta,\rho). \qquad (4.8)
$$

Given $\theta \in [0, 2\pi]$ there exists a positive integer $\bar{k}$ such that $\theta \in (\alpha_{\bar{k}-1}, \alpha_{\bar{k}}]$ and, therefore $z_i(\theta,\rho) = z_i^{\bar{k}}(\theta,\rho)$. Moreover using the expressions (4.17) and (4.8) we can write (4.6) into the form

$$
\begin{aligned}
z_1^1(\theta,\rho) = & \int_0^\theta \Big(F_1(\phi,\varphi(\phi,\rho)) + \partial F_0(\phi,\varphi(\phi,\rho)) z_1(\phi,\rho) \Big) d\phi, \\
z_i^1(\theta,\rho) = & \; i! \int_0^\theta \Big(F_i(\phi,\varphi(\phi,\rho)) \\
& + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2!2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_{i-l}(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m(\phi,\rho)^{b_m} \Big) d\phi, \qquad (4.9) \\
z_i^{\bar{k}}(\theta,\rho) = & \; z_i^{\bar{k}-1}(\alpha_{\bar{k}-1},\rho) + i! \int_{\alpha_{\bar{k}-1}}^\theta \Big(F_i(\phi,\varphi(\phi,\rho)) \\
& + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2!2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_{i-l}(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m(\phi,\rho)^{b_m} \Big) d\phi.
\end{aligned}
$$

In the above equality we are denoting

$$\partial^L F_{i-l}(\phi,\varphi(\phi,\rho)) = \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\phi)\partial^L F^j_{i-l}(\phi,\varphi(\phi,\rho)).$$

Proceeding recursively on $\bar{k}$ we obtain

$$\begin{aligned}
z_1(\theta,\rho) = & \int_0^\theta \Big(F_1(\phi,\varphi(\phi,\rho)) + \partial F_0(\phi,\varphi(\phi,\rho)) z_1(\phi,\rho) \Big) d\phi, \\
z_i(\theta,\rho) = & \sum_{p=1}^{\bar{k}-1} \int_{\alpha_{p-1}}^{\alpha_p} \Big(F_i^p(\phi,\varphi(\phi,\rho)) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \\
& \cdot \partial^L F^p_{i-l}(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m^p(\phi,\rho)^{b_m} \Big) d\phi + \int_{\alpha_{\bar{k}-1}}^{\theta} \Big(F_i^{\bar{k}}(\phi,\varphi(\phi,\rho)) \\
& + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F^{\bar{k}}_{i-l}(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m^{\bar{k}}(\phi,\rho)^{b_m} \Big) d\phi \\
= & \; i! \int_0^\theta \Big(F_i(\phi,\varphi(\phi,\rho)) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \\
& \cdot \partial^L F_{i-l}(\phi,\varphi(\phi,\rho)) \prod_{m=1}^{l} z_m(\phi,\rho)^{b_m} \Big) d\phi.
\end{aligned} \tag{4.10}$$

Computing the derivative in the variable θ of the expressions (4.10) and (4.3) for $i = 1$ we see that the functions $z_1(\theta,\rho)$ and $y_1(\theta,\rho)$ satisfy the same differential equation. Moreover for each $i = 2,\cdots,k$, the integral equations (4.3) and (4.10) which provides respectively y_i and z_i are defined by the same recurrence. Therefore we conclude that y_i and z_i satisfy the same differential equations for $i = 1,2,\cdots,k$, which are linear with variable coefficients (that is the Existence and Uniqueness Theorem holds). Now, it only remains to prove that their initial conditions coincide. Let $i \in \{1,2,\ldots,k\}$, since $y_i(0,\rho) = 0$ and, by (4.9), $z_i(0,\rho) = 0$, it follows that the initial conditions are the same. Hence $y_i(\theta,\rho) = z_i(\theta,\rho)$, which concludes the Proposition. □

Note that when $F_0 \neq 0$ the recurrence defined in (4.6) is actually an integral equation. Moreover in order to implement an algorithm to compute the averaged function, it may be easier to write each z_i^j in terms of the partial Bell polynomials, which are already implemented in algebraic manipulators as Mathematica and Maple. For each pair of nonnegative integers (p,q), the partial Bell polynomial is defined as

$$B_{p,q}(x_1,x_2,\ldots,x_{p-q+1}) = \sum_{\widetilde{S}_{p,q}} \frac{p!}{b_1! b_2! \cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!} \right)^{b_j}, \tag{4.11}$$

where $\widetilde{S}_{p,q}$ is the set of all $(p-q+1)$-tuple of nonnegative integers $(b_1,b_2,\ldots,b_{p-q+1})$ satisfying $b_1 + 2b_2 + \cdots + (p-q+1)b_{p-q+1} = p$, and $b_1 + b_2 + \cdots + b_{p-q+1} = q$. In the next proposition, following (NOVAES, 2017), we solve the integral equation (4.6) to provide the explicit recurrence formula for z_i^j in terms of the Bell polynomials.

Proposition 4.3.2. For each $j = 1,2,\dots,n$ let $\eta_j(\theta,\rho)$ be defined as

$$\eta_j(\theta,\rho) = \int_{\alpha_{j-1}}^{\theta} \partial F_0^j(\phi,\varphi(\phi,\rho))d\phi.$$

Then for $i = 1,2,\dots,k$ and $j = 1,2,\dots,n$ the recurrence (4.6) can be written as follows

$$z_1^1(\theta,\rho) = e^{\eta_1(\theta,\rho)} \int_0^{\theta} e^{-\eta_1(\phi,\rho)} F_1^1(\phi,\varphi(\phi,\rho))d\phi,$$

$$z_1^j(\theta,\rho) = e^{\eta_j(\theta,\rho)} \left(z_1^{j-1}(\alpha_{j-1},\rho) + \int_{\alpha_{j-1}}^{\theta} e^{-\eta_j(\phi,\rho)} F_1^j(\phi,\varphi(\phi,\rho))d\phi \right), \quad \text{for } j = 2,\dots$$

$$\begin{aligned} z_i^1(\theta,\rho) = e^{\eta_1(\theta,\rho)} i! \int_0^{\theta} e^{-\eta_1(\phi,\rho)} \Bigg[& F_i^1(\phi,\varphi(\phi,\rho)) \\ & + \sum_{l=1}^{i-1}\sum_{m=1}^{l} \frac{1}{l!} \partial^m F_{i-l}^1(\theta,\varphi(\theta,\rho)) B_{l,m}(z_1^1,z_2^1,\dots,z_{l-m+1}^1) \\ & + \sum_{m=2}^{i} \frac{1}{i!} \partial^m F_0^1(\theta,\varphi(\theta,\rho)) B_{i,m}(z_1^1,z_2^1,\dots,z_{i-m+1}^1) \Bigg] d\phi, \qquad \text{for } i = 2,\dots \end{aligned}$$

$$\begin{aligned} z_i^j(\theta,\rho) = e^{\eta_j(\theta,\rho)} \Bigg(z_i^{j-1}(\alpha_{j-1},\rho) + i! \int_{\alpha_{j-1}}^{\theta} e^{-\eta_j(\phi,\rho)} \Bigg[& F_i^j(\phi,\varphi(\phi,\rho)) \\ & + \sum_{l=1}^{i-1}\sum_{m=1}^{l} \frac{1}{l!} \partial^m F_{i-l}^j(\theta,\varphi(\theta,\rho)) B_{l,m}(z_1^j,z_2^j,\dots,z_{l-m+1}^j) \\ & + \sum_{m=2}^{i} \frac{1}{i!} \partial^m F_0^j(\theta,\varphi(\theta,\rho)) B_{i,m}(z_1^j,z_2^j,\dots,z_{i-m+1}^j) \Bigg] d\phi \Bigg), \qquad \text{for } i,j = 2,\dots. \end{aligned}$$

Proof. We shall prove this proposition for $i = 1,2,\dots,k$, and $j = 1$. The other cases will follow in a similar way.

For $i = j = 1$, the integral equation (4.6) is equivalent to the following Cauchy problem:

$$\frac{\partial z_1^1}{\partial \theta}(\theta,\rho) = F_1^1(\theta,\varphi(\theta,\rho)) + \partial F_0^1(\theta,\varphi(\theta,\rho))\, z_1^1 \ \text{ with } \ z_1^1(0,\rho) = 0.$$

Solving the above linear differential equation we get

$$z_1^1(\theta,\rho) = e^{\eta_1(\theta,\rho)} \int_0^{\theta} e^{-\eta_1(\phi,\rho)} F_1^1(\phi,\varphi(\phi,\rho))d\phi.$$

Now for $i = 2, \ldots, k$ and $j = 1$ the recurrence (4.6) can be written in terms of the partial Bell polynomials as (for more details, see (NOVAES, 2017)):

$$\begin{aligned} z_i^1(\theta,\rho) = & \; i! \int_0^\theta \Bigg(F_i^1(\phi,\varphi(\phi,\rho)) \\ & + \sum_{l=1}^{i} \sum_{m=1}^{l} \frac{1}{l!} \partial^m F_{i-l}^1(\phi,\varphi(\phi,\rho)) B_{l,m}(z_1^1, z_2^1, \ldots, z_{l-m+1}^1) \Bigg) d\phi. \end{aligned} \tag{4.12}$$

We note that the function z_i^1 appears in the right hand side of (4.12) only if $l = i$ and $m = 1$. In this case $B_{i,1}(z_1^1, z_2^1, \ldots, z_i^1) = z_i^1$ for every $i \geq 1$. So we can rewrite (4.12) as the following integral equation

$$\begin{aligned} z_i^1(\theta,\rho) = & \; i! \int_0^\theta \Bigg(F_i^1(\phi,\varphi(\phi,\rho)) \\ & + \sum_{l=1}^{i-1} \sum_{m=1}^{l} \frac{1}{l!} \partial^m F_{i-l}^1(\phi,\varphi(\phi,\rho)) B_{l,m}(z_1^1, z_2^1, \ldots, z_{l-m+1}^1) \\ & + \sum_{m=2}^{i} \frac{1}{i!} \partial^m F_0^1(\phi,\varphi(\phi,\rho)) B_{i,m}(z_1^1, z_2^1, \ldots, z_{i-m+1}^1) \\ & + \frac{1}{i!} \partial F_0^1(\phi,\varphi(\phi,\rho)) B_{i,1}(z_1^1, z_2^1, \ldots, z_i^1) \Bigg) d\phi, \end{aligned}$$

which is equivalent to the following Cauchy problem:

$$\begin{aligned} \frac{\partial z_i^1}{\partial \theta}(\theta,\rho) = & \; i! \Bigg[F_i^1(\theta,\varphi(\theta,\rho)) + \frac{1}{i!} \partial F_0^1(\theta,\varphi(\theta,\rho)) z_i^1 \\ & + \sum_{l=1}^{i-1} \sum_{m=1}^{l} \frac{1}{l!} \partial^m F_{i-l}^1(\theta,\varphi(\theta,\rho)) B_{l,m}(z_1^1, z_2^1, \ldots, z_{l-m+1}^1) \\ & + \sum_{m=2}^{i} \frac{1}{i!} \partial^m F_0^1(\theta,\varphi(\theta,\rho)) B_{i,m}(z_1^1, z_2^1, \ldots, z_{i-m+1}^1) \Bigg], \\ z_i^1(0,\rho) = & \; 0. \end{aligned}$$

Solving the above linear differential equation we obtain the expressions of $z_i^1(\theta,\rho)$, for $i = 2, \ldots, k$, given in the statement of the proposition. □

4.4 Statement and Proof of the Main Result

Considering the study presented above we present the next theorem. In short, it says that the averaged functions still controlling the number of 2π-periodic solution $r(\theta,\varepsilon)$ of system (4.5).

Theorem 4.4.1. Assume that $(H1)$ holds and that for some $l \in \{1, 2, \ldots, k\}$ the functions defined in (4.2) satisfy $f_s = 0$ for $s = 1, 2, \ldots, l-1$ and $f_l \neq 0$. If there exists $\rho^* \in D$ such that $f_l(\rho^*) = 0$ and $f_l'(\rho^*) \neq 0$, then for $|\varepsilon| \neq 0$ sufficiently small there exists a 2π-periodic solution $r(\theta, \varepsilon)$ of system (4.5) such that $r(0, \varepsilon) \to \rho^*$ when $\varepsilon \to 0$.

Remark 4.4.2. The assumption $D \subset \mathbb{R}_+$ is not restrictive. In fact, if one consider D as being an open subset of $\mathbb{R}^n$ the conclusion of Theorem 4.4.1 still holds by assuming that the Jacobian matrix $Jf_l(\rho^*)$ is nonsingular, that is $\det(Jf_l(\rho^*)) \neq 0$. In this case the derivative $\partial^L G(\phi, \rho)$ is a symmetric L-multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^n$, denoted as $\prod_{j=1}^{L} y_j \in \mathbb{R}^{nL}$ (see (LLIBRE; NOVAES; TEIXEIRA, 2014b)).

For the particular class of systems (4.5) Theorem 4.4.1 generalizes the main results of (LLIBRE; MEREU; NOVAES, 2015; LLIBRE JAUME NOVAES, 2015; LLIBRE; NOVAES; TEIXEIRA, 2015), increasing the order of the averaging theory. It also generalizes the main results of (ITIKAWA; LLIBRE; NOVAES, 2017; WEI; ZHANG, 2018) dealing now with nonvanishing unperturbed systems and allowing more zones of continuity. For piecewise smooth systems (4.5), the first return map is given, necessarily, by a composition of several maps. When one consider only two zones of continuity, the displacement function can be studied straightforwardly and then used, instead of the first return map, to obtain the averaged functions (4.2). In this case, dealing with composition of maps can be avoided. However when one consider more zones of continuity, as we shall see, it is inevitable.

Before to prove Theorem 4.4.1 we introduce a class of autonomous piecewise smooth differential system that can be studied via it. More specifically, we shall see that this class of autonomous systems can be transformed into the standard form (4.5). The construction performed in the sequel has been done in (LLIBRE; MEREU; NOVAES, 2015) for a particular class of systems.

Let $n > 1$ be a positive integer, $\alpha_0 = 0$, $\alpha_n = 2\pi$ and $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{T}^{n-1}$ a $(n-1)$-tuple of angles such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n = 2\pi$. Let $\mathscr{X}(x, y; \varepsilon) = (X_1, X_2, \ldots, X_n)$ be a n-tuple of smooth vector fields defined on an open bounded neighborhood $U \subset \mathbb{R}^2$ of the origin and depending on a small parameter ε in the following way

$$X_j(x, y; \varepsilon) = \sum_{i=0}^{k} \varepsilon^i X_i^j(x, y) \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{4.13}$$

For $j = 1, \ldots, n$ let L_j be the intersection between the domain U with the ray starting at the origin and passing through the point $(\cos \alpha_j, \sin \alpha_j)$, and take $\Sigma = \bigcup_{j=1}^{n} L_j$. We note that Σ splits the set $U \backslash \Sigma \subset \mathbb{R}^2$ in n disjoint open sectors. We denote the sector delimited by L_j and L_{j+1}, in counterclockwise sense, by C_j, for $j = 1, 2, \ldots, n-1$, and by C_n the sector delimited by L_n and L_1.

Now let $Z_{\mathscr{X},\alpha} : U \to \mathbb{R}^2$ be a discontinuous piecewise smooth vector field defined as $Z_{\mathscr{X},\alpha}(x, y; \varepsilon) = X_j(x, y; \varepsilon)$ when $(x, y) \in C_j$, and consider the following planar discontinuous piecewise smooth differential system

$$(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;\varepsilon). \tag{4.14}$$

The above notation means that at each sector C_j we are considering the smooth differential system

$$(\dot{x},\dot{y})^T = X_j(x,y;\varepsilon). \tag{4.15}$$

As our main hypothesis we shall assume that there exists a period annulus $\mathscr{A}$ surrounding the origin, fulfilled by crossing periodic solutions of the unperturbed system $(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;0)$.

Theorem 4.4.1 deals with periodic nonautonomous differential systems in the standard form (4.1). Therefore in order to use the averaging theory for studying system (4.14) it has to be written in the standard form. A possible approach for doing this is to consider the polar change of variables $x = r\cos\theta$ and $y = r\sin\theta$. However the appropriate change of variables may depend on the initial system (4.14). In general, for each $j = 1,2,\ldots,n$, after a suitable change of variables system (4.15) reads

$$r'(\theta) = \frac{\dot{r}(t)}{\dot{\theta}(t)} = \sum_{i=0}^{k} \varepsilon^i F_i^j(\theta,r) + \varepsilon^{k+1} R^j(\theta,r,\varepsilon). \tag{4.16}$$

Now $\theta \in [\alpha_{j-1},\alpha_j]$, $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}$ and $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ are C^{k+1} functions depending on the vector fields X_i^j, and they are 2π-periodic in the first variable, being D an open bounded interval of $\mathbb{R}_+$ and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$. Therefore denoting

$$\begin{aligned} F_i(\theta,r) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) F_i^j(\theta,r), \; i = 0,1,\ldots,k, \quad \text{and} \\ R(\theta,r,\varepsilon) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1},\alpha_j]}(\theta) R^j(\theta,r,\varepsilon), \end{aligned} \tag{4.17}$$

system (4.14) reads like (4.5).

Proof of Theorem 4.4.1

The proof of Theorem 4.4.1 is based on a preliminary result (see Lemma 4.4.3) which expands the solutions of the discontinuous differential equation (4.5) in powers of ε.

From hypothesis $(H1)$ the solution $\varphi(\theta,\rho)$ of the unperturbed system reads

$$\varphi(\theta,\rho) = \begin{cases} \varphi_1(\theta,\rho) & \text{if } 0 = \alpha_0 \le \theta \le \alpha_1, \\ \vdots & \\ \varphi_j(\theta,\rho) & \text{if } \alpha_{j-1} \le \theta \le \alpha_j, \\ \vdots & \\ \varphi_n(\theta,\rho) & \text{if } \alpha_{n-1} \le \theta \le \alpha_n = 2\pi, \end{cases}$$

such that, for each $j = 1,2,\dots,n$, φ_j is the solution of the unperturbed system with the initial condition $\varphi_j(\alpha_{j-1},\rho) = \varphi_{j-1}(\alpha_{j-1},\rho)$.

Now for $j = 1,2,\dots,n$ let $\xi_j(\theta,\theta_0,\rho_0,\varepsilon)$ be the solution of the discontinuous differential equation (4.16) such that $\xi_j(\theta_0,\theta_0,\rho_0,\varepsilon) = \rho_0$. We then define the recurrence

$$r_j(\theta,\rho,\varepsilon) = \xi_j(\theta,\alpha_{j-1},r_{j-1}(\alpha_{j-1},\rho,\varepsilon),\varepsilon), \quad j = 2,\dots,n,$$

with initial condition $r_1(\theta,\rho,\varepsilon) = \xi_1(\theta,0,\rho,\varepsilon)$. From hypothesis $(H1)$ it is easy to see that each $r_j(\theta,\rho,\varepsilon)$ is defined for every $\theta \in [\alpha_{j-1},\alpha_j]$. Therefore $r(\cdot,\rho,\varepsilon) : [0,2\pi] \to \mathbb{R}$ defined as

$$r(\theta,\rho,\varepsilon) = \begin{cases} r_1(\theta,\rho,\varepsilon) & \text{if } 0 = \alpha_0 \le \theta \le \alpha_1, \\ r_2(\theta,\rho,\varepsilon) & \text{if } \alpha_1 \le \theta \le \alpha_2, \\ \vdots & \\ r_j(\theta,\rho,\varepsilon) & \text{if } \alpha_{j-1} \le \theta \le \alpha_j, \\ \vdots & \\ r_n(\theta,\rho,\varepsilon) & \text{if } \alpha_{n-1} \le \theta \le \alpha_n = 2\pi, \end{cases}$$

is the solution of the differential equation (4.5) such that $r(0,\rho,\varepsilon) = \rho$. Moreover the equalities hold

$$r_1(0,\rho,\varepsilon) = \rho \text{ and } r_j(\alpha_{j-1},\rho,\varepsilon) = r_{j-1}(\alpha_{j-1},\rho,\varepsilon), \tag{4.18}$$

for $j = 1,2,\dots,n$. Clearly $r_j(\theta,\rho,0) = \varphi_j(\theta,\rho)$ for all $j = 1,2,\dots,n$.

Lemma 4.4.3. For $j \in \{1,2,\dots,n\}$ and $\theta_\rho^j > \alpha_j$, let $r_j(\cdot,\rho,\varepsilon) : [\alpha_{j-1},\theta_\rho^j)$ be the solution of (4.16). Then

$$r_j(\theta,\rho,\varepsilon) = \varphi_j(\theta,\rho) + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} z_i^j(\theta,\rho) + \mathcal{O}(\varepsilon^{k+1}),$$

where $z_i^j(\theta,\rho)$ is defined in (4.6).

Proof. Fix $j \in \{1,2,\dots,n\}$, from the continuity of the solution $r_j(\theta,\rho,\varepsilon)$ and by the compactness of the set $[\alpha_{j-1},\alpha_j] \times \overline{D} \times [-\varepsilon_0,\varepsilon_0]$ it is easy to obtain that

$$\int_{\alpha_{j-1}}^{\theta} R^j(\phi,r_j(\phi,\rho,\varepsilon),\varepsilon)d\phi = \mathcal{O}(\varepsilon), \quad \theta \in [\alpha_{j-1},\alpha_j].$$

Thus integrating the differential equation (4.16) from α_{j-1} to θ, we get

$$r_j(\theta,\rho,\varepsilon) = r_j(\alpha_{j-1},\rho,\varepsilon) + \sum_{i=0}^{k} \varepsilon^i \int_{\alpha_{j-1}}^{\theta} F_i^j(\phi,r_j(\phi,\rho,\varepsilon))d\phi + \mathcal{O}(\varepsilon^{k+1}). \tag{4.19}$$

Note that in the above expression the value of the initial condition $r_j(\alpha_{j-1},\rho,\varepsilon)$ is not substituted yet.

