Numerical Computation of Invariant Objects with Wavelets

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Motivation

We are interested in approximate, via expansions of a truncated base of wavelets, *complicated objects* semianalitically. From such approximation, we want to predict and understand changes in the geometry or dynamical properties (among others) of such objects.

As a testing ground of our developed techniques, we will be focused on skew products of the form

$$\mathfrak{F}_{\sigma,\varepsilon}\begin{pmatrix}\theta_n \\ x_n\end{pmatrix} = \begin{cases}\theta_{n+1} & = R_\omega(\theta_n) = \theta_n + \omega \pmod 1, \\ x_{n+1} & = F_{\sigma,\varepsilon}(\theta_n, x_n),\end{cases} \tag{1}$$

here $x \in \mathbb{R}^+, \theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

The [GOPY]-Keller model: a testing ground

In the System (1), we take $F_{\sigma,\varepsilon}(\theta, x) = f_\sigma(x) g_\varepsilon(\theta)$ (*multiplicative forcing*) with

1. $f_\sigma : [0,\infty) \longrightarrow [0,\infty) \in \mathcal{C}^1$, bounded, strictly increasing, strictly concave and verifying $f(0) = 0$.
2. $g_\varepsilon : \mathbb{S}^1 \longrightarrow [0,\infty)$ bounded and continuous.

Fixing ideas, we will use $\omega = \frac{\sqrt{5}-1}{2}$ and the following one-parameter family of skew products (with $x \equiv 0$ invariant)

$$\mathfrak{F}_{\sigma,\varepsilon(\sigma)} \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} = \begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = 2\sigma \tanh(x_n)(\overbrace{\varepsilon}^{\varepsilon(\sigma)} + |\cos(2\pi\theta_n)|), \end{cases} \tag{2}$$

where

$$\varepsilon(\sigma) = \begin{cases} (\sigma - 1.5)^2 & \text{when } 1.5 < \sigma \le 2, \\ 0 & \text{when } 1 < \sigma \le 1.5. \end{cases}$$

The toy model is similar to the [GOPY] model.

[GOPY] Grebogi, Celso *et al.*, *Strange attractors that are not chaotic*, Phys. D 13 1984 1–2 261–268.

The [GOPY]-Keller model: a testing ground

In this testing ground we want to approximate the attractor, φ, of the above system (if it exists).

Pinching condition $\Rightarrow$ SNA's creation

When $g_\varepsilon = 0$ at some point it is called the *pinched case*, whereas if g_ε is strictly positive it is called the *non-pinched case*.

In the pinched case, any $\mathfrak{F}_{\sigma,\varepsilon}$–invariant set has to be 0 *on a point* and, hence, *on a dense set* (in fact on a *residual set*). This is because the circle $x \equiv 0$ is invariant and the θ-projection of every invariant object must be invariant under R_ω.

Our main goal: work with wavelet approximations

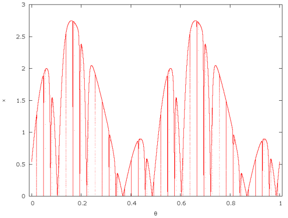
Compute φ in terms of wavelet coefficients to recover the appearance of the residual set from such coefficients.

The [GOPY]-Keller model: a testing ground

In the next slide will appear a theorem due to Keller [Kel] that makes the above informal ideas rigorous. Before stating it we need to introduce the constant σ:

Since the line $x = 0$ is invariant, by Birkhoff Ergodic Theorem, the vertical Lyapunov exponent on the circle $x \equiv 0$ is the logarithm of

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log g_\varepsilon(\theta) d\theta \right) < \infty.$$



A particular instance of the Keller-GOPY attractor

The parameterization $\varepsilon(\sigma)$ *controls* the Lyapunov Exponent and the pinched point at the *same time*.

Keller's Theorem (shortened)

There exists an upper semicontinuous map $\varphi\colon \mathbb{S}^1 \longrightarrow [0,\infty)$ whose graph is invariant under the Model (2). Moreover,

1. if $\sigma > 1$ and $g_\varepsilon(\theta_0) = 0$ for some θ_0 then the set $\{\theta\colon \varphi(\theta) > 0\}$ has full Lebesgue measure and the set $\{\theta\colon \varphi(\theta) = 0\}$ is residual,
2. if $\sigma > 1$ and $g_\varepsilon > 0$ then φ is positive and continuous; if g_ε is $\mathcal{C}^1$ then so is φ,
3. if $\sigma \neq 1$ then $|x_n - \varphi(\theta_n)| \to 0$ exponentially fast for almost every θ and every $x > 0$.

[Kel] Keller, Gerhard, *A note on strange nonchaotic attractors*, Fund. Math. 151 1996 2 139–148.

On the use of wavelets

Notice that the invariant objects that we want to compute are expressed as graphs of functions (from $\mathbb{S}^1$ to $\mathbb{R}$).

The standard approach to compute with such objects is to use finite Fourier approximations to get expansions as:

$$\varphi \sim a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\theta) + b_n \sin(n\theta) \right).$$

Since the topology and geometry of these objects is extremely complicate, the regularity and periodicity of the Fourier basis make this approach too costly.

