Study of the period function of a biparametric family of centers

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Contents

Introduction

A critical point p of a planar differential system is called *center* if it has a punctured neighbourhood that consist entirely of periodic orbits surrounding p. The largest neighbourhood with this property is called *period annulus* and henceforth it will be denoted by $\mathscr{P}$. The *period function* is the function that assigns to each periodic orbit of $\mathscr{P}$ its period. The present work is an study of the period function of an specific biparametric family of planar vector fields which has a center at the origin for all value of the parameter and has two principal motivations.

The first one is to give an introduction in the study of period functions for planar vector fields. More concretely, we will introduce general results about analytic systems in the plane and we will emphasise them for potential systems. The second motivation is that the family chosen for this study comes from a work of Y. Miyamoto and K. Yagasaki [13]. The authors in [13] proved that period function associated to the center at $x = 1$ of the differential equation

$$\ddot{x} = x - x^p$$

with $p \in \mathbb{N}$ is monotone increasing in all the parameter space. Lately, Yagasaki [18] extended this result for $p \in \mathbb{R}$ with $p > 1$ by using a monotonicity criterion of Chicone [2]. These two works motived us to study that family in a more extended way. More concretely, we will study the family of differential equations

$$\ddot{x} = x^q - x^p$$

with $p > q$ and restricting them to not take the value -1 in order to have the same expression for the potential function of the system for all the treated parameters. Moreover, for convenience we translate the center at the origin to make the computations easier so finally the potential system under consideration will be

$$X_\mu \begin{cases} \dot{x} = -y \\ \dot{y} = (1+x)^p - (1+x)^q \end{cases} , \quad \mu = (q,p), \ p > q$$

with $p, q \neq -1$. Notice that the family X_μ is a generalization of the one studied by Miyamoto and Yagasaki, which corresponds to $q = 1$.

In the first chapter, we will introduce all the general tools that we will use in following chapters for study the period function. We will present the main background needed to follow this work and also we will introduce in a more concrete way the family of systems under consideration. Particularly, we will show an original result on the computation of the commutator vector field in a neighbourhood of the center for potential systems that have an isochronous at the origin (Proposition 1.2).

The second chapter is dedicated to study the monotonicity of the period function. We will use the well known Schaaf's criterion [16] in order to prove that the period function of the family X_μ is monotone increasing in some regions and monotone decreasing in others (Theorem 2.2). Particularly, the result of Yagasaki in [18] is included in that proof.

In the third chapter, we will present the notion of period constants and the relation with the isochronous centers, the centers such that all the periodic orbits have the same period. We will also introduce the bifurcation theory to study the appearance of critical periods for perturbations of the system in the family. There will be two different kind of bifurcation that we will study: bifurcations from the center and bifurcations from the period annulus. In the case of bifurcations from the origin itself, we will prove a result that ensures that at most one critical period bifurcates from the center (Propositions 3.2 and 3.3). On the other hand, the result for bifurcations from the period annulus also will give us an upper bound of one critical period that bifurcates from the isochrones. Moreover, the upper bound will be taken if we perturb the system in a specific way (Theorem 3.7).

The fourth chapter is dedicated to study the period function in the outer boundary of the period annulus. The study near the outer boundary is generally much more difficult than the one in the center. In this chapter we will present a general result for the value of the extension at the outer boundary of the period function in potential systems (Theorems 4.1 and 4.2) and we will apply it in the family under consideration (Theorem 4.3).

In the final chapter, we will sum up the main dynamical results about the period function of the family and we will give a conjecture for the whole dynamic behaviour of the period function.