In the sequel we shall expand the right hand side of the above equality in Taylor series in ε around $\varepsilon = 0$. To do that we first recall the Faá di Bruno's Formula about the l-th derivative of a composite function. Let g and h be sufficiently smooth functions then

$$\frac{d^l}{d\alpha^l} g(h(\alpha)) = \sum_{S_l} \frac{l!}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} g^{(L)}(h(\alpha)) \prod_{j=1}^{l} \left(h^{(j)}(\alpha)\right)^{b_j},$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$. So expanding $F_i^j(\phi, r_j(\phi,\rho,\varepsilon))$ in Taylor series in ε around $\varepsilon = 0$ we obtain

$$\begin{aligned} F_i^j(\phi, r_j(\phi,\rho,\varepsilon)) = \;& F_i^j(\phi, r_j(\phi,\rho,0)) \\ & + \sum_{l=1}^{k-i} \frac{\varepsilon^l}{l!} \left(\frac{\partial^l}{\partial\varepsilon^l} F_i^j(\phi, r_j(\phi,\rho,\varepsilon))\right)\Big|_{\varepsilon=0} + \mathscr{O}(\varepsilon^{k-i+1}). \end{aligned} \tag{4.20}$$

From the Faá di Bruno's Formula we compute

$$\begin{aligned} \frac{\partial^l}{\partial\varepsilon^l} F_i^j(\phi, r_j(\phi,\rho,\varepsilon))\Big|_{\varepsilon=0} = \;& \sum_{S_l} \frac{l!}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \\ & \cdot \partial^L F_i^j(\phi, \varphi_j(\phi,\rho)) \prod_{m=1}^{l} w_m^j(\phi,\rho)^{b_m}, \end{aligned} \tag{4.21}$$

where

$$w_m^j(\phi,\rho) = \frac{\partial^m}{\partial\varepsilon^m} r_j(\phi,\rho,\varepsilon)\Big|_{\varepsilon=0}.$$

Substituting (4.21) in (4.20) we have

$$\begin{aligned} F_i^j(\phi, r_j(\phi,\rho,\varepsilon)) = \;& F_i^j(\phi, \varphi_j(\phi,\rho)) \\ & + \sum_{l=1}^{k-i} \sum_{S_l} \frac{\varepsilon^l}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \partial^L F_i^j(\phi, \varphi_j(\phi,\rho)) \prod_{m=1}^{l} w_m^j(\phi,\rho)^{b_m}, \end{aligned} \tag{4.22}$$

for $i = 0, 1, ..., k-1$. Moreover for $i = k$ we have that

$$F_k^j(\phi, r_j(\phi,\rho,\varepsilon)) = F_k^j(\phi, \varphi_j(\phi,\rho)) + \mathscr{O}(\varepsilon). \tag{4.23}$$

Substituting (4.22) and (4.23) in (4.19) we get

$$\begin{aligned} r_j(\theta,\rho,\varepsilon) = \;& r_j(\alpha_{j-1},\rho,\varepsilon) + \int_{\alpha_{j-1}}^{\theta} \left(\sum_{i=0}^{k} \varepsilon^i F_i^j(\phi, \varphi_j(\phi,\rho)) d\phi \right. \\ & + \sum_{i=0}^{k-1} \sum_{l=1}^{k-i} \varepsilon^{l+i} \sum_{S_l} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \\ & \left. \cdot \partial^L F_i^j(\phi, \varphi_j(\phi,\rho)) \prod_{m=1}^{l} w_m^j(\phi,\rho)^{b_m} \right) d\phi + \mathscr{O}(\varepsilon^{k+1}). \end{aligned} \tag{4.24}$$

Denote

$$Q_j(\phi,\rho,\varepsilon)=\sum_{i=0}^{k-1}\sum_{l=1}^{k-i}\varepsilon^{l+i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_i^j(\phi,\varphi_j(\phi,\rho))\prod_{m=1}^{l}w_m^j(\phi,\rho)^{b_m}.$$

After some transformations of the indexes i and l we obtain

$$Q_j(\phi,\rho,\varepsilon)=\sum_{i=1}^{k}\varepsilon^i\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}^j(\phi,\varphi_j(\phi,\rho))\prod_{m=1}^{l}w_m^j(\phi,\rho)^{b_m}. \tag{4.25}$$

Therefore from (4.24) and (4.25) we have

$$r_j(\theta,\rho,\varepsilon)=r_j(\alpha_{j-1},\rho,\varepsilon)+\sum_{i=0}^{k}\varepsilon^i I_i^j(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1}), \tag{4.26}$$

where for $i=0,\ldots,k$ and $j=1,2,\ldots,n$ we are taking

$$\begin{aligned} I_0^j(\theta,\rho)= & \int_{\alpha_{j-1}}^{\theta}F_0^j(\phi,\varphi_j(\phi,\rho))d\phi, \quad j=1,2,\ldots \\ I_i^j(\theta,\rho)= & \int_{\alpha_{j-1}}^{\theta}\Bigg(F_i^j(\phi,\varphi_j(\phi,\rho))+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \\ & \cdot\partial^L F_{i-l}^j(\phi,\varphi_j(\phi,\rho))\prod_{m=1}^{l}w_m^j(\phi,\rho)^{b_m}\Bigg)d\phi, \quad i,j=1,2,\ldots \end{aligned} \tag{4.27}$$

Note that for $i=1,\ldots,k$ and $j=2,\ldots,n$ the following recurrence holds

$$\begin{aligned} w_i^j(\theta,\rho)= & \frac{\partial^i}{\partial\varepsilon^i}r_j(\theta,\rho,\varepsilon)\Big|_{\varepsilon=0} \\ = & \frac{\partial^i}{\partial\varepsilon^i}r_{j-1}(\alpha_{j-1},\rho,\varepsilon)\Big|_{\varepsilon=0}+i!I_i^j(\theta,\rho) \\ = & w_i^{j-1}(\alpha_{j-1},\rho)+i!I_i^j(\theta,\rho), \end{aligned} \tag{4.28}$$

with the initial condition

$$w_i^1(\theta,\rho)=\frac{\partial^i r_1}{\partial\varepsilon^i}(\theta,\rho,\varepsilon)\Big|_{\varepsilon=0}=\frac{\partial^i}{\partial\varepsilon^i}\left(\rho+\sum_{q=0}^{k}\varepsilon^q I_q^1(\theta,\rho)\right)\Bigg|_{\varepsilon=0}=i!I_i^1(\theta,\rho). \tag{4.29}$$

Putting (4.28) and (4.29) together we obtain

$$w_i^j(\theta,\rho)=i!\left(I_i^1(\alpha_1,\rho)+I_i^2(\alpha_2,\rho)+\cdots+I_i^{j-1}(\alpha_{j-1},\rho)+I_i^j(\theta,\rho)\right).$$

for $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$.

Claim 1. For $j=1,2,\ldots,n$ we have

$$r_j(\theta,\rho,\varepsilon)=\varphi_j(\theta,\rho)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^j(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1}).$$

This claim will be proved by induction on j. Let $j=1$. Since φ_1 is the solution of (4.16) for $\varepsilon=0$ and $j=1$ with the initial condition $\varphi_1(0,\rho)=\rho$ we get

$$\varphi_1(\theta,\rho)=\rho+\int_0^\theta F_0^1(\theta,\varphi_1(\phi,\rho))d\phi.$$

Hence from (4.26), (4.18) and (4.29) it follows that

$$\begin{aligned}
r_1(\theta,\rho,\varepsilon) &= \rho+\sum_{i=0}^{k}\varepsilon^i I_i^1(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1})\\
&= \rho+\int_0^\theta F_0^1(\theta,\varphi_1(\phi,\rho))d\phi+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^1(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1})\\
&= \varphi_1(\theta,\rho)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^1(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1}).
\end{aligned}$$

Therefore the claim is proved for $j=1$.

Now using induction we shall prove the claim for $j=j_0$ assuming that it holds for $j=j_0-1$, that is

$$r_{j_0-1}(\theta,\rho,\varepsilon)=\varphi_{j_0-1}(\theta,\rho)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^{j_0-1}(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1}). \tag{4.30}$$

Since φ_{j_0} is the solution of (4.16) for $\varepsilon=0$ and $j=j_0$ with the initial condition $\varphi_{j_0}(\alpha_{j_0-1},\rho)=\varphi_{j_0-1}(\alpha_{j_0-1},\rho)$ we get

$$\varphi_{j_0}(\theta,\rho)=\varphi_{j_0-1}(\alpha_{j_0-1},\rho)+\int_{\alpha_{j_0-1}}^\theta F_0^1(\theta,\varphi_j(\phi,\rho))d\phi=\varphi_{j_0-1}(\alpha_{j_0-1},\rho)+I_0^{j_0}(\theta,\rho). \tag{4.31}$$

From (4.26), (4.18) and (4.28) we have

$$\begin{aligned}
r_{j_0}(\theta,\rho,\varepsilon) &= r_{j_0-1}(\alpha_{j_0-1},\rho,\varepsilon)+\sum_{i=0}^{k}\varepsilon^i I_i^{j_0}(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1})\\
&= r_{j_0-1}(\alpha_{j_0-1},\rho,\varepsilon)+I_0^{j_0}(\theta,\rho)+\sum_{i=1}^{k}\varepsilon^i\frac{w_i^{j_0}(\theta,\rho)-w_i^{j_0-1}(\alpha_{j-1},\rho)}{i!}+\mathscr{O}(\varepsilon^{k+1}).
\end{aligned}$$

Finally using (4.30) and (4.31) the above expression becomes

$$\begin{aligned}
r_{j_0}(\theta,\rho,\varepsilon)= \quad & \varphi_{j_0-1}(\alpha_{j_0-1},\rho)+I_0^{j_0}(\theta,\rho)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^{j_0-1}(\alpha_{j_0-1},\rho)\\
&+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}(w_i^{j_0}(\theta,\rho)-w_i^{j_0-1}(\alpha_{j_0-1},\rho))+\mathscr{O}(\varepsilon^{k+1})\\
= \quad & \varphi_{j_0}(\theta,\rho)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^{j_0}(\theta,\rho)+\mathscr{O}(\varepsilon^{k+1}).
\end{aligned}$$

This proves Claim 1.

The proof of Lemma 4.4.3 ends by proving the following claim.

Claim 2. The equality $w_i^j = z_i^j$ holds for $i = 1,2,\dots,k$ and $j = 1,2,\dots,n$.

Computing the derivative in the variable θ of the expressions (4.6) and (4.29), for $i = j = 1$, we see, respectively, that the functions $z_1^1(\theta,\rho)$ and $w_1^1(\theta,\rho)$ satisfy the same differential equation. Moreover for each $i = 1,2,\dots,k$ the integral equations (4.6) and (4.28) (and the equivalent differential equations), which provides respectively z_i^j and w_i^j, are defined by the same recurrence for $j = 2,\dots,n$. Therefore we conclude that the functions $z_i^j(\theta,\rho)$ and $w_i^j(\theta,\rho)$ satisfy the same differential equations for $i = 1,2,\dots,k$ and $j = 1,2,\dots,n$.

It remains to prove that their initial conditions coincide. Let $i \in \{1,2,\dots,k\}$. For $j = 1$ we have from (4.29) and (4.6) that $w_i^1(0,\rho) = 0 = z_i^1(0,\rho)$. For $j = 2,\dots,n$ the initial conditions are defined by the recurrence $z_i^j(\alpha_{j-1},\rho) = z_i^{j-1}(\alpha_{j-1},\rho)$ (see (4.6)), which is the same recurrence for the initial conditions of $w_i^j(\alpha_{j-1},\rho)$. Indeed from (4.28) and (4.27) we see that for $j = 2,\dots,n$ we have $w_i^j(\alpha_{j-1},\rho) = w_i^{j-1}(\alpha_{j-1},\rho) + i! I_i^j(\alpha_{j-1},\rho) = w_i^{j-1}(\alpha_{j-1},\rho)$. Therefore $z_i^j(\alpha_{j-1},\rho) = w_i^j(\alpha_{j-1},\rho)$ for every $i = 1,2,\dots,k$ and $j = 1,2,\dots,n$.

Hence Claim 2 follows from the uniqueness property of the solutions of the differential equations. $\square$

Using the results above the proof of Theorem 4.4.1 follows easily:

Proof of Theorem 4.4.1. Since $\varphi(\theta,\rho)$ is 2π-periodic, using Lemma 4.4.3 we have

$$\begin{aligned} r_n(2\pi,\rho,\varepsilon) &= \varphi_n(2\pi,\rho) + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} z_i^n(2\pi,\rho) + \mathscr{O}(\varepsilon^{k+1}) \\ &= \rho + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} z_i^n(2\pi,\rho) + \mathscr{O}(\varepsilon^{k+1}). \end{aligned}$$

Therefore from (4.7) the following equality holds

$$r_n(2\pi,\rho,\varepsilon) = \rho + \varepsilon f_1(\rho) + \varepsilon^2 f_2(\rho) + \cdots + \varepsilon^k f_k(\rho) + \mathscr{O}(\varepsilon^{k+1}). \tag{4.32}$$

Consider the displacement function

$$f(\rho,\varepsilon) = r(2\pi,\rho,\varepsilon) - \rho = r_n(2\pi,\rho,\varepsilon) - \rho.$$

Clearly for some $\varepsilon = \bar{\varepsilon} \in (-\varepsilon_0,\varepsilon_0)$ discontinuous differential equation (4.5) admits a periodic solution passing through $\bar{\rho} \in D$ if and only if $f(\bar{\rho},\bar{\varepsilon}) = 0$. From (4.32) we have that

$$f(\rho,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i f_i(\rho) + \mathscr{O}(\varepsilon^{k+1}).$$

By hypotheses $f_l(\rho^*) = 0$ and $f_l'(\rho^*) \neq 0$. Using the Implicit Function Theorem for the function $\mathscr{F}(\rho,\varepsilon) = f(\rho,\varepsilon)/\varepsilon^l$ we guarantee the existence of a differentiable function $\rho(\varepsilon)$ such that $\rho(0) = \rho^*$ and $f(\rho(\varepsilon),\varepsilon) = 0$ for every $|\varepsilon| \neq 0$ sufficiently small. This completes the proof of Theorem 4.4.1. $\square$

4.4.1 Three applications for Theorem 4.4.1

In this subsection we present three applications of Theorem 4.4.1). In the first two examples we use the averaged functions (4.7) up to order 7 to provide lower bounds for the maximum number of limit cycles admitted by some piecewise linear systems with four zones. The first system is a piecewise linear perturbation of the linear center $(\dot{x},\dot{y}) = (-y,x)$, and the second one is a piecewise linear perturbation of a discontinuous piecewise constant center. As usual, the expressions of the higher order averaged functions are extensive (see (ITIKAWA; LLIBRE; NOVAES, 2017; LLIBRE; NOVAES; TEIXEIRA, 2014b)), so we shall omit them here. We emphasize that our goal in these first two examples, by taking particular classes of perturbations, is to illustrate the using of the higher order averaged functions.

In the third example we study the quadratic isochronous center $(\dot{x},\dot{y}) = (-y+x^2, x+xy)$ perturbed inside a particular family of piecewise quadratic system with n zones. Using the first order averaged function (4.7) we provide lower bounds, depending on n, for the maximum number of limit cycles admitted by this system. We emphasize that our goal in this last example, again by taking a particular class of perturbation, is to illustrate the using of Theorem 4.4.1 to study discontinuous piecewise smooth nonlinear system with many zones.

The next proposition, proved in (COLL; GASULL; PROHENS, 2005), is needed to deal with our examples.

Proposition 4.4.4. Consider n linearly independent functions $h_i : I \to \mathbb{R}$, $i = 1,2,\ldots,n$.

(i) Given $n-1$ arbitraries values of $a_i \in I$, $i = 1,2,\ldots,n-1$ there exist n constants β_k, $i = 1,2,\ldots,n$ such that

$$h(x) \doteq \sum_{k=1}^{n} \beta_k h_k(x), \tag{4.33}$$

is not the zero function and $h(a_i) = 0$ for $i = 1,2,\ldots,n-1$.

(ii) Furthermore, if all h_i are analytical functions on I and there exists $j \in \{1,2,\ldots,n\}$ such that $h_j|_I$ has constant sign, it is possible to get an h given by (4.33), such that it has at least $n-1$ simple zeroes in I.

Example 4.4.5 (Nonsmooth perturbation of the linear center)**.** The bifurcation of limit cycles from smooth and nonsmooth perturbations of the linear center $(\dot{x},\dot{y}) = (-y,x)$ is a fairly studied problem in the literature, see for instance (BUZZI; PESSOA; TORREGROSA, 2013; CARDIN; TORREGROSA, 2016). Here we apply Theorem 4.4.1 to study these limit cycles when the linear center is perturbed inside a particular of piecewise linear system with 4 zones. Following the notation introduced in (4.13) we take

$$
\begin{aligned}
X_0^j(x,y) &= \big(-y,x\big), \text{ for } j=1,\dots,n, \text{ and}\\
X_i^j(x,y) &= \big(a_{ij}x+b_j,0\big), \text{ for } j=1,\dots,n, \text{ and } i=1,\dots,k.
\end{aligned}
\tag{4.34}
$$

with $a_{ij},b_{ij}\in\mathbb{R}$ for all i,j. We consider the discontinuous piecewise smooth differential system $(\dot x,\dot y)^T=Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$ (see (4.14)) where $\mathscr{X}=\big(X_1,\dots,X_4\big)$ (see (4.13)) and $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(0,\pi/2,\pi,3\pi/2)$.

First of all, in order to apply Theorem 4.4.1 to study the limit cycles of $(\dot x,\dot y)^T=Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$, we shall write it into the standard form (4.5). To do that we consider the polar coordinates $x=r\cos\theta$, $y=r\sin\theta$. So the set of discontinuity becomes $\Sigma=\{\theta=0\}\cup\{\theta=\alpha_1\}\cup\{\theta=\alpha_2\}\cup\{\theta=\alpha_3\}$ and in each sector C_j (see (4.15)), $j=1,2,3,4$, the differential system $(\dot x,\dot y)^T=Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$ reads

$$
\begin{aligned}
\dot r(t) &= \sum_{i=1}^{7}\varepsilon^i(a_{ij}r\cos^2\theta+b_{ij}\cos\theta),\\
\dot\theta(t) &= 1-\frac{1}{r}\sum_{i=1}^{7}\varepsilon^i(a_{ij}r\cos\theta\sin\theta+b_{ij}\sin\theta).
\end{aligned}
$$

Note that $\dot\theta(t)\neq 0$ for $|\varepsilon|$ sufficiently small, thus we can take θ as the new independent time variable by doing $r'(\theta)=\dot r(t)/\dot\theta(t)$. Then

$$
r'(\theta)=\frac{\dot r(t)}{\dot\theta(t)}=\sum_{i=1}^{7}\varepsilon^iF_i^j(\theta,r)+\varepsilon^{k+1}R^j(\theta,r,\varepsilon),\quad \text{for}\quad j=1,2,3,4,\tag{4.35}
$$

where F_i^j is the coefficient of ε^i in the Taylor series in ε of $\dot r(t)/\dot\theta(t)$ around $\varepsilon=0$.

From here we shall use the averaged functions (4.7) up to order 7 to study the isolated periodic solutions of the piecewise smooth differential equation defined by (4.35) or, equivalently, the limit cycles of the piecewise smooth differential system $(\dot x,\dot y)^T=Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$ defined by (4.34). As we have said before, due to the complexity of the expressions of the higher order averaged functions we shall not provided them explicitly. So we first describe the methodology to obtain lower bounds for the number of their zeros, and consequently for the number of limit cycles of (4.34).

Assume that one have computed the list of averaged functions f_i, $i=1,\dots,k$, and that they are polynomials. The first step is to established a lower bound for the number of zeros that f_1 can have. To do that, one can build a vector M_1 where each entry s of M_1 is given by the coefficient of r^s of the function f_1. Clearly M_1 is a function on the parameter variable $v_1=\{a_{1j}:\ j=1,\dots,4\}\cup\{b_{1j}:\ j=1,\dots,4\}$, $M_1=M_1(v_1)$. Let v_1^* be an election of the parameters such that $M_1(v_1^*)$ vanishes. So taking the derivative $D_{v_1}M_1(v_1^*)$, a lower bound for the number of zeros of f_i will be given by the rank of the matrix $D_{v_1}M_1(v_1^*)$ decreased by 1. For this case, v_1^* can be taken as being the null vector. In our first example system (4.35), the averaged function f_1 reads

$$\begin{aligned} f_1(r) = &\int_0^{\frac{\pi}{2}} F_1^1(\theta,r)d\theta + \int_{\frac{\pi}{2}}^{\pi} F_1^2(\theta,r)d\theta + \int_{\pi}^{\frac{3\pi}{2}} F_1^3(\theta,r)d\theta + \int_{\frac{3\pi}{2}}^{2\pi} F_1^4(\theta,r)d\theta \\ = &\frac{\pi}{4}r(a_{11}+a_{12}+a_{13}+a_{14}) + b_{11} - b_{12} - b_{13} + b_{14}. \end{aligned}$$

Clearly f_1 has at most one positive root and there exist parameters a_{1j}'s and b_{1j}'s for which this zero exists. The analysis of the rank of the matrix $D_{v_1}M_1(v_1^*)$ is not needed here. However, since it is not the case for averaged functions of higher order, we shall performed this analysis here. Accordingly

$$M_1(v_1) = \Big(b_{11} - b_{12} - b_{13} + b_{14}\,,\, \frac{1}{4}\pi(a_{11}+a_{12}+a_{13}+a_{14})\Big),$$

$$D_{v_1}M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ \frac{\pi}{4} & \frac{\pi}{4} & \frac{\pi}{4} & \frac{\pi}{4} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We note that the matrix $D_{v_1}M_1$ has maximum rank 2. Expanding f_1 around $v_1 = v_1^*$, we see that f_1 is written as a combination of two linear independent functions plus higher order terms in $v_1 - v_1^*$ Applying Theorem 4.4.1 for $l = 1$ we obtain at least one limit cycle for the differential system (4.34).

The next step is to choose parameters to assure that $f_1(r) \equiv 0$. In our example $a_{11} = -(a_{12}+a_{13}+a_{14})$ and $b_{11} = b_{12}+b_{13}-b_{14}$. To continue the analysis we repeat the above procedure: build a vector M_2 where each entry s of M_2 is given by the coefficient of r^s of the function f_2; define the parameter vector $v_2 = \{a_{1j} : i = 1,2,\ j = 1,\dots,4\} \cup \{b_{1j} : i = 1,2,\ j = 1,\dots,4\}$ such that $M_2 = M_2(v_2)$; let v_2^* be an election of the parameters such that $M_2(v_2^*)$ vanishes; and take the derivative $D_{v_2}M_2(v_2^*)$. Again a lower bound for the number of zeros of f_2 will be given by the rank of the matrix $D_{v_2}M_2(v_2^*)$ decreased by 1. In our example

$$\begin{aligned} f_2(r) = \; & r^2\,[\pi(a_{21}+a_{22}+a_{23}+a_{24}) + 2(a_{12}+a_{13})(a_{13}+a_{14})] \\ & + r\,[\pi(a_{12}+a_{13})(b_{13}-b_{14}) - 4(a_{14}b_{12} + (a_{12}+a_{14})b_{13} \\ & + a_{13}(b_{12}+2b_{13}-b_{14}) - a_{12}b_{14} - b_{21} + b_{22} + b_{23} - b_{24})] \\ & + 4(b_{12}+b_{13})(b_{13}-b_{14}), \end{aligned}$$

The function f_2 is a polynomial of degree 2 in r. It is easy to see that we can choose v_2^* such that the matrix $D_{v_2}M_2(v_2^*)$ has maximum rank again, that is 3. Expanding f_2 around $v_2 = v_2^*$, we see that f_2 is written as a combination of three linear independent functions plus higher order terms in $v_2 - v_2^*$. Applying Theorem 4.4.1 for $l = 2$ we obtain at least two limit cycles for the differential system (4.34).

In general, after estimating a lower bound for the number of zeros of f_{l-1} we chose parameters to assure that $f_{l-1}(r) \equiv 0$. Then we follow the above steps: build a vector M_l where each entry s of M_l is given by the coefficient of r^s of the function f_l; define the parameter vector $v_l = \{a_{ij} : i = 1,\dots l, j = 1,\dots,4\} \cup \{b_{ij} : i = 1,\dots l, j = 1,\dots,4\}$ such that $M_l = M_l(v_l)$; let v_l^* be an election of the parameters such that $M_l(v_l^*)$ vanishes; and take the derivative $D_{v_l}M_l(v_l^*)$. As above a lower bound for the number of zeros of f_l will be given by the rank of the matrix $D_{v_l}M_l(v_l^*)$ decreased by 1.

In what follows, using the procedure described above, we provide a table showing the lower bound $N(l)$, $l = 1,\dots,7$, for the maximum number of limit cycles of the piecewise smooth differential system $(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$, defined by (4.34), obtained by studying the averaged function of order l.

l	1	2	3	4	5	6	7
$N(l)$	1	2	2	3	3	3	3

Example 4.4.6 (Nonsmooth perturbation of a piecewise constant center)**.** Consider the discontinuous piecewise constant differential system

$$(\dot{x},\dot{y})^T = X(x,y) = \begin{cases} X_1(x,y) & \text{if } x>0 \text{ and } y>0, \\ X_2(x,y) & \text{if } x<0 \text{ and } y>0, \\ X_3(x,y) & \text{if } x<0 \text{ and } y<0, \\ X_4(x,y) & \text{if } x>0 \text{ and } y<0, \end{cases} \tag{4.36}$$

where

$$X_1(x,y) = \begin{cases} -1+\sum_{i=1}^{7}\varepsilon^i(a_{i1}x+b_{i1}), \\ 1, \end{cases} \qquad X_2(x,y) = \begin{cases} -1+\sum_{i=1}^{7}\varepsilon^i(a_{i2}x+b_{i2}), \\ -1, \end{cases}$$

$$X_3(x,y) = \begin{cases} 1+\sum_{i=1}^{7}\varepsilon^i(a_{i3}x+b_{i3}), \\ -1, \end{cases} \qquad X_4(x,y) = \begin{cases} 1+\sum_{i=1}^{7}\varepsilon^i(a_{i4}x+b_{i4}), \\ 1, \end{cases}$$

with $a_{ij}, b_{ij} \in \mathbb{R}$ for all $i \in \{1,2,\dots,7\}$ and $j \in \{1,2,3,4\}$.

First of all, in order to apply Theorem 4.4.1 to study the limit cycles of the differential system (4.36), we shall write it into the standard form (4.5). Again, to do that we consider polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. So the set of discontinuity becomes $\Sigma = \{\theta = 0\} \cup \{\theta = \alpha_1\} \cup \{\theta = \alpha_2\} \cup \{\theta = \alpha_3\}$, with $\alpha_0 = 0, \alpha_1 = \pi/2, \alpha_2 = \pi, \alpha_3 = 3\pi/2$, and $\alpha_4 = 2\pi$, and for each $j = 1,2,3,4$ the differential system $(\dot{x},\dot{y}) = X_j(x,y)$ reads

$$\dot{r}(t) = g_j(\theta) + \sum_{i=1}^{7} \varepsilon^i (a_{ij} r\cos^2\theta + b_{ij}\cos\theta),$$
$$\dot{\theta}(t) = \frac{1}{r}\left(\widehat{g}_j(\theta) - \sum_{i=1}^{7} \varepsilon^i (a_{ij} r\cos\theta\sin\theta + b_{ij}\sin\theta)\right),$$

where

$$\begin{aligned}
g_1(\theta) &= \sin\theta - \cos\theta & \widehat{g}_1(\theta) &= \sin\theta + \cos\theta,\\
g_2(\theta) &= -(\sin\theta + \cos\theta) & \widehat{g}_2(\theta) &= \sin\theta - \cos\theta,\\
g_3(\theta) &= -\sin\theta + \cos\theta & \widehat{g}_3(\theta) &= -(\sin\theta + \cos\theta),\\
g_4(\theta) &= \sin\theta + \cos\theta & \widehat{g}_4(\theta) &= -\sin\theta + \cos\theta.
\end{aligned}$$

Note that for each $j = 1,2,3,4$ and $\alpha_{j-1} \le \theta \le \alpha_j$, we have that $\dot{\theta}(t) \neq 0$ for $|\varepsilon|$ sufficiently small, thus we can take θ as the new independent time variable by doing $r'(\theta) = \dot{r}(t)/\dot{\theta}(t)$. Then

$$r'(\theta) = \frac{\dot{r}(t)}{\dot{\theta}(t)} = \sum_{i=0}^{7} \varepsilon^i F_i^j(\theta,r) + \varepsilon^{k+1} R^j(\theta,r,\varepsilon), \tag{4.37}$$

where F_i^j is the coefficient related to ε^i in Taylor series in ε of $\dot{r}(t)/\dot{\theta}(t)$ around $\varepsilon = 0$.

From here we shall use the averaged functions (4.7) up to order 7 to study the isolated periodic solutions of the piecewise smooth differential equation defined by (4.37) or, equivalently, the limit cycles of the piecewise smooth differential system (4.36). Following the same methodology described in Example 4.4.5, we provide a table showing the lower bound $N(l)$, $l = 1,\dots,7$, for the maximum number of limit cycles of (4.36) obtained by studying the averaged function of order l.

l	1	2	3	4	5	6	7
$N(k)$	1	2	2	2	2	2	2

Example 4.4.7 (Nonsmooth perturbation of an isochronous quadratic center)**.** In this section we consider the quadratic isochronous center $(\dot{x},\dot{y}) = (-y+x^2, x+xy)$ perturbed inside a class of

piecewise quadratic system with n zones. For this system we take

$$X_0^j(x,y) = \big(-y+x^2, x+xy\big), \text{ for } j=1,\ldots,n, \text{ and}$$

$$X_i^j(x,y) = \big(a_jx^2+b_jx+c_j, 0\big), \text{ for } j=1,\ldots,n,$$

where a_j, b_j and c_j are real numbers for all $j \in \{1,2,\ldots,n\}$. We consider the discontinuous piecewise smooth differential system $(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$ (see (4.14)) where $\mathscr{X} = \big(X_1,\ldots,X_n\big)$ (see (4.13)) and $\alpha = (\alpha_j)_{j=0}^{n-1} = (2j\pi/n)_{j=0}^{n-1}$.

As before, in order to apply Theorem 4.4.1 to study the limit cycles of $(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$, we shall write it into the standard form (4.5). To do that we consider a first change of coordinates $x = -u/(v-1)$, $y = -v/(v-1)$ (see (CHAVARRIGA; SABATINI, 1999)). Note that this change keeps fixed all straight lines passing through the origin and therefore does not change the set of discontinuity. In each sector C_j (see (4.15)), $j = 1,2,3,4$, the differential system $(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$ reads

$$\begin{aligned} \dot{u} &= -v+\varepsilon\left(u\left(b_j-\frac{a_j}{v-1}u\right)+c_j(1-v)\right), \\ \dot{v} &= u. \end{aligned} \tag{4.38}$$

Now, as a second change of variables, we consider the polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$. Taking θ as the new independent time variable by doing $r'(\theta) = \dot{r}(t)/\dot{\theta}(t)$, system (4.38) becomes

$$r'(\theta) = \varepsilon F^j(\theta,r) + \mathscr{O}(\varepsilon^2),$$

where

$$F^j(\theta,r) = \cos\theta\left(c_j + r\left(-c_j\sin\theta + \cos\theta\left(b_j + \frac{a_j r\cos\theta}{1-r\sin\theta}\right)\right)\right).$$

for $j = 1,\ldots,n$.

In this new coordinates the piecewise smooth differential system $(\dot{x},\dot{y})^T = Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$ reads

$$r'(\theta) = \varepsilon F(\theta,r) + \mathscr{O}(\varepsilon^2), \tag{4.39}$$

where

$$F(\theta,r) = \sum_{j=1}^{n} \chi_{[\frac{2(j-1)\pi}{n},\frac{2j\pi}{n}]}(\theta) F^j(\theta,r).$$

Computing the first order averaged function f_1 of (4.39) we obtain

$$
\begin{aligned}
f_1(r) = & \sum_{j=1}^{n} \int_{\frac{2(j-1)\pi}{n}}^{\frac{2j\pi}{n}} F^j(\theta,r)d\theta \\
= & \frac{1}{4}\Bigg[\left(\sum_{j=1}^{n} 4(a_j+cj)\left(\sin\left(\frac{2j\pi}{n}\right)-\sin\left(\frac{2(j-1)\pi}{n}\right)\right)\right) \\
& +r\left(\sum_{j=1}^{n}\left(\frac{4\pi}{n}+\sin\left(\frac{4j\pi}{n}\right)-\sin\left(\frac{4(j-1)\pi}{n}\right)\right)b_j\right. \\
& \left.+(a_j-c_j)\left(\cos\left(\frac{4(j-1)\pi}{n}\right)-\cos\left(\frac{4j\pi}{n}\right)\right)\right) \\
& +\frac{(r^2-1)}{r}\left(\sum_{j=1}^{n} 4a_j \ln\left(1-r\sin\left(\frac{2(j-1)\pi}{n}\right)\right)\right) \\
& +\frac{(r^2-1)}{r}\left(\sum_{j=1}^{n} -4a_j \ln\left(1-r\sin\left(\frac{2j\pi}{n}\right)\right)\right)\Bigg].
\end{aligned}
$$

Since $\sin\left(\frac{2(j-1)\pi}{n}\right)=0$ for $j=1$, and $\sin\left(\frac{2j\pi}{n}\right)=0$ for $j=n$, the above expression simplifies as

$$
\begin{aligned}
f_1(r) = & \frac{1}{4}\Bigg[\left(\sum_{j=1}^{n} 4(a_j+cj)\left(\sin\left(\frac{2j\pi}{n}\right)-\sin\left(\frac{2(j-1)\pi}{n}\right)\right)\right) \\
& +r\left(\sum_{j=1}^{n}\left(\frac{4\pi}{n}+\sin\left(\frac{4j\pi}{n}\right)-\sin\left(\frac{4(j-1)\pi}{n}\right)\right)b_j\right. \\
& \left.+(a_j-c_j)\left(\cos\left(\frac{4(j-1)\pi}{n}\right)-\cos\left(\frac{4j\pi}{n}\right)\right)\right) \\
& +\frac{(r^2-1)}{r}\left(\sum_{j=2}^{n} 4(a_j-a_{j-1}) \ln\left(1-r\sin\left(\frac{2(j-1)\pi}{n}\right)\right)\right)\Bigg].
\end{aligned}
$$

Note that f_1 is written as a linear combination of $n+1$ functions of the family

$$
\mathscr{F} = \left\{1, r, h_j(r) \doteq \frac{(r^2-1)}{r}\ln\left(1-r\sin\left(\frac{2(j-1)\pi}{n}\right)\right) : j=2,3,\ldots,n\right\}.
$$

It is easy to see that this combination is linearly independent.

Regarding the functions h_j's we have the following properties

(1) Let $j \in \{2,3,\ldots,n\}$. Then $h_j(r) \equiv 0$ if and only if n is even and $j = 1+n/2$.

(2) Let $j_1, j_2 \in \{2,3,\ldots,n\}$. Then $h_{j_1}(r) \equiv h_{j_2}(r)$ if and only if n is even and $(j_1+j_2-2) \in \{n/2, 3n/2\}$.

From the above properties we first conclude that if n is odd then the function f_1 is a linearly independent combination of $n+1$ linearly independent functions. From Proposition 4.4.4 we can find parameters such that f_1 has n simple zeros.

If $n=2$ then $f_1(r)=\pi(b_1+b_2)r/2$ which has no simple positive zeros. From now on we assume that n is even and greater than 2. From property (1) we already know that $h_{j_0}\equiv 0$ for $j_0=1+n/2$. From property (2) it remains to analyze how many pairs of integers (j_1,j_2), $2\leq j_1<j_2\leq n$, satisfy the equations $2(j_1+j_2-2)=n$ and $2(j_1+j_2-2)=3n$.

Let $\overline{n}$ be a positive integer. If $n=4\overline{n}$ then both equations $2(j_1+j_2-2)=n$ and $2(j_1+j_2-2)=3n$ have $n/4-1$ solutions. If $n=4\overline{n}+2$ then both equations $2(j_1+j_2-2)=n$ and $2(j_1+j_2-2)=3n$ have $(n-2)/4$ solutions. Therefore we conclude that:

- If $n=4\overline{n}$ then $\#\mathscr{F}=\dfrac{n}{2}+2$;
- If $n=4\overline{n}+2$ then $\#\mathscr{F}=\dfrac{n}{2}+1$;

Denote by N the maximum number of limit cycles of $(\dot{x},\dot{y})^T=Z_{\mathscr{X},\alpha}(x,y;\varepsilon)$. Applying Proposition 4.4.4 and Theorem 4.4.1 we conclude that:

(i) If n is odd then $N\geq n$;

(ii) If $n=2$ then $N\geq 0$ (no information!);

(iii) If $n=4k$ then $N\geq\dfrac{n}{2}+1$;

(iv) If $n=4k+2$ then $N\geq\dfrac{n}{2}$.

CHAPTER

5

AVERAGING THEORY AND LYAPUNOV–SCHMIDT REDUCTION FOR NONSMOOTH SYSTEMS

In chapter 4 we presented the theory for study the periodic solutions of the systems on the form

$$x' = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

where $F_i : \mathbb{S}^1 \times D \to \mathbb{R}^m$ and $R : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^m$ are piecewise C^{k+1} functions and T-periodic in the variable t and we assume that the solutions of the unperturbed system $x' = F_0(t,x)$ were all T–periodic. In other words the unperturbed system has a d-dimensional submanifold of periodic solutions with $d = m$. However when $d < m$ the averaging theory is not enough to study the persistence of periodic solutions when $|\varepsilon| \neq 0$ is small. Now, in this chapter we present the theory for the case $d < m$ using the Lyapunov-Schmidt reduction method and the averaging theory.

Consider the unperturbed system $x' = F_0(t,x)$ and its set of initial conditions whose orbits are periodic denoted here by $\mathscr{Z}$. Assume that the set $\mathscr{Z}$ is a d-dimensional submanifold of $\mathbb{R}^m$ such that $\dim(\mathscr{Z}) = d < m$. In this case only the averaging theory is not enough to study the number of limit cycles of the systems and other techniques need to be employed together, as the *Lyapunov-Schmidt reduction method*. In the case that F_i are smooth functions we have the works (BUICA; FRANÇOISE; LLIBRE, 2007; CÂNDIDO; LLIBRE; NOVAES, 2017; GINÉ *et al.*, 2016). If the functions F_i are not smooth or even continuous we have the works (LLIBRE; NOVAES, 2015; LLIBRE JAUME NOVAES, 2015), where the authors studied some classes of these systems.

In what follows we describe how to use the averaging theory and Lyapunov-Schmidt reduction method for computing isolated periodic solutions of the piecewise smooth differential systems. Then, we set the class of non-autonomous discontinuous piecewise smooth differential equations that we are interested as well as our main result (Theorem 5.5.1).

5.1 Lyapunov-Schmidt reduction

Consider the function

$$g(z,\varepsilon)=\sum_{i=0}^{k}\varepsilon^i g_i(z)+\mathscr{O}(\varepsilon^{k+1}),\tag{5.1}$$

where $g_i:D\to\mathbb{R}^m$ is a C^{k+1} function, $k\geq 1$, for $i=0,1,\dots,k$, in which D an open bounded subset of $\mathbb{R}^m$. For $d<m$, let V be an open bounded subset of $\mathbb{R}^d$ and $\beta:\overline{V}\to\mathbb{R}^{m-d}$ a C^{k+1} function such that

$$\mathscr{Z}=\{z_\alpha=(\alpha,\beta(\alpha)):\alpha\in\overline{V}\}\subset D.\tag{5.2}$$

The main hypothesis is

(H_α) the function g_0 vanishes on the d-dimensional submanifold $\mathscr{Z}$ of D.

In (CÂNDIDO; LLIBRE; NOVAES, 2017) the authors used the Lyapunov-Schmidt reduction method to develop the *bifurcation functions of order* i, for $i=0,1,\dots,k$, which for $|\varepsilon|\neq 0$ sufficiently small control the existence of branches of zeros $z(\varepsilon)$ of system (5.1) that bifurcate from $z(0)\in\mathscr{Z}$. In this subsection we present the results developed in that work and those that we shall need later on.

First we present some notation. Consider the projections onto the first d coordinates and onto the last $m-d$ coordinates denoted by $\pi:\mathbb{R}^d\times\mathbb{R}^{m-d}\to\mathbb{R}^d$ and $\pi^\perp:\mathbb{R}^d\times\mathbb{R}^{m-d}\to\mathbb{R}^{m-d}$, respectively. Also, for a point $z\in\mathscr{Z}$ we write $z=(a,b)\in\mathbb{R}^d\times\mathbb{R}^{m-d}$.

Let L be a positive integer, let $x=(x_1,x_2,\dots,x_m)\in D$, $t\in\mathbb{R}$ and $y_j=(y_{j1},\dots,y_{jm})\in\mathbb{R}^m$ for $j=1,\dots,L$. Given $G:\mathbb{R}\times D\to\mathbb{R}^m$ a sufficiently smooth function, for each $(t,x)\in\mathbb{R}\times D$ we denote by $\partial^L G(t,x)$ a symmetric L–multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^m$, which we denote as $\bigodot_{j=1}^L y_j\in\mathbb{R}^{mL}$. The definition of this L–multilinear map is

$$\partial^L G(t,x)\bigodot_{j=1}^{L}y_j=\sum_{i_1,\dots,i_L=1}^{n}\frac{\partial^L G(t,x)}{\partial x_{i_1}\dots\partial x_{i_L}}y_{1i_1}\dots y_{Li_L}.$$

We define ∂^0 as the identity functional.

The bifurcation functions $f_i:\overline{V}\to\mathbb{R}^d$ of order i are defined for $i=0,1,\dots,k$ as

$$f_i(\alpha)=\pi g_i(z_\alpha)+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{c_1!\,c_2!2!^{c_2}\dots c_l!\,l!^{c_l}}\partial_b^L\pi g_{i-l}(z_\alpha)\bigodot_{j=1}^{l}\gamma_j(\alpha)^{c_j},\tag{5.3}$$

where the $\gamma_i : V \to \mathbb{R}^{m-d}$, for $i = 1, 2, \ldots, k$, are defined recursively as

$$\begin{aligned}\gamma_1(\alpha) = & \quad -\Delta_\alpha^{-1}\pi^\perp g_1(z_\alpha) \quad \text{and} \\ \gamma_i(\alpha) = & \quad -i!\Delta_\alpha^{-1}\Bigg(\sum_{S_i'} \frac{1}{c_1!\,c_2!2!^{c_2}\ldots c_{i-1}!(i-1)!^{c_{i-1}}}\partial_b^{I'}\pi^\perp g_0(z_\alpha)\bigodot_{j=1}^{i-1}\gamma_j(\alpha)^{c_j} \\ & + \sum_{l=1}^{i-1}\sum_{S_l}\frac{1}{c_1!\,c_2!2!^{c_2}\ldots c_l!l!^{c_l}}\partial_b^L\pi^\perp g_{i-l}(z_\alpha)\bigodot_{j=1}^{l}\gamma_j(\alpha)^{c_j}\Bigg).\end{aligned} \tag{5.4}$$

We denote by S_l the set of all l-tuples of non-negative integers $(c_1, c_2, \ldots, c_l)$ such that $c_1 + 2c_2 + \ldots + lc_l = l$, $L = c_1 + c_2 + \ldots + c_l$, and by S_i' the set of all $(i-1)$-tuples of non-negative integers $(c_1, c_2, \ldots, c_{i-1})$ such that $c_1 + 2c_2 + \ldots + (i-1)c_{i-1} = i$, $I' = c_1 + c_2 + \ldots + c_{i-1}$ and $\Delta_\alpha = \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)$.

About the zeros of the function (5.1) the authors proved in (CÂNDIDO; LLIBRE; NOVAES, 2017) the following result.

Theorem 5.1.1. Let Δ_α denote the lower right corner $(m-d) \times (m-d)$ matrix of the Jacobian matrix $Dg_0(z_\alpha)$. Additionally to hypothesis (H_α) we assume that

(i) for each $\alpha \in \overline{V}$, $\det \Delta_\alpha \neq 0$; and

(ii) $f_1 = f_2 = \ldots = f_{k-1} = 0$ and f_k is not identically zero.

If there exists $\alpha^* \in V$ such that $f_k(\alpha^*) = 0$ and $\det(Df_k(\alpha^*)) \neq 0$, then there exists a branch of zeros $z(\varepsilon)$ with $g(z(\varepsilon), \varepsilon) = 0$ and $|z(\varepsilon) - z_{\alpha^*}| = \mathscr{O}(\varepsilon)$.

5.2 The averaging theory

In section 4.1 of Chapter 4 we present the averaged functions for systems defined in $\mathbb{S}^1 \times D$ with D an open subset of $\mathbb{R}$. Now we deal with systems where D is an open subset of $\mathbb{R}^m$, so we present the again the averaged functions but for higher dimension. They have basically the same expression, the difference is that now we use the “product” of L vectors of $\mathbb{R}^m$, denoted by $\bigodot_{j=1}^L y_j \in \mathbb{R}^{mL}$.

In (CÂNDIDO; LLIBRE; NOVAES, 2017), using Theorem 5.1.1, the authors studied high order bifurcation of periodic solutions of the following T-periodic C^{k+1} with $k \geq 1$ differential system

$$x' = F(t,x,\varepsilon) = F_0(t,x) + \sum_{i=1}^k \varepsilon^i F_i(t,x) + \mathscr{O}(\varepsilon^{k+1}), \quad (t,z) \in \mathbb{S}^1 \times D, \tag{5.5}$$

where the prime denotes the derivative with respect to the independent variable t, usually called the time. In their work they assumed that the manifold $\mathscr{Z}$, defined in (5.2), is such that all solutions of the unperturbed system

$$x' = F_0(t,x),$$

starting at points of $\mathscr{Z}$ are T-periodic and $\dim \mathscr{Z} \leq m$.

Consider the variational equation

$$y' = \frac{\partial F_0}{\partial x}(t,x(t,z,0))y, \tag{5.6}$$

where $x(t,z,0)$ denotes the solution of system (5.5) when $\varepsilon = 0$, and we denote a fundamental matrix of system (5.6) by $Y(t,z)$. The *averaged function of order i* of system (5.5) is defined as

$$g_i(z) = Y^{-1}(T,z)\frac{y_i(T,z)}{i!}, \tag{5.7}$$

where

$$\begin{aligned}
y_1(t,z) = {} & Y(t,z)\int_0^t Y(s,z)^{-1}F_1\left(s,x(s,z,0)\right)ds,\\
y_i(t,z) = {} & i!Y(t,z)\int_0^t Y(s,z)^{-1}\Big(F_i\left(s,x(s,z,0)\right)\\
& +\sum_{S_i'}\frac{1}{b_1!\,b_2!2!^{b_2}\ldots b_{i-1}!(i-1)!^{b_{i-1}}}\partial^{I'}F_0(s,x(s,z,0))\bigodot_{j=1}^{i-1}y_j(s,z)^{b_j}\\
& +\sum_{l=1}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\ldots b_l!l!^{b_l}}\partial^{L}F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Big)ds.
\end{aligned} \tag{5.8}$$

Using the functions g_i stated in (5.7) are defined the functions f_i and γ_i given by (5.3) and (5.4), respectively. Under some assumptions and with Theorem 5.1.1 it was proved that the simple zeros of the functions f_i provide the existence of isolated periodic solutions of the differential system (5.5). By a simple zero of a function f we mean a point a such that $f(a) = 0$ and $\det(Df(a)) \neq 0$, where $Df(a)$ denotes the Jacobian matrix of f at the point a.

Remark 5.2.1. The functions $y_i(t,z)$ could be defined recurrently by an integral equation as done in other works (see (ITIKAWA; LLIBRE; NOVAES, 2017; LLIBRE; NOVAES; TEIXEIRA, 2014b; LLIBRE; NOVAES; TEIXEIRA, 2014a)). Indeed, we define

$$\begin{aligned}
y_1(t,z) = {} & \int_0^t \Big(F_1\left(s,x(s,z,0)\right)+\partial F_0(s,x(s,z,0))y_1(s,z)\Big)ds,\\
y_i(t,z) = {} & i!\int_0^t \Big(F_i\left(s,x(s,z,0)\right)+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\ldots b_l!l!^{b_l}}\\
& \cdot\partial^L F_{i-l}\left(s,x(s,z,0)\right)\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Big)ds, \text{ for } i=2,\ldots,k,
\end{aligned} \tag{5.9}$$

and it is not difficult to see that solving this integral equations we obtain the formulae (5.8).

5.3 Standard form

Let $n > 1$ be a positive integer. For $i = 0, 1, \ldots, k$ and $j = 1, 2, \ldots, n$ let $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}^m$ and $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ be functions C^{k+1} where D is an open subset of $\mathbb{R}^m$ and $\mathbb{S}^1 \equiv \mathbb{R}/(T\mathbb{Z})$. We define

$$F_i(t,x) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(t) F_i^j(t,x), \; i = 0,1,...,k, \quad \text{and} \quad R(t,x,\varepsilon) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(t) R^j(t,x,\varepsilon).$$

Consider the discontinuous and T-periodic differential system

$$x' = F(t,x,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{5.10}$$

and the submanifold $\mathscr{Z}$ of periodic solutions of the unperturbed system

$$x' = F_0(t,x). \tag{5.11}$$

The set Σ of discontinuity of system (5.10) is given by

$$\Sigma = (\{t = 0 \equiv T\} \cup \{t = t_1\} \cup \ldots \cup \{t = t_{n-1}\}) \cap \mathbb{S}^1 \times D,$$

where $0 < t_1 < t_2 < \ldots < t_{n-1} < T$.

For each $j = 1, 2, \ldots, n$ and $t \in [t_{j-1}, t_j]$ we have the differential system

$$x' = \sum_{i=0}^{k} \varepsilon^i F_i^j(t,x) + \varepsilon^{k+1} R^j(t,x,\varepsilon). \tag{5.12}$$

To continue we need to give some definition about system (5.10). For each $z \in D$ and ε sufficiently small we denote by $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^m$ the solution of system (5.10) such that $x(0,z,\varepsilon) = z$, where $[0, t_{(z,\varepsilon)})$ is the interval of definition for the solution $x(t,z,\varepsilon)$.

Consider the submanifold $\mathscr{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \bar{V}\}$, where V is an open bounded subset of $\mathbb{R}^m$, and $\beta_0 : \overline{V} \to \mathbb{R}^{d-m}$ is a C^k function with $k \geq 1$. Notice that for each $z_\alpha \in \mathscr{Z}$, $(t_i, x(t_i, z_\alpha, 0) \in \Sigma^c)$, for $i \in \{0, 1, \ldots, k\}$. Indeed, for each $j = 1, 2, \ldots, n$, the set of discontinuity can be locally described by $h_j^{-1}(0)$, where $f : \mathbb{S}^1 \times D \to \mathbb{R}$ is $h_j(t,x) = t - t_j$. It is known that to show that we are in the crossing region it is sufficient to prove that $\langle \nabla h_j(t,x), F^j(t,x) \rangle \langle \nabla h_j(t,x), F^{j+1}(t,x) \rangle > 0$, where $\nabla h_j(t,x)$ denotes the gradient vector of the function $h_j(t,x)$. Here, $\nabla h_j(t,x) = (1,0)$ and $\langle \nabla h_j(t,x), F^j(t,x) \rangle \langle \nabla h_j(t,x), F^{j+1}(t,x) \rangle = 1 > 0$.

In Chapter 4 the averaging theory was developed assuming $\dim(\mathscr{Z}) = m$. Here, we are interested in the case $\dim(\mathscr{Z}) < m$. Accordingly, we shall extend the averaged functions (5.7) and the bifurcation functions (5.3) obtained in (CÂNDIDO; LLIBRE; NOVAES, 2017) to this class of discontinuous differential system, providing then sufficient conditions in order to control which periodic solutions of $\mathscr{Z}$, with $\dim \mathscr{Z} = d < m$, persists to $\varepsilon \neq 0$ sufficiently small.

For system (5.11) we consider the fundamental matrix $Y(t,z)$ of the variational system

$$y' = \frac{\partial}{\partial x} F_0(t, x(t,z,0))y, \tag{5.13}$$

where Y is an $m \times m$ matrix. Notice that, for each $j = 1, 2, \ldots, n$, if $x_j(t,z,\varepsilon)$ denotes the solution of (5.12) for $t_{j-1} \leq t \leq t_j$, the function $t \mapsto (\partial x_j / \partial z)(t,z,0)$ is a solution of (5.13) for $t_{j-1} \leq t \leq t_j$. Recall that the right product of a solution of the variational equation (5.13) by constant matrix is still a solution of (5.13). Therefore, the solution $Y(t,z)$ can be built as follows:

$$Y(t,z) = \begin{cases} Y_1(t,z) & \text{if } 0 = t_0 \leq t \leq t_1, \\ Y_2(t,z) & \text{if } t_1 \leq t \leq t_2, \\ \vdots & \\ Y_n(t,z) & \text{if } t_{n-1} \leq t \leq t_n = T, \end{cases}$$

where

$$\begin{aligned} Y_1(t,z) = &\ \frac{\partial x_1}{\partial z}(t,z,0), \quad \text{and} \\ Y_j(t,z) = &\ \frac{\partial x_j}{\partial z}(t,z,0) \left(\frac{\partial x_j}{\partial z}(t_{j-1},z,0) \right)^{-1} Y_{j-1}(t_{j-1},z), \quad \text{for } j = 2, 3, \ldots, n. \end{aligned} \tag{5.14}$$

The derivatives $\partial^j F_i(s,z)$, which appears in (5.3), are computed as follows:

$$\frac{\partial^j F_i}{\partial z}(s,z) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(s) \frac{\partial^j F_i^j}{\partial z^j}(s,z).$$

The main result of this chapter says that the simple zeros of the bifurcation functions (5.3) also controls the branching of isolated periodic solutions of the nonsmooth system (5.10). Before we enunciate it we provide an alorithm for the bifurcation functions.

5.4 An algorithm for the bifurcation functions

In this section we will provide an algorithm for computing the averaged functions, defined in (5.9), for the nonsmooth case. Their expressions are defined recurrently and using Bell polynomials, which can be implemented more easily and were previously defined in (4.11). Using them it follows that if g and h are sufficiently smooth functions, then we have that

$$\frac{d^l}{dx^l} g(h(x)) = \sum_{m=1}^{l} g^{(m)}(h(x)) B_{l,m}(h'(x), h''(x), \ldots, h^{(l-m+1)}(x)),$$

where $B_{l,m}$ is the partial Bell polynomial.

5.4.1 Averaged Functions

In this section we develop a recurrence to compute the averaged function in the particular case of the discontinuous differential equation (5.10). So, consider the functions $w_i^j : (t_{j-1}, t_j] \times D \to \mathbb{R}^m$ defined recurrently for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$, as

$$\begin{aligned}
w_1^1(t,z) =\ & \int_0^t \Big(F_1^1(s, x(s,z,0)) + \partial F_0^1(s, x(s,z,0)) w_1^1(s,z) \Big) ds, \\
w_i^1(t,z) =\ & i! \int_0^t \Big(F_i^1(s, x(s,z,0)) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^1(s, x(s,z,0)) \bigodot_{m=1}^{l} w_m^1(s,z)^{b_m} \Big) ds, \\
w_i^j(t,z) =\ & w_i^{j-1}(t_{j-1}, z) + i! \int_{t_{j-1}}^t \Big(F_i^j(s, x(s,z,0)) + \\
& \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^j(s, x(s,z,0)) \bigodot_{m=1}^{l} w_m^j(s,z)^{b_m} \Big) ds.
\end{aligned} \tag{5.15}$$

Since $F_0 \neq 0$ the recurrence defined in (5.15) is an integral equation and the next lemma solves it using Bell polynomials.

Lemma 5.4.1. For $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$ the recurrence (5.15) can be written as follows

$$\begin{aligned}
w_1^1(t,z) =\ & Y_1(t,z) \int_0^t Y_1^{-1}(s,z) F_1^1(s, x(s,z,0)) ds, \\
w_1^j(t,z) =\ & Y_j(t,z) \Big(Y_j^{-1}(t_{j-1}, z) w_1^{j-1}(t_{j-1}, z) + \int_{t_{j-1}}^t Y_j^{-1}(s,z) F_1^j(s, x(s,z,0)) ds \Big), \\
w_i^1(t,z) =\ & Y_1(t,z) \int_0^t Y_1^{-1}(s,z) \Big(i! F_i^1(s, x(s,z,0)) + \sum_{m=2}^{i} \partial^m F_0^1(s, x(s,z,0)).B_{i,m}(w_1^1, \dots, w_{i-m+1}^1), \\
& + \sum_{l=1}^{i-1} \sum_{m=1}^{l} \frac{i!}{l!} \partial^m F_{i-l}^1(s, x(s,z,0)).B_{l,m}(w_1^1, \dots, w_{l-m+1}^1) \Big) ds, \\
w_i^j(t,z) =\ & Y_j(t,z) \Big[Y_j^{-1}(t_{j-1}, z) w_i^{j-1}(t_{j-1}, z) + \int_{t_{j-1}}^t Y_j^{-1}(s,z) \Big(i! F_i^j(s, x(s,z,0)) \\
& + \sum_{m=2}^{i} \partial^m F_0^j(s, x(s,z,0)).B_{i,m}(w_1^j, \dots, w_{i-m+1}^j), \\
& + \sum_{l=1}^{i-1} \sum_{m=1}^{l} \frac{i!}{l!} \partial^m F_{i-l}^j(s, x(s,z,0)).B_{l,m}(w_1^j, \dots, w_{l-m+1}^j) \Big) ds. \Big]
\end{aligned}$$

Proof. The idea of the proof is to relate the integral equations (5.15) to the Cauchy problem and then solve it. For example, if $i = j = 1$ the integral equation is equivalent to the following

Cauchy problem

$$\frac{\partial w_1^1}{\partial t}(t,z) = F_1^1\left(t,x(t,z,0)\right) + \partial F_0^1\left(t,x(t,z,0)\right) w_1^1 \quad \text{with} \quad w_1^1(0,z) = 0,$$

and solving this linear differential equation we get the expression of $w_1^1(t,z)$ described in the statement of the lemma. For more details see Proposition 4.3.2 of Chapter 4. □

Now, we provide a formula for the averaged functions (5.7) for the class of discontinuous differential systems studied in this chapter.

Proposition 5.4.2. For $i = 1,2,\dots,k$, the averaged function (5.7) of order i is

$$g_i(z) = Y_n^{-1}(T,z)\frac{w_i^n(T,z)}{i!}.$$

Proof. For each $i = 1,2,\dots,k$ we define

$$w_i(t,z) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(t) w_i^j(t,z).$$

Given $t \in [0,T]$ there exists a positive integer $\bar{k}$ such that $t \in (t_{\bar{k}-1},t_{\bar{k}}]$ and, therefore, $w_i(t,z) = w_i^{\bar{k}}(t,z)$. By the proof of Proposition 4.3.1 we obtain

$$\begin{aligned} w_1(t,z) &= \int_0^t \Big(F_1(s,x(s,z,0)) + \partial F_0(s,x(s,z,0))w_1(s,z)\Big)ds,\\ w_i(t,z) &= i!\int_0^\theta \Big(F_i(s,x(s,z,0)) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\cdot \partial^L F_{i-l}(s,x(s,z,0))\bigodot_{m=1}^{l} w_m(s,z)^{b_m}\Big)ds. \end{aligned} \tag{5.16}$$

Since by Remark 5.2.1 we can consider the functions (5.8) given implicitly, we compute the derivatives in the variable t of the functions (5.16) and (5.9) for $i = 1$, and we see that the functions $w_1(t,z)$ and $y_1(t,z)$ satisfy the same differential equation. Moreover, for each $i = 2,\dots,k$, the integral equations (5.9) and (5.16), which provide respectively y_i and w_i, are defined by the same recurrence. Then the functions y_i and w_i satisfy the same differential equations for $i = 1,2,\dots,k$, and their initial conditions coincide. Indeed, let $i \in \{1,2,\dots,k\}$, since $y_i(0,z) = 0$ and, by (5.16), $w_i(0,z) = 0$, it follows that the initial conditions are the same. Applying the Existence and Uniqueness Theorem on the solutions of the differential system we get $y_i(t,z) = w_i(t,z)$, for all $i \in \{1,2,\dots,k\}$. □

5.4.2 Bifurcation Functions

In this section we shall write the bifurcation functions (5.3) and the functions $\gamma_i(\alpha)$ given by (5.4) in terms of Bell polynomials.

Claim 3. The bifurcation function (5.3) is given by

$$f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^{i}\sum_{m=1}^{l}\frac{1}{l!}\partial_b^m \pi g_{i-l}(z_\alpha) B_{l,m}(\gamma_1(\alpha),\dots,\gamma_{l-m+1}(\alpha)),$$

where

$$\begin{aligned}\gamma_1(\alpha) = &\ -\Delta_\alpha^{-1}\pi^{\perp} g_1(z_\alpha) \quad \text{and}\\ \gamma_i(\alpha) = &\ -\Delta_\alpha^{-1}\Bigg(\sum_{l=0}^{i-1}\frac{i!}{l!}\sum_{m=1}^{l}\partial_b^m \pi^{\perp} g_{i-l}(z_\alpha) B_{l,m}(\gamma_1(\alpha),\dots,\gamma_{l-m+1}(\alpha))\\ &+\sum_{m=2}^{i}\partial_b^m \pi^{\perp} g_0(z_\alpha) B_{i,m}(\gamma_1(\alpha),\dots,\gamma_{i-m+1}(\alpha))\Bigg).\end{aligned}$$

Proof. The expressions (5.3) was obtained in (CÂNDIDO; LLIBRE; NOVAES, 2017) using the Faá di Bruno's formula for the L-th derivative of a composite function. This claim follows just by applying the version of the Faá di Bruno's formula in terms of the Bell polynomials. □

5.5 The main theorem and its proof

The main theorem of this chapter says that the simple zeroes of the bifurcation functions (5.3) controls the number of limit cycles of system (5.10).

Theorem 5.5.1. Let Δ_α denote the lower right corner $(m-d)\times(m-d)$ matrix of the matrix $Id - Y^{-1}(T,z)$. We assume that the functions defined by (5.3) and (5.7) satisfy $f_1 = f_2 = \dots = f_{k-1} = 0$ and that for each $\alpha \in \overline{V}$, $\det(\Delta_\alpha) \neq 0$. If there exists $\alpha^* \in V$ such that $f_k(\alpha^*) = 0$, and that $\det(Df_k(\alpha^*)) \neq 0$, then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of (5.10) such that $|\varphi(0,\varepsilon) - z_{\alpha^*}| = \mathscr{O}(\varepsilon)$.

For $j = 1,2,\dots,n$ let $\xi_j(t,t_0,z_0,\varepsilon)$ be the solution of the discontinuous differential system (5.12) such that $\xi_j(t_0,t_0,z_0,\varepsilon) = z_0$. Then, we define the recurrence

$$\begin{aligned}x_1(t,z,\varepsilon) &= \xi_1(t,0,z,\varepsilon)\\ x_j(t,z,\varepsilon) &= \xi_j(t,t_{j-1},x_{j-1}(t_{j-1},z,\varepsilon),\varepsilon), \quad j = 2,\dots,n.\end{aligned}$$

Since we are working in the cross region it is easy to see that, for $|\varepsilon| \neq 0$ sufficiently small, each $x_j(t,z,\varepsilon)$ is defined for every $t \in [t_{j-1},t_j]$. Therefore $x(\cdot,z,\varepsilon) : [0,T] \to \mathbb{R}$ is defined as

$$x(t,z,\varepsilon) = \begin{cases} x_1(t,z,\varepsilon) & \text{if } 0 = t_0 \leq t \leq t_1,\\ x_2(t,z,\varepsilon) & \text{if } t_1 \leq t \leq t_2,\\ \vdots & \\ x_j(t,z,\varepsilon) & \text{if } t_{j-1} \leq t \leq t_j,\\ \vdots & \\ x_n(t,z,\varepsilon) & \text{if } t_{n-1} \leq t \leq t_n = T.\end{cases}$$

Notice that $x(t,z,\varepsilon)$ is the solution of the differential equation (5.11) such that $x(0,z,\varepsilon)=z$. Moreover, the following equality hold

$$x_j(t_{j-1},z,\varepsilon)=x_{j-1}(t_{j-1},z,\varepsilon),$$

for $j=1,2,\dots,n$.

The next lemma expands the solution $x_j(\cdot,z,\varepsilon)$ around $\varepsilon=0$.

Lemma 5.5.2. For $j\in\{1,2,\dots,n\}$ and $t_z^j>t_j$, let $x_j(\cdot,z,\varepsilon):[t_{j-1},t_j)$ be the solution of (5.12). Then

$$x_j(t,z,\varepsilon)=x_j(t,z,0)+\sum_{i=1}^{k}\frac{\varepsilon^i}{i!}w_i^j(t,z)+\mathscr{O}(\varepsilon^{k+1}).$$

Proof. First, fixed $j\in\{1,2,\dots,n\}$, we use the continuity of the solution $x_j(t,z,\varepsilon)$ and the compactness of the set $[t_{j-1},t_j]\times\overline{D}\times[-\varepsilon_0,\varepsilon_0]$ to get that

$$\int_{t_{j-1}}^{t}R^j(s,x_j(s,z,\varepsilon),\varepsilon)ds=\mathscr{O}(\varepsilon),\quad t\in[t_{j-1},t_j].$$

Thus, integrating the differential equation (5.12) from t_{j-1} to t, we get

$$\begin{aligned} x_j(t,z,\varepsilon)&=x_j(t_{j-1},z,\varepsilon)+\sum_{i=0}^{k}\varepsilon^i\int_{t_{j-1}}^{t}F_i^j(s,x_j(s,z,\varepsilon))ds+\mathscr{O}(\varepsilon^{k+1}),\quad\text{and}\\ x_j(t,z,0)&=x_j(t_{j-1},z,0)+\int_{t_{j-1}}^{t}F_0^j(s,x_j(s,z,0))ds.\end{aligned}\tag{5.17}$$

By the differentiable dependence of the solutions of a differential system on its parameters the function $\varepsilon\mapsto x_j(t,z,\varepsilon)$ is a C^{k+1} map. Then, the next step is to compute the Taylor expansion of $F_i^j(t,x_j(t,z,\varepsilon))$ around $\varepsilon=0$ and for this we use the Faá di Bruno's Formula about the l-th derivative of a composite function, which guarantees that if g and h are sufficiently smooth functions then

$$\frac{d^l}{d\alpha^l}g(h(\alpha))=\sum_{S_l}\frac{l!}{b_1!\,b_2!2!^{b_2}\dots b_l!l!^{b_l}}g^{(L)}(h(\alpha))\bigodot_{j=1}^{l}\Big(h^{(j)}(\alpha)\Big)^{b_j},$$

where S_l is the set of all l-tuples of non-negative integers $(b_1,b_2,\dots,b_l)$ satisfying $b_1+2b_2+\dots+lb_l=l$, and $L=b_1+b_2+\dots+b_l$.

For each $i=0,1,...,k-1$, expanding $F_i^j(s,x_j(s,z,\varepsilon))$ around $\varepsilon=0$ we get

$$\begin{aligned} F_i^j(s,x_j(s,z,\varepsilon))=\;&F_i^j(s,x_j(s,z,0))+\\ &\sum_{l=1}^{k-i}\sum_{S_l}\frac{\varepsilon^l}{b_1!\,b_2!2!^{b_2}\dots b_l!l!^{b_l}}\partial^L F_i^j(s,x_j(s,z,0))\bigodot_{m=1}^{l}r_m^j(s,z)^{b_m},\end{aligned}\tag{5.18}$$

where

$$r_m^j(s,z) = \frac{\partial^m}{\partial \varepsilon^m} x_j(s,z,\varepsilon)\Big|_{\varepsilon=0},$$

and for $i = k$

$$F_k^j(s,x_j(s,z,\varepsilon)) = F_k^j(s,x_j(s,z,0)) + \mathscr{O}(\varepsilon). \tag{5.19}$$

Substituting (5.18) and (5.19) in (5.17) we get

$$\begin{aligned} x_j(t,z,\varepsilon) &= x_j(t_{j-1},z,\varepsilon) + \int_{t_{j-1}}^{t} \left(\sum_{i=0}^{k} \varepsilon^i F_i^j(s,x_j(s,z,0))ds \right. \\ &\left. + \sum_{i=0}^{k-1}\sum_{l=1}^{k-i} \varepsilon^{l+i} \sum_{S_l} \frac{1}{b_1!\,b_2!2!^{b_2}\ldots b_l!\,l!^{b_l}} \cdot \partial^L F_i^j(s,x_j(s,z,0)) \bigodot_{m=1}^{l} r_m^j(s,z)^{b_m} \right) ds + \mathscr{O}(\varepsilon^{k+1}). \end{aligned}$$

Then, the proof of the lemma ends using the next two claims.

Claim 4. For $j = 1,2,\ldots,n$ we have

$$x_j(t,z,\varepsilon) = x_j(t,z,0) + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} r_i^j(t,z) + \mathscr{O}(\varepsilon^{k+1}).$$

Claim 5. The equality $r_i^j = w_i^j$ holds for $i = 1,2,\ldots,k$ and $j = 1,2,\ldots,n$.

The proof of Claims 4 and 5 can be done following the steps described in the proof of Claims 1 and 2, in Chapter 4, respectively. □

Proof of Theorem 5.5.1. Consider the *displacement function*

$$h(z,\varepsilon) = x(T,z,\varepsilon) - z = x_n(T,z,\varepsilon) - z \tag{5.20}$$

It is easy to see that $x(\cdot,\bar{z},\bar{\varepsilon})$ is a T-periodic solution if and only if $h(\bar{z},\bar{\varepsilon}) = 0$. Moreover, to study the zeros of (5.20) is equivalent to study the zeros of

$$g(z,\varepsilon) = Y_n^{-1}(T,z)h(z,\varepsilon). \tag{5.21}$$

From Lemma 5.5.2 we have that

$$x_n(T,z,\varepsilon) = x_n(T,z,0) + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} w_i^n(T,z) + \mathscr{O}(\varepsilon^{k+1}), \tag{5.22}$$

for all $(t,z) \in \mathbb{S}^1 \times D$. Replacing (5.22) in (5.21) it follows that

$$\begin{aligned} g(z,\varepsilon) =\; & Y_n^{-1}(T,z)\left(x_n(T,z,0) - z + \sum_{i=1}^{k} \frac{\varepsilon^i}{i!} w_i^n(T,z) + \mathscr{O}(\varepsilon^{k+1}) \right) \\ =\; & Y_n^{-1}(T,z)(x_n(T,z,0) - z) + \sum_{i=1}^{k} g_i(z) + \mathscr{O}(\varepsilon^{k+1}) \\ =\; & \sum_{i=0}^{k} g_i(z) + \mathscr{O}(\varepsilon^{k+1}), \end{aligned} \tag{5.23}$$

where $g_0(z) = Y_n^{-1}(T,z)(x_n(T,z,0) - z)$.

From hypothesis (H) the function $g_0(z)$ vanishes on the submanifold $\mathscr{Z}$, therefore hypothesis (H_α) holds for the function (5.23). In order to take the derivative of $g_0(z)$ with respect to the variable z we have the next claim.

Claim 6. For every $j \in \{1,2,\dots,n\}$

$$Y_j(t_j,z) = \frac{\partial x_j}{\partial z}(t_j,z,0).$$

The proof will be done by induction on j. For $j=1$ the claim is exactly the definition. Suppose that the claim is valid for $j = j_0 - 1$ and we shall prove it for $j = j_0$. Since $x_j(t_{j-1},z,\varepsilon) = x_{j-1}(t_{j-1},z,\varepsilon)$ for all $j = 1,2,\dots,n$ we have

$$\begin{aligned}
Y_{j_0}(t_{j_0},z) &= \frac{\partial x_{j_0}}{\partial z}(t_{j_0},z,0)\left(\frac{\partial x_{j_0}}{\partial z}(t_{j_0-1},z,0)\right)^{-1} Y_{j_0-1}(t_{j_0-1},z) \\
&= \frac{\partial x_{j_0}}{\partial z}(t_{j_0},z,0)\left(\frac{\partial x_{j_0-1}}{\partial z}(t_{j_0-1},z,0)\right)^{-1} \frac{\partial x_{j_0-1}}{\partial z}(t_{j_0-1},z,0) \\
&= \frac{\partial x_{j_0}}{\partial z}(t_{j_0},z,0).
\end{aligned}$$

Hence if $z \in \mathscr{Z}$ then

$$\begin{aligned}
\frac{\partial g_0}{\partial z}(z) &= Y^{-1}(T,z)\left(\frac{\partial x}{\partial z}(T,z,0) - Id\right) \\
&= Y^{-1}(T,z)(Y(T,z) - Id) \\
&= Id - Y^{-1}(T,z),
\end{aligned}$$

which has by assumption its lower right corner $(m-d)\times(m-d)$ matrix Δ_α nonsingular. From here, the result follows from Proposition 5.4.2 and Theorem 5.1.1. □

5.6 Examples

This section is devoted to present some applications of Theorem 5.5.1. The first one is as 3D piecewise smooth system for which the plane $y=0$ is the discontinuous manifold and admits a surface $z = f(x,y)$ foliated by periodic solutions. The second one is a 3D piecewise smooth system for which the algebraic variety $xy = 0$ is the discontinuous set and the plane $z=0$ has a piecewise constant center. For these systems, we compute some of the bifurcations functions in order to study the persistence of periodic solutions.

Nonsmooth perturbation of a 3D system

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differential functions such that $g(x,y) = f(x,y) + x\partial_y f(x,y) - y\partial_x f(x,y)$. Consider the nonsmooth vector field

$$X_\varepsilon(x,y,z) = \begin{cases} X_\varepsilon^+(x,y,z), & y > 0 \\ \\ X_\varepsilon^-(x,y,z), & y < 0 \end{cases} \tag{5.24}$$

where

$$X_\varepsilon^+(x,y,z) = \big(-y + \varepsilon(a_0 + a_1 z) + \varepsilon^2(a_2 + a_3 z),\ x,\ -z + g(x,y)\big), \quad \text{and}$$

$$X_\varepsilon^-(x,y,z) = \big(-y,\ x + \varepsilon b_1 z + \varepsilon^2(b_2 + b_3)z,\ -z + g(x,y)\big),$$

with $a_0, a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$. Denote the discontinuous se by $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : y = 0\}$.

Notice that the surface $z = g(x,y)$ is an invariant set of the unperturbed vector field X_0. Indeed, considering the function $\hat{f}(x,y,z) = z - f(x,y)$, we get

$$\langle \nabla \hat{f}(x,y,z), X_0(x,y,z)\rangle\big|_{z=f(x,y)} = 0.$$

Moreover, since $X_0(x,y,f(x,y)) = \big(-y, x, x\partial_y f(x,y) - y\partial_x f(x,y)\big)$ we conclude that the invariant set $z = f(x,y)$ is foliate by periodic solutions.

Next result gives suficient conditions in order to guarantee the persistence of some periodic solution. Consider the function

$$f_1(r) = a_1 \int_0^\pi f(r\cos\phi, r\sin\phi)\cos\phi\, d\phi + b_1 \int_\pi^{2\pi} f(r\cos\phi, r\sin\phi)\sin\phi\, d\phi. \tag{5.25}$$

Theorem 5.6.1. Consider the piecewise vector field (5.24). Then, for each $r* > 0$, such that $f_1(r^*) = 0$ and $f_1'(r^*) \neq 0$, there exists a crossing limit cycle $\varphi(t,\varepsilon)$ of X of period $T_\varepsilon = 2\pi + \mathcal{O}(\varepsilon)$ such that $\varphi(t,\varepsilon) = (x^*, y^*, f(x^*, y^*)) + \mathcal{O}(\varepsilon)$ with $|(x^*, z^*)| = r^*$.

In order to apply Theorem 5.5.1 for proving Theorem 5.6.1 we need to write system (5.24) in the standard form. Considering cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the set of discontinuity becomes $\Sigma = \{\theta = 0\} \cup \{\theta = t_1\}$ with $t_0 = 0, t_1 = \pi$ and $t_2 = 2\pi$. The differential system $(\dot{x}, \dot{y}, \dot{z}) = X_\varepsilon^+(x,y,z)$ in cylindrical coordinates writes

$$\begin{aligned} r'(t) &= \varepsilon(a_0 + a_1 z)\cos\theta + \varepsilon^2(a_2 + a_3 z)\cos\theta, \\ z'(t) &= g(r\cos\theta, r\sin\theta) - z, \\ \theta'(t) &= 1 - \varepsilon\frac{(a_0 + a_1 z)\sin\theta}{r} - \varepsilon^2\frac{(a_2 + a_3 z)\sin\theta}{r}, \end{aligned}$$

and the differential system $(\dot{x},\dot{y},\dot{z}) = X_\varepsilon^-(x,y,z)$ becomes

$$\begin{aligned} r'(t) &= \varepsilon b_1 z\sin\theta + \varepsilon^2(b_2+b_3z)\sin\theta,\\ z'(t) &= g(r\cos\theta, r\sin\theta) - z,\\ \theta'(t) &= 1+\varepsilon\frac{b_1 z\cos\theta}{r} + \varepsilon^2\frac{(a_2+a_3z)\cos\theta}{r}. \end{aligned} \tag{5.26}$$

Notice that, for each $j=1,2$ and $t_{j-1}\le\theta\le t_j$, we have $\dot{\theta}(t)\neq 0$ for $|\varepsilon|\neq 0$ sufficiently small. Thus, in a sufficiently small neighborhood of the origin we can take θ as the new independent time variable. Accordingly, system (5.26) becomes

$$\begin{aligned} \dot{r}(\theta) &= \frac{r'(t)}{\theta'(t)} = F_{01}(\theta,r,z)+\varepsilon F_{11}(\theta,r,z)+\varepsilon^2F_2(\theta,r,z)+\mathscr{O}_1(\varepsilon^3),\\ \dot{z}(\theta) &= \frac{z'(t)}{\theta'(t)} = F_{02}(\theta,r,z)+\varepsilon F_{12}(\theta,r,z))+\varepsilon^2F_{22}(\theta,r,z)+\mathscr{O}_2(\varepsilon^3). \end{aligned}$$

Considering the notation of Theorem 5.5.1 we have $F_i(\theta,r,z) = (F_{i1}(\theta,r,z),F_{i2}(\theta,r,z))$ for each $i\in\{1,2\}$. Moreover, for each $i\in\{1,2\}$ the function $F_i(\theta,r,z)$ is written in the form $F_i(\theta,r,z) = \sum_{j=1}^2\chi_{[t_{j-1},t_j]}(\theta)F_i^j(\theta,r,z)$.

Defining $\tilde{f}(\theta,r) = f(r\cos\theta, r\sin\theta)$ and $\tilde{g}(\theta,r) = g(r\cos\theta,r\sin\theta)$ we write explicitly the expressions of F_0, F_1^j and F_2^j for $j\in\{1,2\}$,

$$\begin{aligned} F_0(\theta,r,z) =&\ (0,\ \tilde{g}(\theta,r)-z),\\ F_1^1(\theta,r,z) =&\ \left((a_0+a_1z)\cos\theta,\ \frac{(a_0+a_1z)\sin\theta}{r}(\tilde{g}(\theta,r)-z)\right),\\ F_1^2(\theta,r,z) =&\ \left(b_1z\sin\theta,\ -\frac{b_1z\cos\theta}{r}(\tilde{g}(\theta,r)-z)\right),\\ F_2^1(\theta,r,z) =&\ \left((a_2+a_3z)\cos\theta+\frac{(a_0+a_1z)^2\sin\theta\cos\theta}{r},\ \frac{\sin\theta}{r^2}\left((a_0+a_1z)^2\sin\theta\right.\right.\\ &\left.\left.+(a_2+a_3z)r\right)(\tilde{g}(\theta,r)-z)\right),\\ F_2^2(\theta,r,z) =&\ \left((b_2+b_3z)\sin\theta-\frac{b_1^2z^2\sin\theta\cos\theta}{r},\ \frac{\cos\theta}{r^2}\left(b_1^2z\cos\theta-(b_2+b_3z)r\right)(\tilde{g}(\theta,r)-z)\right). \end{aligned}$$

The unperturbed systems is smooth and its solution $(r(\theta,r_0,z_0),z(\theta,r_0,z_0))$ with initial condition (r_0,z_0) is given by

$$r(\theta) = \bar{r}(\theta,r_0,z_0) = r_0,\quad z(\theta) = \bar{z}(\theta,r_0,z_0) = e^{-\theta}\left(z_0+\int_0^\theta e^s\tilde{g}(s,r_0)ds\right). \tag{5.27}$$

Consequently, a fundamental matrix solution of (5.13) is given by

$$Y(\theta,r_0,z_0)=\frac{\partial(\bar r,\bar z)}{\partial(r_0,z_0)}(\theta,r_0,z_0)=\begin{pmatrix}1 & 0\\ G(\theta,r_0) & e^{-\theta}\end{pmatrix},$$

where $G(\theta,r_0)$ is the derivative of $\bar z(\theta,r_0,z_0)$ with respect to the variable r_0. Notice that, from (5.27), $G(\theta,r_0)$ does not depend on z_0.

Let $\varepsilon_0>0$ be a real positive number and consider the set $\mathscr{Z}\subset\mathbb{R}^2$ such that $\mathscr{Z}=\{(r,\tilde f(0,r)):r>\varepsilon_0\}$. Notice that for $(r_0,z_0)=(r_0,\tilde f(0,r_0))\in\mathscr{Z}$ we have $z(\theta,r_0,z_0)=\tilde f(\theta,r_0)=f(r_0\cos\theta,r_0\sin\theta)$. Indeed, let $w(\theta)=f(r_0\cos\theta,r_0\sin\theta)$. So

$$\begin{aligned}
w'(\theta) &= \partial_x f(r_0\cos\theta,r_0\sin\theta)(-r_0\sin\theta)+\partial_y f(r_0\cos\theta,r_0\sin\theta)(r_0\cos\theta)\\
&= g(r_0\cos\theta,r_0\sin\theta)-f(r_0\cos\theta,r_0\sin\theta)\\
&= g(r_0\cos\theta,r_0\sin\theta)-w(\theta)\\
&= \tilde g(\theta,r_0)-w(\theta).
\end{aligned}$$

The second equality holds because $g(x,y)=f(x,y)+x\partial_y f(x,y)-y\partial_x f(x,y)$. Hence, for $(r_0,z_0)\in\mathscr{Z}$ the solution $z(\theta,r_0,z_0)$ is 2π-periodic. Moreover,

$$Id-Y^{-1}(2\pi,r,z)=\begin{pmatrix}0 & 0\\ \star & 1-e^{2\pi}\end{pmatrix}.$$

Consequently, $\Delta_\alpha=1-e^{2\pi}\neq 0$. Accordingly, all the hypotheses of Theorem 5.5.1 are satisfied.

Proof of Theorem 5.6.1. Denote by (r,z_r) a point in $\mathscr{Z}$, that is $z_r=\tilde f(0,r)$. Notice that the bifurcation function of first order is $f_1(r)=\pi g_1(r,z_r)$, where g_1 is defined in (5.7). Indeed, from definition $f_1(r)=\pi g_1(r,z_r)+\dfrac{\partial\pi g_0}{\partial b}(r,z_r)\gamma_1(r)$. But

$$g_0(r,z)=Y^{-1}(2\pi,r,z)((r,z(2\pi,r,z)))-(r,z(0,r,z)))=(0,\star),$$

and then $\pi g_0\equiv 0$. Moreover,

$$\begin{aligned}
w_1^1(\theta,r,z)=\Bigg(&a_0\sin\theta+a_1\int_0^\theta z(\phi)\cos\phi\,d\phi,\ G(\theta,r)\left(a_0\sin\theta+a_1\int_0^\theta z(\phi)\cos\phi\,d\phi\right)-\\
&e^{-\theta}\int_0^\theta\left(e^{\phi}G(\phi,r)(a_0+a_1z(\phi))\cos\phi+\sin\phi\,\frac{e^{\phi}(\tilde g(\phi,r)-z(\phi))(a_0+a_1z(\phi))}{r}\right)d\phi\Bigg),
\end{aligned}$$

$$
\begin{aligned}
w_1^2(\theta,r,z) =& Y(\theta,r,z)\left[Y^{-1}(\pi,r,z)w_1^1(\pi,r,z)+\int_\pi^\theta Y^{-1}(\phi,r,z)F_1^2(\phi,r(\phi),z(\phi))d\phi\right]\\
=& Y(\theta,r,z)\Bigg(a_1\int_0^\pi z(\phi)\cos\phi d\phi+b_1\int_\pi^\theta z(\phi)\sin\phi d\phi,\\
&\int_0^\pi \frac{e^\phi((a_0+a_1z(\phi))(\sin\phi(g(r\cos\phi,r\sin\phi)-z(\phi))-r\cos\phi\, G(\phi,r))}{r}d\phi\\
&+\int_\pi^\theta -\frac{b_1e^\phi z(\phi)(\cos\phi(g(r\cos\phi,r\sin\phi)-z(\phi))+r\sin\phi\, G(\phi,r))}{r}d\phi\Bigg).
\end{aligned}
$$

Since $g_1(r,z)=Y^{-1}(2\pi,r,z)w_1^2(2\pi,r,z)$ and $f_1(r)=\pi g_1(r,z_r)$ it follows that

$$
f_1(r)=a_1\int_0^\pi f(r\cos\phi,r\sin\phi)\cos\phi d\phi+b_1\int_\pi^{2\pi} f(r\cos\phi,r\sin\phi)\sin\phi d\phi. \tag{5.28}
$$

So, from Theorem 5.5.1, each positive simple zero of (5.25) provides an isolated periodic solution of system (5.24). This concludes this proof. □

The next result is an application of Theorem 5.6.1. We shall use in its statement the concept of Bessel functions, which are defined as the canonical solutions $y(x)$ of Bessel's differential equation

$$
x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0,\quad \alpha\in\mathbb{C}.
$$

This equation has two linearly independent solutions. Using Frobenius' method we obtain one of these solutions, which is called a *Bessel function of the first kind*, and is denoted by $J_\alpha(x)$. More details about this function can be found in (WATSON, 1995).

Corollary 5.6.2. Consider the piecewise vector field (5.24).

(a) If $f(x,y)=\cos x$, then the piecewise smooth vector field X admits a sequence of limit cycles $\varphi_i(t,\varepsilon)$ of X of period T_ε such that $T_\varepsilon=2\pi+\mathcal{O}(\varepsilon)$, $\varphi_n(t,\varepsilon)=(x_n^*,y_n^*,\cos(x_n^*))+\mathcal{O}(\varepsilon)$, and $|(x_n^*,z_n^*)|=n\pi/2$.

(b) If $f(x,y)=\sin x$, then the piecewise smooth vector field X admits a sequence of limit cycles $\varphi_i(t,\varepsilon)$ of X of period T_ε such that $T_\varepsilon=2\pi+\mathcal{O}(\varepsilon)$, $\varphi_i(t,\varepsilon)=(x_n^*,y_n^*,\sin(x_n^*))+\mathcal{O}(\varepsilon)$, and $|(x_n^*,z_n^*)|=r_n^*$, where each r_n is a zero of the Bessel Function of First Kind, $J_1(r)$.

Proof. For $f(x,y)=\cos x$, the bifurcation function (5.28) reads $f_1(r)=-(2b_1\sin r)/r$, and for $f(x,y)=\cos(x)$, the bifurcation function (5.28) reads $f_1(r)=a_1\pi J_1(r)$. Therefore the result follows directly from Theorem 5.6.1. □

Notice that Theorem 5.6.1 cannot be applied when f_1 is identically zero, which is the case when $f(x,y)=2x^2-y^2$ for instance. For these cases we define the function

$$
\begin{aligned}
f_2(r)=\int_0^{\pi}\Bigg(& a_1\cos s\Bigg(G(s,r)\int_0^s\cos\phi(a_0+a_1\tilde{f}(\phi,r))d\phi\\
&-e^{-s}\int_0^s e^{\phi}(a_0+a_1\tilde{f}(\phi,r))(r\cos\phi\, G(\phi,r)+(\tilde{f}(\phi,r)-\tilde{g}(\phi,r))d\phi\\
&+a_2+a_3\tilde{f}(\phi,r)+\frac{\sin s}{r}(a_0+a_1\tilde{f}(s,r))^2\Bigg)\Bigg)ds\\
&+\frac{e^{-2\pi}(1+e^{\pi})}{2(1-e^{2\pi})}(a_1e^{\pi}-b_1)\Bigg[\int_0^{\pi}e^{\phi}G(\phi,r)\cos\phi(a_0+a_1\tilde{f}(\phi,r))d\phi\\
&+\int_0^{\pi}\frac{e^{\phi}\sin\phi}{r}(a_0+a_1\tilde{f}(\phi,r))(\tilde{g}(\phi,r)-\tilde{f}(\phi,r))d\phi\\
&+b_1\int_{\pi}^{2\pi}e^{\phi}G(\phi,r)\sin\phi\tilde{f}(\phi,r)d\phi+\frac{b_1}{r}\int_{\pi}^{2\pi}e^{\phi}\cos\phi(\tilde{g}(\phi,r)-\tilde{f}(\phi,r))d\phi\Bigg]\\
&+\int_{\pi}^{2\pi}\Bigg(\frac{2}{r}(-b_1^2\cos s(\tilde{f}(s,r))^2+\sin s(b_2+b_3\tilde{f}(s,r)))\\
&+2b_1\sin s\Bigg(G(s,r)\int_0^{\pi}\cos\phi(a_0+a_1\tilde{f}(\phi,r))+b_1G(s,r)\int_{\pi}^{s}\sin\phi\tilde{f}(\phi,r)d\phi\\
&+e^{-s}\Bigg(\int_0^{\pi}-e^{\phi}\cos\phi\, G(\phi,r)(a_0+a_1\tilde{f}(\phi,r))+\frac{e^{\phi}\sin\phi}{r}(\tilde{g}(\phi,r)-\tilde{f}(\phi,r))d\phi\\
&+b_1\int_{\pi}^{s}e^{\phi}\left(\frac{\cos\phi}{r}(\tilde{f}(\phi,r)-\tilde{g}(\phi,r))-G(\phi,r)\sin\phi\right)d\phi\Bigg)\Bigg)\Bigg)ds.
\end{aligned}
\tag{5.29}
$$

Theorem 5.6.3. Consider the piecewise vector field (5.24). Assume that $f_1\equiv 0$. Then, for each $r*>0$, such that $f_2(r^*)=0$ and $f_2'(r^*)\neq 0$, there exists a crossing limit cycle $\varphi(t,\varepsilon)$ of X of period T_ε such that $T_\varepsilon=2\pi+\mathscr{O}(\varepsilon)$, $\varphi(t,\varepsilon)=(x^*,y^*,f(x^*,y^*))+\mathscr{O}(\varepsilon)$, and $|(x^*,z^*)|=r^*$.

Proof. As we saw before $\pi g_0\equiv 0$. So, from (5.3), we compute the bifurcation function of order 2 as

$$
f_2(r)=\frac{\partial\pi g_1}{\partial b}(r,z_r)\gamma_1(r)+\pi g_2(r,z_r),
\tag{5.30}
$$

where $\gamma_1(r)=-\dfrac{1}{1-e^{2\pi}}\pi^{\perp}g_1(r,z_r)$ and

$$
\begin{aligned}
\pi^{\perp}g_1(r,z_r)=&\int_0^{\pi}\frac{e^{\phi}((a_0+a_1\tilde{f}(\phi,r))(\sin\phi(g(r\cos\phi,r\sin\phi)-\tilde{f}(\phi,r))-r\cos\phi G(\phi,r,z))}{r}d\phi\\
&-b_1\int_{\pi}^{2\pi}\frac{e^{\phi}\tilde{f}(\phi,r)(\cos\phi(g(r\cos\phi,r\sin\phi)-\tilde{f}(\phi,r))+r\sin\phi G(\phi,r,z))}{r}d\phi.
\end{aligned}
$$

From Proposition 5.4.2, we have $g_2(r,z_r)=Y^{-1}(2\pi,r,z)w_2^2(2\pi,r,z)/2$, where $w_i^j(2\pi,r,z)$ is given in Lemma 5.4.1. All these functions may be computed to get (5.30) as (5.29). Again, from

Theorem 5.5.1, each positive simple zero of (5.29) provides an isolated periodic solution of system (5.24). This concludes this proof. □

The next result is an application of Theorem 5.6.3.

Corollary 5.6.4. Consider the piecewise vector field (5.24) and let $f(x,y) = 2x^2 - y^2$. Assuming $a_1^2 + b_1^2 \neq 0$ define

$$\begin{aligned} A_0 &= \frac{-80b_2(1-e^{\pi})}{(1+e^{\pi})4(15a_1b_1 - b_1^2 - 14a_1^2) - 5\pi(1-e^{\pi})(b_1^1 + 10a_1^2)}, \\ A_1 &= \frac{40a_0((1+e^{\pi})(b_1 - a1) - a1\pi(1-e^{\pi})}{(1+e^{\pi})4(15a_1b_1 - b_1^2 - 14a_1^2) - 5\pi(1-e^{\pi})(b_1^1 + 10a_1^2)}, \end{aligned} \tag{5.31}$$

and $D = -4A_1^3 - 27A_0^2$.

(i) If $D > 0$ then the piecewise smooth vector field admits at least one limit cycle. Moreover, if $A_1 < 0$ and $A_0 > 0$, then the piecewise smooth vector field admits at least two limit cycles;

(ii) If $D \leq 0$ and $A_0 < 0$, then the piecewise smooth vector field admits at least one limit cycle.

Moreover, in both cases we have a limit cycle $\varphi(t,\varepsilon)$ of X of period T_ε such that $T_\varepsilon = 2\pi + \mathscr{O}(\varepsilon)$, $\varphi(t,\varepsilon) = (x_n^*, y_n^*, 2(x_n^*)^2 - (y_n^*)^2) + \mathscr{O}(\varepsilon)$, and $|(x_n^*, z_n^*)| = r_n^*$.

Proof. For $f(x,y) = 2x^2 - y^2$ the bifurcation function (5.29) becomes

$$\begin{aligned} f_2(r) = -2b_2 &+ \frac{a_0\left((e^{\pi}(1-\pi) + 1 + \pi)a_1 - (1+e^{\pi})b_1\right)}{e^{\pi} - 1} r \\ &+ \frac{\left(-(e^{\pi}(56 - 50\pi) + 56 + 50\pi)a_1^2 + 60(1+e^{\pi})a_1b_1 - (e^{\pi}(4-5\pi) + 4 + 5\pi)b_1^2\right)}{40(e^{\pi} - 1)} r^3. \end{aligned} \tag{5.32}$$

Dividing f_2 by $a_1^2 + b_1^2 \neq 0$, we see that the equation $f_2(r) = 0$ is equivalent to $\tilde{f}_2(r) \doteq A_0 + A_1 r + r^3 = 0$, where A_0 and A_1 are given in (5.31).

Notice that $\tilde{f}_2(r)$ is a polynomial function of degree 3, so it has at least one real root and can be written as $\tilde{f}_2(r) = r^3 - (r_1 + r_2 + r_3)r^2 + (r_1r_2 + r_1r_3 + r_2r_3)r - r_1r_2r_3$, where $r_i, i = 1,2,3$ are the zeros of the polynomial. Moreover, the sign of its discriminant $D = -4A_1^3 - 27A_0^2$ carries information about its number of real roots.

If $D > 0$ the polynomial $\tilde{f}_2(r)$ has three simple real roots r_1, r_2 and r_3. Since the polynomial has no quadratic term, it follows that $r_1 + r_2 + r_3 = 0$ and then at least one of these roots must be positive. Moreover, if $A_1 < 0$ and $A_0 > 0$ then there are two changes of sign between the terms of the polynomial and then by *Descartes Sign Theorem* we get the two positive roots.

If $D \leq 0$ then there is a pair of complex roots or a double real root. In both cases the condition $A_0 < 0$ implies that at least one root is positive.

Now, from Theorem 5.5.1, each positive simple zero of (5.32) provides an isolated periodic solution of system (5.24). This concludes this proof. $\square$

Nonsmooth perturbation of a nonsmooth center

In this example we consider a discontinuous differential system in $\mathbb{R}^3$ defined in 4 zones ($n = 4$). Consider the nonsmooth vector field

$$X(u,v,w) = \begin{cases} X_1(u,v,w) & \text{if } u > 0 \text{ and } v > 0, \\ X_2(u,v,w) & \text{if } u < 0 \text{ and } v > 0, \\ X_3(u,v,w) & \text{if } u < 0 \text{ and } v < 0, \\ X_4(u,v,w) & \text{if } u > 0 \text{ and } v < 0, \end{cases} \tag{5.33}$$

where

$$\begin{aligned} X_1(u,v,w) &= (-1+\varepsilon(a_1x+b_1),\, 1,\, -w+\varepsilon(c_1x+d_1)), \\ X_2(u,v,w) &= (-1+\varepsilon(a_2x+b_2),\, -1,\, -w+\varepsilon(c_2x+d_2)), \\ X_3(u,v,w) &= (1+\varepsilon(a_3x+b_3),\, -1,\, -w+\varepsilon(c_3x+d_3)), \\ X_4(u,v,w) &= (1+\varepsilon(a_4x+b_4),\, 1,\, -w+\varepsilon(c_4x+d_4)), \end{aligned}$$

with $a_j, b_j, c_j, d_j \in \mathbb{R}$ for all j.

Writing in cylindrical coordinates $u = r\cos\theta$, $v = r\sin\theta$, $w = w$, the set of discontinuity is $\Sigma = \{\theta = 0\} \cup \{\theta = t_1\} \cup \{\theta = t_2\} \cup \{\theta = t_3\}$ with $t_0 = 0, t_1 = \pi/2, t_2 = \pi, t_3 = 3\pi/2$ and $t_4 = 2\pi$. For each $j = 1,2,3,4$ the differential system $(\dot{u},\dot{v},\dot{w}) = X_j(u,v,w)$ in cylindrical coordinates writes

$$\begin{aligned} r'(t) &= g_j(\theta) + \sum_{i=1}^{k} \varepsilon^i (a_{ij} r\cos^2\theta + b_{ij}\cos\theta), \\ w'(t) &= -w + \sum_{i=1}^{k} \varepsilon^i (c_{ij} r\cos\theta + d_{ij}\cos\theta), \\ \theta'(t) &= \frac{1}{r}\left(\widehat{g}_j(\theta) - \sum_{i=1}^{k} \varepsilon^i (a_{ij} r\cos\theta\sin\theta + b_{ij}\sin\theta)\right), \end{aligned}$$

where

$$
\begin{aligned}
g_1(\theta) &= \sin\theta - \cos\theta, & \widehat{g}_1(\theta) &= \sin\theta + \cos\theta,\\
g_2(\theta) &= -(\sin\theta + \cos\theta), & \widehat{g}_2(\theta) &= \sin\theta - \cos\theta,\\
g_3(\theta) &= -\sin\theta + \cos\theta, & \widehat{g}_3(\theta) &= -(\sin\theta + \cos\theta),\\
g_4(\theta) &= \sin\theta + \cos\theta, & \widehat{g}_4(\theta) &= -\sin\theta + \cos\theta.
\end{aligned}
$$

Notice that, for each $j = 1,2,3,4$ and $t_{j-1} \le \theta \le t_j$, $\dot{\theta}(t) \neq 0$ for $|\varepsilon|$ sufficiently small. Thus, in a sufficiently small neighborhood of the origin we can take θ as the new independent time variable by doing $r'(\theta) = \dot{r}(t)/\dot{\theta}(t)$ and $w'(\theta) = \dot{w}(t)/\dot{\theta}(t)$. Taking θ as the new independent time variable we have

$$
\begin{aligned}
r'(\theta) &= F_{01}^j(\theta,z) + \varepsilon F_{11}^j(\theta,z) + \mathcal{O}_1(\varepsilon^2),\\
w'(\theta) &= F_{02}^j(\theta,z) + \varepsilon F_{12}^j(\theta,z) + \mathcal{O}_2(\varepsilon^2).
\end{aligned}
\tag{5.34}
$$

Here, $z = (r,w)$ and the prime denotes the derivative with respect to θ. The expressions of F_{01}^j and F_{02}^j for $j = 1,2,3,4$ are given by

$$
F_{01}^1 = \frac{r(\sin\theta - \cos\theta)}{\sin\theta + \cos\theta},\ F_{02}^1 = \frac{-rw}{\sin\theta + \cos\theta},\ F_{01}^2 = \frac{r(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta},\ F_{02}^2 = \frac{rw}{\cos\theta - \sin\theta},
$$

$$
F_{01}^3 = \frac{r(\sin\theta - \cos\theta)}{\sin\theta + \cos\theta},\ F_{02}^3 = \frac{rw}{\sin\theta + \cos\theta},\ F_{01}^4 = \frac{r(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta},\ F_{02}^4 = \frac{-rw}{\cos\theta - \sin\theta}.
$$

The expressions of F_{11}^j and F_{12}^j for $j = 1,2,3,4$ are also easily computed. Nevertheless, we shall omit these expressions because of their size.

For each $j \in \{1,2,3,4\}$, the differential system (5.34) is 2π-periodic in the variable θ and is written in the standard form with

$$
F_i^j(\theta,z) = \left(F_{i1}^j(\theta,z), F_{i2}^j(\theta,z)\right),
$$

for $i = 0,1$. Now, for each $j \in \{1,2,3,4\}$ we compute the solution $x_j(\theta,z,0)$ of the unperturbed system

$$
\dot{r}(\theta) = F_{01}^j(\theta,z), \quad \dot{w}(\theta) = F_{02}^j(\theta,z).
$$

and this solution is

$$x_1(\theta,z,0) = \left(\frac{r}{\sin\theta+\cos\theta}, we^{-\frac{r\sin\theta}{\sin\theta+\cos\theta}}\right),$$

$$x_2(\theta,z,0) = \left(\frac{-r}{\cos\theta-\sin\theta}, we^{-\frac{r\sin\theta}{\cos\theta-\sin\theta}-2r}\right),$$

$$x_3(\theta,z,0) = \left(\frac{-r}{\sin\theta+\cos\theta}, we^{-\frac{r\sin\theta}{\sin\theta+\cos\theta}-2r}\right),$$

$$x_4(\theta,z,0) = \left(\frac{r}{\cos\theta-\sin\theta}, we^{-\frac{r\sin\theta}{\cos\theta-\sin\theta}-4r}\right).$$

We note that in each quadrant the denominators of these four solutions never vanish.

Let $0 < r_0 < r_1$ be positive real numbers and consider the set $\mathscr{Z} \subset \mathbb{R}^2$ such that $\mathscr{Z} = \{(\alpha,0) : r_0 < \alpha < r_1\}$. The solution $x(\theta,z,0)$ of the unperturbed system $x'(\theta) = F_0(\theta,z)$ satisfies $x(\theta,z,0) = x_j(\theta,z,0)$, for $\theta \in [t_{j-1},t_j]$, and $x(2\pi,z,0) - x(0,z,0) = (0,z(1-e^{-4r}))$. Consequently, for each $z_\alpha \in \mathscr{Z}$, the solution $x(\theta,z,0)$ is 2π-periodic and system (5.33) satisfies hypothesis (H). Moreover, the fundamental matrix $Y(\theta,z)$ is given by

$$Y(\theta,z) = \begin{cases} Y_1(\theta,z) & \text{if } 0 = t_0 \le \theta \le \pi/2, \\ Y_2(\theta,z) & \text{if } \pi/2 \le \theta \le \pi, \\ Y_3(\theta,z) & \text{if } \pi \le \theta \le 3\pi/2, \\ Y_4(\theta,z) & \text{if } 3\pi/2 \le \theta \le 2\pi, \end{cases}$$

where $Y_j(t,z)$ are defined by (5.14). So

$$Y_1(\theta,z) = \begin{pmatrix} \frac{1}{g_4(\theta)} & 0 \\ -\frac{e^{-\frac{r\sin\theta}{g_4(\theta)}} w\sin\theta}{g_4(\theta)} & e^{-\frac{r\sin\theta}{g_4(\theta)}} \end{pmatrix},$$

$$Y_4(\theta,z) = \begin{pmatrix} \frac{1}{g_3(\theta)} & 0 \\ -\frac{e^{-\frac{r\sin\theta}{g_3(\theta)}-4r} w(\sin\theta+4g_3(\theta))}{g_3(\theta)} & e^{-\frac{r\sin\theta}{g_3(\theta)}-4r} \end{pmatrix}.$$

Hence,

$$Y_1(0,z)^{-1} - Y_4(2\pi,z)^{-1} = \begin{pmatrix} 0 & 0 \\ -4w & 1-e^{4r} \end{pmatrix},$$

and then $\det(\Delta_\alpha) = 1 - e^{4r} \neq 0$ if $z_\alpha = (\alpha,0) \in \mathscr{Z}$. Thus, we can compute the bifurcation functions (5.3) for system (5.33). For doing this we first obtain the functions (5.15) corresponding to this system,

$$
\begin{aligned}
g_0(\theta,z) = \quad & (0, w(1-e^{4r})),\\
w_1^4(2\pi,z) = \quad & \Big(\frac{1}{2}r(r(a_1+a_2+a_3+a_4)+2(b_1-b_2-b_3+b_4)),\\
& \frac{1}{3}e^{-4r}(-r^2w(6a_1+3a_2+2a_3)-3r(w(4b_1-2b_2-b_3)\\
& +e^{2r}(-e^{2r}c_4+c_2+c_3)+c_1)+3(e^r-1)(e^r(c_2+d_2)\\
& +e^{2r}(c_3-d_3)+e^{3r}(d_4-c_4)+c_1+d_1))\Big),
\end{aligned}
$$

and

$$
g_1(z) = Y_4(2\pi,z)^{-1}w_1^4(2\pi,z). \tag{5.35}
$$

So, the bifurcation function (5.3) corresponding to the function (5.35) becomes

$$
f_1(\alpha) = \frac{1}{2}\alpha(\alpha(a_1+a_2+a_3+a_4)+2(b_1-b_2-b_3+b_4)),
$$

which has a simple zero α^*. So, from Theorem 5.5.1, we get the existence of an isolated periodic solution of system (5.34) for ε sufficiently small.

CHAPTER

6

FINAL CONSIDERATIONS

The object of study of this thesis was the invariant curves for some families of differential systems. This study was done in two parts: invariant algebraic cubics for planar polynomial systems and periodic solutions for non–smooth differential systems. In fact, considering the planar differential system

$$\dot{x} = P(x,y), \ \ \dot{y} = Q(x,y), \tag{6.1}$$

where P and Q are polynomial in the variables x, y of degree two, first we investigated the conditions under the parameters of P and Q in order that system (6.1) had an invariant cubic. Secondly, under the conditions obtained, we drew the non–equivalent and realizable phase portraits for systems that had a Darbouxian integral. To draw the phase phase portraits it was used classical results on qualitative theory, as the theorems that classify semi–hyperbolic and non–elementary singular points, founded in (DUMORTIER; LLIBRE; ARTÉS, 2006). The greatest difficulty in this study was to work with the number of parameters that the systems presented. We obtained systems with many parameters and in some cases we had to study their singularities (and respective topological types) without being able to reduce this number.

In this line of research, future plans are to consider a system of type (6.1), but with P and Q polynomials of degree n arbitrary. The objective is to study the conditions so that it has an invariant cubic and a Darboux invariant. We expect to achieve a classification in terms of normal forms for systems with such conditions.

The second part of this thesis was devoted to studying periodic orbits in systems on the form

$$\dot{x} = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^k F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{6.2}$$

where $F_i(t,x)$ are piecewise smooth functions defined in $\mathbb{S}^1 \times D$, for $i = 0,1,\ldots,k$ and $D \subset \mathbb{R}^n$ was a bounded open subset of R^n. For $k = 1$ in (6.2) we presented the classical averaging theory (averaging theory of first order) and we apply it in some systems defined in the plane, in $\mathbb{R}^3$ and to a center defined in $\mathbb{R}^n$, with n even. For k arbitrary also presented the classical theorems but the main result was the extension of the averaging theory for a family of non–smooth differential systems. The simple zeros of the called averaged functions controlled very well the number of periodic solutions of system (6.2). To prove this, one of the hypothesis in this extension was that the all solutuions of the unperturbed system $\dot{x} = F_0(t,x)$ was T–periodic for each $\rho \in D \subset \mathbb{R}^n$. In other words the manifold $\mathscr{Z}$ of periodic solutions of the unperturbed system had dimension $m = n$.

When this hypothesis is not vallid, that is, the dimension of $\mathscr{Z}$ is $m < n$ then the averaing theory is no sufficient for estimate the number of periodic solutions of system (6.2). So we aslo used the Lyapunov–Schmidt Reduction to obtain the bifurcation functions, whose simple zeros controls this number.

In both cases we give the reader examples and the explicit formulas for the calculation and implementation of the averaged and bifurcation functions. For the examples we used the software Mathematica.

In addition to demonstrating the extensions, calculating the functions and estimating their number of zeros was one of the great difficulties in developing this part of the thesis. It is important to note that many families of non–smooth systems can already be studied using the theories developed in this thesis. We have made a significant contribution to the study of periodic orbits in differential systems as in (6.2). We now expect to study other invariant sets for discontinuous systems, developing new tools and seeking to contribute more and more to the qualitative study of these systems.

APPENDIX

A

NONSMOOTH DIFFERENTIAL SYSTEMS OF KUKLES' TYPE

During the master's degree, we studied a particular discontinuous differential system, which we call the generalized Kukles polynomial differential system. A first version of this work can be found in (RODRIGUES, 2015). After improvements in the results, it was accepted for publication in the year 2018. In this appendix we put the final version entitled "Limit cycles for a class of discontinuous piecewise generalized Kukles" differential systems and published in Nonlinear Dynamics. The objective is to give an estimative to the number of limit cycles which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ by the averaging method of first order when it is perturbed inside a class of discontinuous generalized Kukles differential systems defined in $2l-$zones, $l = 1, 2, 3, \ldots$, in the plane. The classical Kukles system was introduced by Kukles in (KUKLES, 1944) where necessary and sufficient conditions for the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + a_0 y + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3, \end{aligned}$$

to have a center at the origin are presented. The existence of a center or focus for such system was considered by Sadovskii in (SADOVSKII, 2003) when $a_2 a_7 \neq 0$. He also proved that such system can have up till seven limit cycles. Zang et al. (ZANG *et al.*, 2008) studied the number and distribution of limit cycles for a class of reduced Kukles systems under cubic perturbation. Chavarriga et al. (CHAVARRIGA *et al.*, 2004) studied the maximum number of the small amplitude limit cycles for Kukles system which can coexist with some invariant algebraic curves. Llibre and Mereu (LLIBRE; MEREU, 2011) studied the maximum number of limit cycles which can bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using averaging method of first and second orders when the system is perturbed inside of the class of generalized Kukles polynomial differential systems

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -x - \sum_{k\geq 1} \varepsilon^k (f_{n_1}^k(x) + g_{n_2}^k(x)y + h_{n_3}^k(x)y^2 + d_0^k y^3),
\end{aligned}
\tag{A.1}
$$

where for every k the polynomials $f_{n_1}^k, g_{n_2}^k, h_{n_3}^k$ have degree n_1, n_2 and n_3 respectively, $d_0^k \neq 0$ is a real number and ε is a small parameter.

Here our main objective is investigating an estimative to the number of limit cycles given by the averaging method of first order which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ when it is perturbed inside a class of discontinuous generalized Kukles differential systems defined in $2l-$zones, $l = 1, 2, 3, \ldots$, in the plane.

We play with many straight lines of discontinuity passing through the origin and with two different continuous Kukles system (of the form (A.1)) located alternatively in the zones defined by such straight lines. These ideas appear in the study of two distinct classes of discontinuous generalized Lienard polynomial differential equation, see (LLIBRE; MEREU *et al.*, 2013) and (LLIBRE; TEIXEIRA, 2015). In (LLIBRE; TEIXEIRA, 2015) the authors provide lower bounds for the maximum number of limit cycles for the m-piecewise discontinuous polynomial differential equations

$$\dot{x} = y + \operatorname{sign}(g_m(x,y))F(x), \qquad \dot{y} = -x,$$

where the zero set of the function $\operatorname{sign}(g_m(x,y))$ with $m = 2, 4, 6, \ldots$ is the product of $m/2$ straight lines passing through the origin of coordinates and $sign(z)$ denotes the sign function z. Note that such lines divide the plane into zones of angle $2\pi/m$. It is worth mentioning that the division of the plane by straight lines passing through a vertex also appear in (AKHMET; ARUĞASLAN, 2009) where the authors investigate non-smooth planar systems with discontinuous right-hand sides.

A.1 Statement of the main results

Let l be a natural number and h_α the function $h_\alpha : \mathbb{R}^2 \to \mathbb{R}$ given by

$$h_\alpha(x,y) = \prod_{k=0}^{l-1} \left(y - \tan\left(\alpha + \frac{k\pi}{l}\right) x \right), \tag{A.2}$$

for a fixed $\alpha \in \left(-\frac{\pi}{l}, \frac{\pi}{l}\right)$.

The set $h_\alpha^{-1}(0)$ is the product of l straight lines passing through the origin of coordinates dividing the plane in $2l-$ zones with angles π/l.

Consider the discontinuous differential system

$$\dot{X} = \begin{cases} X_1(x,y), & \text{if } h_\alpha(x,y) > 0, \\ X_2(x,y), & \text{if } h_\alpha(x,y) < 0, \end{cases} \tag{A.3}$$

where

$$X_j(x,y) = \begin{pmatrix} y \\ -x - \varepsilon(f_{n_1}^j(x) + g_{n_2}^j(x)y + h_{n_3}^j(x)y^2 + d_0^j y^3) \end{pmatrix}$$

with $f_{n_1}^j(x)$, $g_{n_2}^j(x)$ and $h_{n_3}^j(x)$ polynomials of degrees n_1, n_2 and n_3 respectively, and d_0^j is a nonzero real number for $j = 1,2$.

System (A.3) also can be written as

$$\dot{X} = G_1(x,y) + sign(h_\alpha(x,y))G_2(x,y),$$

where

$$G_1(x,y) = \frac{1}{2}(X_1(x,y) + X_2(x,y))$$

and

$$G_2(x,y) = \frac{1}{2}(X_1(x,y) - X_2(x,y)).$$

The main results are the following

Theorem A.1.1. Suppose $j = 1,2$, $f_{n_1}^j(x)$, $g_{n_2}^j(x)$ and $h_{n_3}^j(x)$ polynomials of degree n_1, n_2 and n_3 (greater or equal to one) respectively, d_0^j a nonzero constant and $l \in \{1,2,3\}$. For $|\varepsilon| > 0$ sufficiently small the averaging theory of first order provides the existence of at most $m(l)$ small limit cycles of the discontinuous piecewise generalized Kukles differential system (A.3) where

i) $m(1) = max\left\{2\left[\frac{n_1}{2}\right], n_2+1, 2\left[\frac{n_3+2}{2}\right], 3\right\};$

ii) $m(2) = max\left\{\left[\frac{n_1-1}{2}\right], \left[\frac{n_2}{2}\right], \left[\frac{n_3+1}{2}\right], 1\right\};$

iii) $m(3) = max\left\{2\left[\frac{n_1}{2}\right], n_2+1, 2\left[\frac{n_3+2}{2}\right], 3\right\} - 1.$

For the smooth case the authors in (LLIBRE; MEREU, 2011) show that the averaging theory of first order provides the existence of at most $max\left\{\left[\frac{n_2}{2}\right], 1\right\}$ small limit cycles to the generalized Kukles polynomial differential systems.

The prove Theorem A.1.1 we use the averaging theory of first order, presented in Chapther 3. First we write

$$f_{n_1}^1(x) = \sum_{i=0}^{n_1} a_i x^i, \;\; g_{n_2}^1(x) = \sum_{i=0}^{n_2} b_i x^i, \;\; h_{n_3}^1(x) = \sum_{i=0}^{n_3} c_i x^i,$$

$$f_{n_1}^2(x)=\sum_{i=0}^{n_1} d_i x^i,\ \ g_{n_2}^2(x)=\sum_{i=0}^{n_2} e_i x^i,\ \ h_{n_3}^2(x)=\sum_{i=0}^{n_3} m_i x^i.$$

Doing the change of coordinates $x=r\cos\theta$, $y=r\sin\theta$ and taking θ as the new independent variable, system (A.3) takes the form

$$\frac{dr}{d\theta}=\varepsilon\sin\theta P_j(r,\theta)+\mathscr{O}(\varepsilon^2), \tag{A.4}$$

where

$$P_1(r,\theta)\ =\sum_{i=0}^{n_1} a_i r^i\cos^i\theta+\sum_{i=0}^{n_2} b_i r^{i+1}\cos^i\theta\sin\theta+\sum_{i=0}^{n_3} c_i r^{i+2}\cos^i\theta\sin^2\theta+d_0^1 r^3\sin^3\theta,$$

and

$$P_2(r,\theta)\ =\sum_{i=0}^{n_1} d_i r^i\cos^i\theta+\sum_{i=0}^{n_2} e_i r^{i+1}\cos^i\theta\sin\theta+\sum_{i=0}^{n_3} m_i r^{i+2}\cos^i\theta\sin^2\theta+d_0^2 r^3\sin^3\theta.$$

Note that system (A.4) satisfies the hypothesis of Theorem 3.2.2 therefore we can estimate the number of limit cycles of system (A.3) estimating the number of zeros of the averaged function (3.7).

Denoting by

$$\varphi_{ijl}(\alpha)=\sum_{k=1}^{l}\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}}\cos^i\theta\sin^j\theta d\theta \qquad \text{and} \qquad \overline{\varphi}_{ijl}(\alpha)=\sum_{k=1}^{l}\int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}}\cos^i\theta\sin^j\theta d\theta,$$

the averaged function (3.7) becomes

$$\begin{aligned} f(r)\ &=\sum_{i=0}^{n_1} r^i\Big[a_i\varphi_{i1l}(\alpha)+d_i\overline{\varphi}_{i1l}(\alpha)\Big]+\sum_{i=0}^{n_2} r^{i+1}\Big[b_i\varphi_{i2l}(\alpha)+e_i\overline{\varphi}_{i2l}(\alpha)\Big]+\\ &\sum_{i=0}^{n_3} r^{i+2}\Big[c_i\varphi_{i3l}(\alpha)+m_i\overline{\varphi}_{i3l}(\alpha)\Big]+r^3\Big[d_0^1\varphi_{04l}(\alpha)+d_0^2\overline{\varphi}_{04l}(\alpha)\Big]. \end{aligned} \tag{A.5}$$

The next lemma yields some relations between the functions φ_{ijl} and $\overline{\varphi}_{ijl}$ for any $i,j\in\mathbb{N}$ when $l\neq 0$.

Lemma A.1.2. For each $i\in\mathbb{N}$ and $l\neq 0$ the following holds

a) $\varphi_{i1l}=-\overline{\varphi}_{i1l}$,

b) $\varphi_{i3l}=-\overline{\varphi}_{i3l}$.

c) If i is odd then $\varphi_{i2l}=-\overline{\varphi}_{i2l}$.

Proof. For each $l \neq 0$ we have that

$$(\varphi_{i1l}+\overline{\varphi}_{i1l})(\alpha)=\sum_{k=1}^{l}\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}}\cos^i\theta\sin\theta d\theta=\int_{\alpha}^{\alpha+2\pi}\cos^i\theta\sin\theta d\theta=0.$$

Analogously,

$$(\varphi_{i3l}+\overline{\varphi}_{i3l})(\alpha)=\sum_{k=1}^{l}\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}}\cos^i\theta\sin^3\theta d\theta=\int_{\alpha}^{\alpha+2\pi}\cos^i\theta\sin^3\theta d\theta=0.$$

Moreover, $(\varphi_{i2l}+\overline{\varphi}_{i2l})(\alpha)=\int_{\alpha}^{\alpha+2\pi}\cos^i\theta\sin^2\theta d\theta$. Assuming $i=2m+1$ item c) follows directly from the formulae 2.511-4 of (GRADSHTEYN; RYZHIK, 2014)

$$\int\cos^{2m+1}\theta\sin^2\theta d\theta=\frac{\sin^3\theta}{2m+3}\left(\cos^{2m}\theta+\sum_{j=1}^{m}A_{j,m}\cos^{2m-2j}\theta\right), \tag{A.6}$$

where

$$A_{m,j}=\frac{2^j m(m-1)...(m-j+1)}{(2m+1)(2m-1)...(2m-2j+3)}. \tag{A.7}$$

□

Remark A.1.3. It follows from the previous lemma

1) $\varphi_{i2l}(\alpha),\overline{\varphi}_{i2l}(\alpha)>0$ for all $l\neq 0$ and $\alpha\in\mathbb{R}$ when i is even.

2) As $\sin^4\theta\geq 0$, $\varphi_{04l}(\alpha),\overline{\varphi}_{04l}(\alpha)>0$ for all $\alpha\in\mathbb{R}$.

In order to establish the number of zeros of the averaged function it remains to study some of the functions $\varphi_{i1l},\varphi_{i3l}$ and φ_{i2l} for some values of $i\in\mathbb{N}$ and $l\neq 0$. But to do it we must to fix the number $l\neq 0$.

A.1.1 Proof of item (i)(case $l=1$).

Fix $\alpha\in(-\pi,\pi)\setminus\{\pm\pi/2\}$. The cases $\alpha=\pm\pi/2$ will be considered separately. If $l=1$ then $\Sigma=h_\alpha^{-1}(0)$, where $h_\alpha(x,y)=y-(\tan\alpha)x$.

The following lemmas are useful to give the maximum number of zeros of the averaged function (A.5) in the case $l=1$.

Lemma A.1.4. $\varphi_{i11}\equiv 0$ when i is odd. Otherwise φ_{i11} does not vanish except to $\alpha=\pm\frac{\pi}{2}$.

Proof. From the definition of the function φ_{i11} we get

$$\varphi_{i11}(\alpha)=\int_{\alpha}^{\alpha+\pi}\cos^i\theta\sin\theta d\theta=-\frac{(-1)^{i+1}\cos^{i+1}\alpha-\cos^{i+1}\alpha}{i+1}=\begin{cases}0 & \text{if } i \text{ is odd,}\\ 2\dfrac{\cos^{i+1}\alpha}{i+1} & \text{if } i \text{ is even.}\end{cases}$$

Then if i is even $\varphi_{i11}(\alpha)$ vanishes if and only if $\alpha=\pm\pi/2$. □

Lemma A.1.5. The functions φ_{i21} do not vanish when i is odd and $\alpha \neq 0$.

Proof. If $i=1$ then $\varphi_{121}(\alpha) = -\frac{2}{3}\sin^3\alpha$. So $\varphi_{121}=0$ if and only if $\alpha=0$.

If $i>1$ is odd we write $i=2m+1$ for some $m \in \mathbb{N}\backslash\{0\}$ and from (A.6) we get

$$\varphi_{(2m+1)21}(\alpha) = -2\frac{\sin^3\alpha}{2m+3}\left(\sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}\alpha + \cos^{2m}\alpha\right).$$

Then $\varphi_{(2m+1)20}(\alpha)=0$ if and only if $\alpha=0$. □

Lemma A.1.6. The function φ_{i31} does not vanish for i even and $\alpha \neq \pm\frac{\pi}{2}$. Moreover $\varphi_{i31} \equiv 0$ if i is odd.

Proof. We have

$$\begin{aligned}\varphi_{i31}(\alpha) &= \tfrac{(-1)^{i+3}\cos^{i+3}\alpha - \cos^{i+3}\alpha}{i+3} - \tfrac{(-1)^{i+1}\cos^{i+1}\alpha - \cos^{i+1}\alpha}{i+1}\\ &= \begin{cases} 0 & \text{if } i \text{ is odd,}\\ 2\left(\dfrac{\cos^{i+1}\alpha}{i+1} - \dfrac{\cos^{i+3}\alpha}{i+3}\right) \neq 0 & \text{if } i \text{ is even.}\end{cases}\end{aligned}$$

So the function $\varphi_{i31}(\alpha)$ vanishes if and only if i odd or $\alpha = \pm\frac{\pi}{2}$. □

Proof of Theorem A.1.1 (i). By the previous lemmas the averaged function is given by

$$\begin{aligned} f(r) &= \sum_{\substack{i=0\\ i \text{ even}}}^{n_1} A_i(\alpha)(a_i-d_i)r^i + \sum_{\substack{i=0\\ i \text{ even}}}^{n_2} B_i(\alpha)(b_i+e_i)r^{i+1}\\ &+ \sum_{\substack{i=1\\ i \text{ odd}}}^{n_2} C_i(\alpha)(b_i-e_i)r^{i+1} + \sum_{\substack{i=0\\ i \text{ even}}}^{n_3} D_i(\alpha)(c_i-m_i)r^{i+2}\\ &+ (E^1(\alpha)d_0^1 + E^2(\alpha)d_0^2)r^3,\end{aligned}$$

where the functions $A_i(\alpha)$ and $D_i(\alpha)$ do not vanish if $\alpha \neq \pm\frac{\pi}{2}$, $B_i(\alpha), E^1(\alpha), E^2(\alpha) > 0$ and $C_i(\alpha) \neq 0$ if $\alpha \neq 0$.

Then the averaged function is a polynomial function of degree m where m is given by

(i) $m = \max\{n_2+1, 3\}$, if $\alpha \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$;

(ii) $m = \max\left\{2\left[\frac{n_1}{2}\right], 2\left[\frac{n_2}{2}\right]+1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$, if $\alpha = 0$;

(iii) $m = \max\left\{2\left[\frac{n_1}{2}\right], n_2+1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$, if $\alpha \notin \{0, \pm\frac{\pi}{2}\}$.

Therefore, the maximum number of zeros of the averaged function is also m.

Moreover it is possible to choose the coefficients $a_i, b_i, c_i, d_i, e_i, m_i$ and d_0^j and α in the expression of the averaged function such that $f(r)$ has exactly $m(1) = max\left\{2\left[\frac{n_1}{2}\right], n_2+1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$ simple positive roots. As the maximum number of zeros is reached we conclude from Theorem 3.7 that system (A.3) provides at most $m(1)$ limit cycles, each one with period near to 2π. $\square$

A.1.2 Proof of item (ii)(case $l = 2$).

In this case $\Sigma = \{(x,y) : (y-(\tan\alpha)x)(y+\tan(\alpha+\frac{\pi}{2})x) = 0\}$, with $\alpha \in (-\pi/2, \pi/2)$.

Lemma A.1.7. $\varphi_{i12} \equiv 0$ if i is even. Otherwise φ_{i12} do not vanish except for $\alpha = \pm\frac{\pi}{4}$.

Proof. It is easy to see that

$$\varphi_{i12}(\alpha) \quad = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 2\dfrac{\cos^{i+1}\alpha - \sin^{i+1}\alpha}{i+1} & \text{if } i \text{ is odd.} \end{cases}$$

So $\varphi_{112}(\alpha) = 0$ if and only $\alpha = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. For any other i odd, $i+1 = 2n$ for some $n \in \mathbb{N}$ and

$$\cos^{2n}\alpha - \sin^{2n}\alpha = \left(\sum_{k+j=2(n-1)} \cos^k\alpha \sin^j\alpha + \sin^{2n-2}\alpha + \cos^{2n-2}\alpha\right)(\cos^2\alpha - \sin^2\alpha).$$

Then the zeros of the function $\varphi_{i12}(\alpha)$ are $\pm\frac{\pi}{4}$ in the established interval. $\square$

Lemma A.1.8. The function $\varphi_{i22} \equiv 0$ if i is odd.

Proof. If $i = 1$ then $\varphi_{122}(\alpha) = 0$. If i is odd, $i > 1$, applying (A.6) to get

$$\begin{aligned}
\varphi_{i22}(\alpha) = \quad & \frac{\sin^3\left(\alpha+\frac{\pi}{2}\right)}{2m+3}\left(\cos^{2m}\left(\alpha+\tfrac{\pi}{2}\right) + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}\left(\alpha+\frac{\pi}{2}\right)\right) \\
& + \frac{\sin^3(\alpha+\frac{3\pi}{2})}{2m+3}\left(\cos^{2m}\left(\alpha+\tfrac{3\pi}{2}\right) + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}\left(\alpha+\frac{3\pi}{2}\right)\right) \\
& - \frac{\sin^3\alpha}{2m+3}\left(\cos^{2m}\alpha + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}\alpha\right) \\
& - \frac{\sin^3(\alpha+\pi)}{2m+3}\left(\cos^{2m}(\alpha+\pi) + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}(\alpha+\pi)\right),
\end{aligned}$$

where $A_{m,j}$ is given in (A.7).

Using trigonometric relations we have

$$\begin{aligned}
\varphi_{i22}(\alpha) = & \frac{\cos^3\alpha}{2m+3}\left(\sin^{2m}\alpha + \sum_{j=1}^{m} A_{m,j}\sin^{2m-2j}\alpha\right) - \frac{\sin^3\alpha}{2m+3}\left(\cos^{2m}\alpha + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}\alpha\right) \\
& - \frac{\cos^3\alpha}{2m+3}\left(\sin^{2m}\alpha + \sum_{j=1}^{m} A_{m,j}\sin^{2m-2j}\alpha\right) + \frac{\sin^3\alpha}{2m+3}\left(\cos^{2m}\alpha + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}\alpha\right) = 0.
\end{aligned}$$

□

Lemma A.1.9. The function φ_{i32} is identically null if i is even. Otherwise its zeros are $\alpha = \pm\frac{\pi}{4}$.

Proof. From straightforward calculations we conclude that $\varphi_{i32}(\alpha) \equiv 0$ for any even i. Otherwise

$$\varphi_{i32}(\alpha) = \frac{2}{i+3}(\sin^{i+3}\alpha - \cos^{i+3}\alpha) + \frac{2}{i+1}(-\sin^{i+1}\alpha + \cos^{i+1}\alpha).$$

Besides $\alpha = \pm\frac{\pi}{4}$ are roots of the function $\varphi_{i32}(\alpha)$ when i is odd. To show that these are the unique roots in the established interval we study the sign of its derivative

$$\varphi'_{i32}(\alpha) = -2(\cos^i\alpha\sin^3\alpha + \cos^3\alpha\sin^i\alpha).$$

Note that for $\alpha \in (-\frac{\pi}{2}, 0)$, $\varphi'_{i32}(\alpha) > 0$ and hence $\varphi_{i32}(\alpha)$ is strictly increasing. For $\alpha \in (0, \frac{\pi}{2})$, $\varphi'_{i32}(\alpha) < 0$ and hence $\varphi_{i32}(\alpha)$ is strictly decreasing. So the unique roots of $\varphi_{i32}(\alpha)$ in the established interval are $\alpha = \pm\frac{\pi}{4}$. □

Proof of Theorem A.1.1 (ii). It follows from Lemmas A.1.7 – A.1.9 that the averaged function is given by

$$\begin{aligned} f(r) &= \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} A_i(\alpha)(a_i - d_i)r^i + \sum_{\substack{i=0 \\ i \text{ even}}}^{n_2} (B_i(\alpha)b_i + \overline{B}_i(\alpha)e_i)r^{i+1} \\ &+ \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} C_i(\alpha)(c_i - m_i)r^{i+2} + (E^1(\alpha)d_0^1 + E^2(\alpha)d_0^2)r^3, \end{aligned}$$

where $A_i(\alpha)$ and $C_i(\alpha)$ are not zero except when $\alpha = \pm\frac{\pi}{4}$ and $B_i(\alpha), \overline{B}_i(\alpha)$ are not zero for any α.

Then the averaged function $f(r)$ is a polynomial of degree m where m is given by

1) $m = max\left\{2\left[\frac{n_1-1}{2}\right]+1, 2\left[\frac{n_2}{2}\right]+1, 2\left[\frac{n_3+1}{2}\right]+1, 3\right\}$, if $\alpha \notin \{-\frac{\pi}{4}, \frac{\pi}{4}\}$;

2) $m = max\left\{2\left[\frac{n_2}{2}\right]+1, 3\right\}$ otherwise.

Moreover if $\alpha \neq \pm\frac{\pi}{4}$ the averaged function is an odd function hence it has at most $\frac{m-1}{2}$ positive roots. From the averaging theory it follows item (ii) of Theorem A.1.1.

Moreover it is not difficult to verify that the averaged function has independent coefficients (the coefficients are polynomial functions in the coefficients of the perturbed system). Therefore the upper bound provides in statement (ii) can be reached. □

A.1.3 Proof of item (iii)(case $l = 3$).

In this case $\Sigma = \{(x,y) : (y-\tan(\alpha)x)(y-\tan(\alpha+\dfrac{2\pi}{3})x)(y-\tan(\alpha+\dfrac{4\pi}{3})x) = 0\}$, with $\alpha \in (-\pi/3, \pi/3)$.

Lemma A.1.10. Concerning function φ_{i13} we have

(i) $\varphi'_{i13}(\alpha) < 0$ in $(0, \frac{\pi}{3})$.

(ii) $\varphi_{i13}(\alpha)$ is an even function.

(iii) $\varphi_{i13}(\alpha)$ is $2\pi/3$–periodic.

(iv) $\varphi_{i13}(\alpha) \neq 0$, if $\alpha \neq \pm\frac{\pi}{6}$ and $i \neq 0$ is even.

Proof. From the definition we get that

$$\begin{aligned}
\varphi_{i13}(\alpha) &= \frac{(-1)^i\cos^{i+1}\alpha + \cos^{i+1}\alpha - (-1)^{i+1}\sin^{i+1}(\alpha-\frac{\pi}{6})}{i+1} \\
&+ \frac{\sin^{i+1}(\alpha-\frac{\pi}{6}) + (-1)^{i+1}\sin^{i+1}(\alpha+\frac{\pi}{6}) - \sin^{i+1}(\alpha+\frac{\pi}{6})}{i+1}.
\end{aligned}$$

Then the function $\varphi_{i13}(\alpha) \equiv 0$ if i is odd. Otherwise,

$$\varphi_{i13}(\alpha) = \frac{2}{i+1}\left(\cos^{i+1}\alpha + \sin^{i+1}(\alpha-\frac{\pi}{6}) - \sin^{i+1}(\alpha+\frac{\pi}{6})\right).$$

Therefore if $i = 0$ then $\varphi_{013}(\alpha) \equiv 0$. If $i \neq 0$ is even we take the derivative of φ_{i13} with respect to α. Using computational tools like Mathematica we get

$$\begin{aligned}
\varphi'_{i13}(\alpha) = &\; -2\cos^i\alpha\sin\alpha + 2^{-i}(\sqrt{3}\cos\alpha + \sin\alpha)(\sqrt{3}\sin\alpha - \cos\alpha)^i \\
&+ 2^{-i}(-\sqrt{3}\cos\alpha + \sin\alpha)(\sqrt{3}\sin\alpha + \cos\alpha)^i.
\end{aligned}$$

If $\alpha \in (0, \frac{\pi}{3})$ we get $\cos^i\alpha\sin\alpha > 0$ and $-\sqrt{3}\cos\alpha + \sin\alpha < 0$ then

$$
\begin{aligned}
& (\sqrt{3}\cos\alpha + \sin\alpha)(\sqrt{3}\sin\alpha - \cos\alpha)^i + (-\sqrt{3}\cos\alpha + \sin\alpha)(\sqrt{3}\sin\alpha + \cos\alpha)^i = \\
& \sqrt{3}\cos\alpha((\sqrt{3}\sin\alpha - \cos\alpha)^i - (\sqrt{3}\sin\alpha + \cos\alpha)^i) \\
& + \sin\alpha((\sqrt{3}\sin\alpha - \cos\alpha)^i + (\sqrt{3}\sin\alpha + \cos\alpha)^i) \\
= \;& -2\sqrt{3}\cos\alpha \sum_{\substack{k=1 \\ k \text{ odd}}}^{i-1} \begin{pmatrix} i \\ k \end{pmatrix} (\sqrt{3})^k \cos^{i-k}\alpha \sin^k\alpha + 2\sin\alpha \sum_{\substack{k=0 \\ k \text{ even}}}^{i} \begin{pmatrix} i \\ k \end{pmatrix} (\sqrt{3})^k \cos^{i-k}\alpha \sin^k\alpha \\
< \;& -2\sqrt{3}\cos\alpha(\sqrt{3}\sin\alpha + \cos\alpha)^i + 2\sin\alpha(\sqrt{3}\sin\alpha + \cos\alpha)^i \\
< \;& 2(-\sqrt{3}\cos\alpha + \sin\alpha)(\sqrt{3}\sin\alpha + \cos\alpha)^i < 0.
\end{aligned}
$$

Hence $\varphi'_{i13}(\alpha) < 0$ if $\alpha \in (0, \frac{\pi}{3})$, so $\varphi_{i13}(\alpha)$ is strictly decreasing in this interval and item (i) is proved.

Moreover

$$
\varphi_{i13}(-\alpha) = \frac{2}{i+1}\left(\cos^{i+1}(\alpha) - \sin^{i+1}(\alpha + \tfrac{\pi}{6}) + \sin^{i+1}(\alpha - \tfrac{\pi}{6})\right) = \varphi_{i13}(\alpha),
$$

and we conclude item (ii), i.e., $\varphi_{i13}(\alpha)$ is an even function.

Because

$$
\varphi_{i13}\left(\alpha + \frac{2\pi}{3}\right) = \frac{2}{i+1}\left(\cos^{i+1}(\alpha) - \sin^{i+1}(\alpha + \frac{\pi}{6}) + \sin^{i+1}(\alpha - \frac{\pi}{6})\right) = \varphi_{i13}(\alpha),
$$

we get that $\varphi_{i13}(\alpha)$ is $\frac{2\pi}{3}$-periodic.

Finally if $i \neq 0$ is even then $\varphi_{i13}(0) = \dfrac{2^{1-i}}{i+1}(2^i - 1) \neq 0$. See the graphic of $\varphi_{i13}(\alpha)$, for i even, $i \neq 0$ in Figure 27.

As φ_{i13} is strictly decreasing in $(0, \frac{\pi}{3})$ and $\varphi_{i13}(\frac{\pi}{6}) = 0$ it follows that $\frac{\pi}{6}$ is the unique root of φ_{i13} in $(0, \frac{\pi}{3})$. Analogously φ_{i13} is strictly increasing in $(-\frac{\pi}{3}, 0)$ and $-\frac{\pi}{6}$ is the unique root in this interval. □

Lemma A.1.11. If i is odd and $\alpha \notin \{0, \pm\frac{\pi}{3}\}$ then the function φ_{i23} does not vanish.

Proof. If $i = 1$ $\varphi_{123}(\alpha) = \frac{1}{2}\sin(3\alpha)$. If $i > 1$ we write $i = 2m + 1$ for some $m \in \mathbb{N}\backslash\{0\}$ and use (A.6) to get

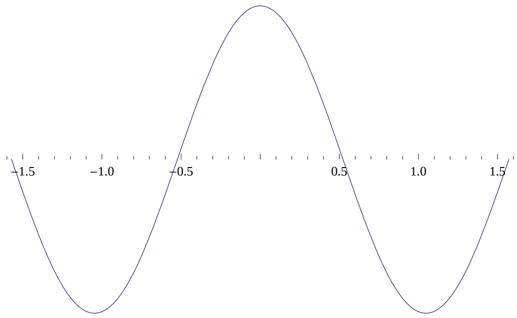

Figure 27 – Graphic of $\varphi_{i13}(\alpha)$, for i even

$$\varphi_{i23}(\alpha) = 2\Bigg[\frac{\sin^3(\alpha+\frac{\pi}{3})}{2m+3}\Bigg(\cos^{2m}(\alpha+\tfrac{\pi}{3}) + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}(\alpha+\frac{\pi}{3})\Bigg)$$
$$+\frac{\sin^3(\alpha+\frac{5\pi}{3})}{2m+3}\Bigg(\cos^{2m}(\alpha+\tfrac{5\pi}{3}) + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}(\alpha+\frac{5\pi}{3})\Bigg)$$
$$+\frac{\sin^3(\alpha+\pi)}{2m+3}\Bigg(\cos^{2m}(\alpha+\pi) + \sum_{j=1}^{m} A_{m,j}\cos^{2m-2j}(\alpha+\pi)\Bigg)\Bigg],$$

where $A_{m,j}$ is defined in (A.7).

Now using the software Mathematica we can evaluate the expressions of $\varphi_{i23}(\alpha)$ with $i = 2m+1$, for some values of m. Below we give some of them ($m = 1,2,...,7$),

(i) $\varphi_{323}(\alpha) = 1/8\sin(3\alpha)$,

(ii) $\varphi_{523}(\alpha) = -61/1.120\sin(3\alpha)$,

(iii)$\varphi_{723}(\alpha) = -31/630\sin(3\alpha) + 1/384\sin(9\alpha)$,

(iv)$\varphi_{923}(\alpha) = -26.123/887.040\sin(3\alpha) + 7/1.536\sin(9\alpha)$,

(v) $\varphi_{11,23}(\alpha) = (1/92.252.160)(-1.612.309\sin(3\alpha) + 389.550\sin(9\alpha))$,

(vi) $\varphi_{13,23}(\alpha) = (1/5.535.129.600)(-60.534.421\sin(3\alpha) + 17.493.735\sin(9\alpha) + 135.135\sin(15\alpha))$,

(vii)$\varphi_{15,23}(\alpha) = (1/376.388.812.800).(-2.715.648.724\sin(3\alpha) + 811.055.280\sin(9\alpha)$
$+ 29.864.835\sin(15\alpha))$.

From the expressions of $\varphi_{i23}(\alpha)$ for $i = 1,2,3,...$ we conclude that $\varphi_{i23}(\alpha) = a_m\sin(3\alpha) + b_m\sin(9\alpha) + c_m\sin(15\alpha) + ...$, with $a_m, b_m, c_m \in \mathbb{R}$. Therefore the roots of these functions are $0, -\frac{\pi}{3}, \frac{\pi}{3}$. □

Proceeding as in the case $l = 2$ we can state the following lemma

Lemma A.1.12. The function φ_{i33} does not vanish for $\alpha \neq \pm\frac{\pi}{6}$ and i even. If i is odd we have $\varphi_{i33} \equiv 0$.

Proof of Theorem A.1.1 (iii). From Lemmas A.1.10 – A.1.12 we conclude that, in the case $l=3$ the averaged function becomes

$$\begin{aligned} f(r) &= \sum_{\substack{i=2 \\ i \text{ even}}}^{n_1} r^i A_i(\alpha)(a_i-d_i) + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_2} r^{i+1} C_i(\alpha)(b_i-e_i) \\ &+ \sum_{\substack{i=0 \\ i \text{ even}}}^{n_2} r^{i+1}(b_i B_i(\alpha)+e_i\overline{B}_i(\alpha)) + \sum_{\substack{i=2 \\ i \text{ even}}}^{n_3} r^{i+2} D_i(\alpha)(c_i-m_i) \\ &+(E^1(\alpha)d_0^1+E^2(\alpha)d_0^2)r^3, \end{aligned}$$

where $B_i(\alpha),\overline{B}_i(\alpha)>0$ for all α, $C_i(\alpha)$ does not vanish if $\alpha \notin \{0,\pm\frac{\pi}{3}\}$ and $A_i(\alpha),D_i(\alpha)$ are different of zero if $\alpha \neq \pm\frac{\pi}{6}$.

Then $f(r)$ is a polynomial function of degree m such that $f(0)=0$ where

1) $m=\max\left\{2\left[\frac{n_1}{2}\right],2\left[\frac{n_2}{2}\right]+1,2\left[\frac{n_3+2}{2}\right],3\right\}$ if $\alpha \in \left\{0,\pm\frac{\pi}{3}\right\}$;

2) $m=\max\{n_2+1,3\}$ if $\alpha \in \left\{\pm\frac{\pi}{6}\right\}$;

3) $m=\max\left\{2\left[\frac{n_1}{2}\right],n_2+1,2\left[\frac{n_3+2}{2}\right],3\right\}$, otherwise.

For $|\varepsilon|>0$ sufficiently small Theorem 3.2.2 provides that there exist at most $m(3)$ limit cycles of system (A.3) bifurcating from the linear center, where

$$m(3)=\max\left\{2\left[\frac{n_1}{2}\right],n_2+1,2\left[\frac{n_3+2}{2}\right],3\right\}-1.$$

Again it is possible to choose the coefficients of the perturbation such that the averaged function $f(r)$ has exactly $m(3)$ positive roots. Then as discussed previously the proof of Theorem A.1.1 is concluded. □

The computations are becoming increasingly complicated as we increase the number of lines. However we also can state a general result as the following.

Theorem A.1.13. Assume that the polynomials $f_{n_1}^j(x)$, $g_{n_2}^j(x)$ and $h_{n_3}^j(x)$ have degree $n_1\geq 1$, $n_2\geq 1$ and $n_3\geq 1$ respectively, d_0^j is a nonzero constant and $l\in\mathbb{N}$, for $j=1,2$. For $|\varepsilon|>0$ sufficiently small the averaging theory of first order provides the existence of at least $m(l)$ small limit cycles of the discontinuous piecewise generalized Kukles differential system (A.3) where

$$m(l)=max\left\{\left[\frac{n_2}{2}\right],1\right\}, \text{if } l\geq 4.$$

Remark A.1.14. In the previous result we obtain a lower bound for the number of limit cycles of system (A.3) because we can not get the exact number of zeros of the averaged function (only an estimative of them).

Using Lemma A.1.2 and Remark A.1.3 we present a general expression for the averaged function (A.5) with many lines of discontinuity

$$\begin{aligned} f(r) &= \sum_{i=0}^{n_1} r^i\left[(a_i-d_i)\varphi_{i1l}(\alpha)\right] + \sum_{\substack{i=0\\ i\text{ even}}}^{n_2} r^{i+1}\left[b_i\underbrace{\varphi_{i2l}(\alpha)}_{>0}+e_i\underbrace{\overline{\varphi}_{i2l}(\alpha)}_{>0}\right] \\ &+ \sum_{\substack{i=1\\ i\text{ odd}}}^{n_2} r^{i+1}\left[(b_i-e_i)\varphi_{i2l}(\alpha)\right] + \sum_{i=0}^{n_3} r^{i+2}\left[(c_i-m_i)\varphi_{i3l}(\alpha)\right] + r^3\left[d_0^1\underbrace{\varphi_{04l}(\alpha)}_{>0}+d_0^2\underbrace{\overline{\varphi}_{04l}(\alpha)}_{>0}\right]. \end{aligned} \tag{A.8}$$

Observe that the degree m of the averaged function (A.8) is $m = \max\{n_1, n_2+1, n_3+2, 3\}$. Moreover, independently if there are or not positive numbers l or angles α such that the functions $\varphi_{i1l}(\alpha)$ and $\varphi_{i3l}(\alpha)$ vanish it is possible to choose coefficients d_i, m_i, d_0^1 and d_0^2 such that the averaged function (A.8) has at least degree $m = \max\left\{2\left[\frac{n_2}{2}\right]+1, 3\right\}$. Since in this case the averaged function $f(r)$ is an odd polynomial function without constant term the maximum number of positive roots of (A.8) is $m(l) = \max\left\{\left[\frac{n_2}{2}\right], 1\right\}$.

Proof of Theorem A.1.13 . Taking $a_i = d_i$, $i = 0, \ldots, n_1$ and $c_j = m_j$, $j = 0, \ldots, n_3$ in (A.8) we get that the number of positive roots of the averaged function depends only of n_2. It is worth to mention that $d_0^1 \neq d_0^2$ guarantee that the original system is discontinuous. Applying Theorem 3.2.2 we conclude the proof. □

A.2 Examples

In this section we illustrate Theorem A.1.1 and Theorem A.1.13 studying the existence of limit cycles for three discontinuous piecewise generalized Kukles polynomial differential systems.

Example A.2.1 (One line of discontinuity)**.** Consider $l = 1$, $\alpha = 0$ and the functions

$$\begin{aligned} f_{n_1}^1(x) &= 1-x, & f_{n_1}^2(x) &= 4+3x, \\ g_{n_2}^1(x) &= 2-x, & g_{n_2}^2(x) &= -3x+\tfrac{22}{\pi}-2, \\ h_{n_3}^1(x) &= -1+2x, & h_{n_3}^2(x) &= \tfrac{7}{2}-4x. \end{aligned}$$

Take the constants $d_0^1 = 1$ and $d_0^2 = \frac{8}{3\pi} - 1$. Under this conditions we have $n_1 = n_2 = n_3 = 1$, $m(1) = 3$ and the discontinuous piecewise system teste

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } y > 0, \\ X_2(x,y) & \text{if } y < 0, \end{cases} \tag{A.9}$$

where $X_i(x,y) = \left(\begin{array}{c} y \\ \\ -x - \varepsilon F_i(x,y) \end{array} \right)$, $i = 1,2$ and

$$F_1(x,y) = (2x-1)y^2 + (2-x)y - x + y^3 + 1,$$

and

$$F_2(x,y) \;\; = \left(\tfrac{7}{2} - 4x\right) y^2 + \left(-3x + \tfrac{22}{\pi} - 2\right) y + 3x + \left(\tfrac{8}{3\pi} - 1\right) y^3 + 4.$$

Therefore the averaged function is $f(r) = r^3 - 6r^2 + 11r - 6$, whose roots are $r = 1,2,3$. It follows from Theorem (A.1.1) that for $|\varepsilon| > 0$ sufficiently small the discontinuous differential system (A.9) has at least three limit cycles. In Figure 28 we can see a local phase portrait of system (A.9) for $\varepsilon = 1/100$ in a neighborhood of the origin. In this Figure the red lines are three concentric limity cycles that are solutions of the system passing through the points $(-2,0), (-1.5,0)$ and $(1,0)$. The dasched line is the line of discontinuity of the system.

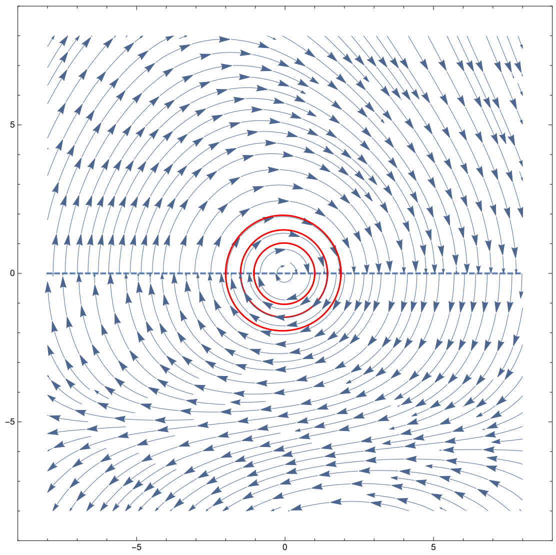

Figure 28 – Local phase portrait of system (A.9) for $\varepsilon = 1/100$ in a neighborhood of the origin. The red lines are three concentric limity cycles that are solutions of the system passing through the points $(-2,0), (-1.5,0)$ and $(1,0)$. The dashed line is the line of discontinuity of the system.

Example A.2.2 (Two lines of discontinuity)**.** Consider $l = 2$, $\alpha = 0$, and the functions

$$f^1_{n_1}(x) = \quad 4+2x-3x^2+2x^3, \qquad f^2_{n_1}(x) = 7+3x+9x^2+2x^3,$$

$$g^1_{n_2}(x) = \quad 3+5x, \qquad g^2_{n_2}(x) = -3+2x,$$

$$h^1_{n_3}(x) = \quad 9+x, \qquad h^2_{n_3}(x) = 5-x.$$

Taking the constants $d^1_0 = 1$ and $d^2_0 = -1$ we have $n_1 = 3, n_2 = n_3 = 1$ and $m(2) = 3$. Therefore the get the discontinuous system

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } xy > 0, \\ X_2(x,y) & \text{if } xy < 0, \end{cases} \tag{A.10}$$

where $X_i(x,y) = \begin{pmatrix} y \\ \\ -x-\varepsilon F_i(x,y) \end{pmatrix}$, $i = 1,2$ and

$$F_1(x,y) = 4+2x-3x^2+2x^3+(3+5x)y+(9+x)y^2+y^3,$$

and

$$F_2(x,y) = 7+3x+9x^2+2x^3+(-3+2x)y+(5-x)y^2-y^3.$$

The averaged function of this system is $f(r) = r^3 - r$, whose roots are $r = 0, 1, -1$. It follows from Theorem A.1.1 that for $|\varepsilon| > 0$ sufficiently small the discontinuous differential system (A.10) has at least one limit cycle.

Example A.2.3 (Three lines of discontinuity)**.** Consider $l = 3$, $\alpha = \frac{\pi}{4}$, the functions

$$f^1_{n_1}(x) = \quad 3+2x+x^2, \qquad f^2_{n_1}(x) = -2+7x+x^2,$$

$$g^1_{n_2}(x) = \quad 2x, \qquad g^2_{n_2}(x) = -\tfrac{12}{\pi}+(2-22\sqrt{2})x,$$

$$h^1_{n_3}(x) = \quad 1+3x+2x^2, \qquad h^2_{n_3}(x) = 1+2x+(2+8\sqrt{2})x^2,$$

and the constants $d^1_0 = d^2_0 = -\frac{8}{\pi}$. In this case $n_1 = n_3 = 2, n_2 = 1, m(3) = 4$ and the discontinuous system is

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } h(x,y) > 0, \\ X_2(x,y) & \text{if } h(x,y) < 0, \end{cases} \tag{A.11}$$

where $h_\alpha(x,y) = (-x+y)((2+\sqrt{3})x+y)((2-\sqrt{3})x+y)$, $X_i(x,y) = \begin{pmatrix} y \\ -x-\varepsilon F_i(x,y) \end{pmatrix}$, $i = 1,2$ with

$$F_1(x,y) = 3+2x+x^2+2xy+(1+3x+2x^2)y^2-\frac{8}{\pi}y^3,$$

and

$$F_2(x,y) = -2+7x+x^2+(-\frac{12}{\pi}+(2-22\sqrt{2})x)y+(1+2x+(2+8\sqrt{2})x^2)y^2-\frac{8}{\pi}y^3.$$

The averaged function is $f(r) = r^4-6r^3+11r^2-6r$, whose roots are $r = 0,1,2,3$. Hence by Theorem A.1.1 it follows that for $|\varepsilon| > 0$ sufficiently small the discontinuous differential system (A.11) has at least three limit cycles.

To illustrate Theorem A.1.13 we present the next example.

Example A.2.4. Consider the plane $\mathbb{R}^2$ divided in 14 zones ($l = 7$), $\alpha = 0$ and the functions

$$f^1_{n_1}(x) = 6x+3x^2-x^3, \qquad f^2_{n_1}(x) = 6x+3x^2-x^3,$$

$$g^1_{n_2}(x) = 3+3x+x^2+2x^3+3x^4, \qquad g^2_{n_2}(x) = -\tfrac{5}{2}+3x-\tfrac{7}{2}x^2-2x^3-2x^4,$$

$$h^1_{n_3}(x) = 1-x+2x^2, \qquad h^2_{n_3}(x) = 1-x+2x^2.$$

So $n_1 = 3, n_2 = 4$ and $n_3 = 2$. If $d^1_0 = -d^2_0 = 1$ we get the discontinuous system

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } h_\alpha(x,y) > 0, \\ X_2(x,y) & \text{if } h_\alpha(x,y) > 0, \end{cases} \tag{A.12}$$

where $X_i(x,y) = \begin{pmatrix} y \\ -x-\varepsilon F_i(x,y) \end{pmatrix}$, $i = 1,2$ and

$$F_1(x,y) = 6x+3x^2-x^3+(3+3x+x^2+2x^3+3x^4)y+(1+3x+2x^2)y^2+y^3,$$

and

$$F_2(x,y) = 6x+3x^2-x^3+(-\frac{5}{2}+3x-\frac{7}{2}x^2-2x^3-2x^4)y+(1-x+2x^2)y^2-y^3.$$

The averaged function becomes $f(r) = \frac{\pi}{16}(r^5-5r^3+4r)$,and their zeros are $-2,-1,0,1,2$. Hence system (A.12) has two limit cycles, guaranteed by Theorem A.1.13.

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