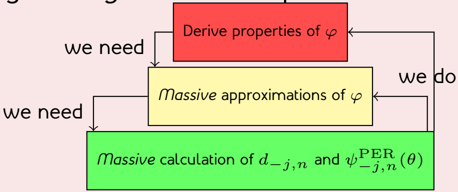
On the use of wavelets

In this case, it seems more natural to use wavelets (an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$) that adapt much better to oscillatory, irregular and highly discontinuous objects.

$$\varphi \sim a_0 + \sum_{j=0}^{N} \sum_{n=0}^{2^j-1} d_{-j,n} \psi_{-j,n}^{\mathrm{PER}}(\theta),$$

where ψ^{PER} is a given wavelet.

Summarizing: given a *generic* skew product we want to



Outline

A primer on wavelets

Let us start by the definition of Multi-resolution Analysis (MRA)

Definition

A sequence of closed subspaces of $\mathscr{L}^2(\mathbb{R})$, $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$, is a *Multi-resolution Analysis* if it satisfies:

1. $\{0\} \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \cdots \subset \mathscr{L}^2(\mathbb{R})$.
2. $\{0\} = \bigcap_{j\in\mathbb{Z}} \mathcal{V}_j$.
3. $\text{clos}\left(\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j\right) = \mathscr{L}^2(\mathbb{R})$.
4. There exists a function $\phi(x)$ whose *integer translates*, $\phi(x-n)$, form an orthonormal basis of $\mathcal{V}_0$. Such function is called the *scaling function*.
5. For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x - 2^j n) \in \mathcal{V}_j$ for each $n \in \mathbb{Z}$.
6. For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x/2) \in \mathcal{V}_{j+1}$.

A primer on wavelets

Consider the bi-indexed family of maps obtained by dilations and translations of $\phi(x)$:

$$\phi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\phi\left(\frac{x-2^j n}{2^j}\right).$$

It is shown that

1. $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{V}_j$ for each $j \in \mathbb{Z}$, and
2. $\phi(x)$ characterizes the whole MRA (see [Mal]).

[Mal] Mallat, Stéphane, *A wavelet tour of signal processing*, Academic Press Inc., San Diego, CA, 1998, xxiv+577.

A primer on wavelets

If we fix an MRA, we know that $\mathcal{V}_j \subset \mathcal{V}_{j-1}$. Then, we define the subspace $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ on $\mathcal{V}_{j-1}$. That is

$$\mathcal{V}_{j-1} = \mathcal{W}_j \oplus \mathcal{V}_j.$$

We are looking for an orthonormal basis of $\mathcal{W}_j$: the *wavelets*. This basis is given, from a function called the *mother wavelet* $\psi(x)$, by

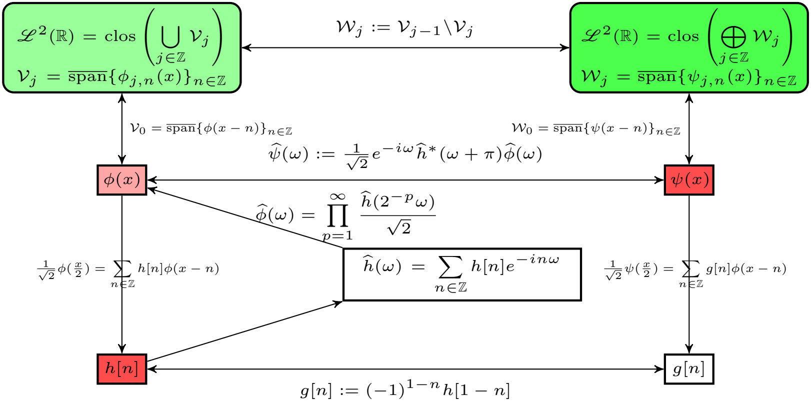
$$\psi_{j,n}(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x - 2^j n}{2^j}\right).$$

The *integer translates*, $\psi(x-n)$, of $\psi(x)$ form an orthonormal basis of $\mathcal{W}_0$. Also, $\psi(x)$ *verifies a relation with* $\phi(x)$. Moreover, from [Mal]:

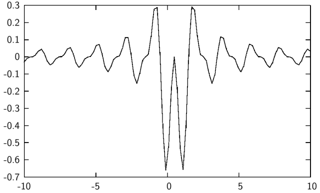
Mallat and Meyer Theorem

- For every $j \in \mathbb{Z}$ the family $\{\psi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of each $\mathcal{W}_j$,
- The wavelets $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}\times\mathbb{Z}}$ are an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$ for all $j, n \in \mathbb{Z}$.

Summarizing



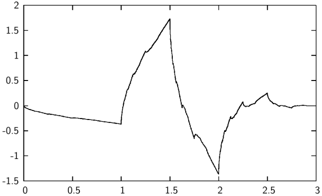
Examples of mother wavelets



Shannon wavelet (no compact support)

$$\psi(x) = \frac{\sin(2\pi(x-1/2))}{2\pi(x-1/2)} - \frac{\sin(\pi(x-1/2))}{\pi(x-1/2)}$$

$$h[n] = \begin{cases} \frac{\sqrt{2}}{2} & \text{if } n = 0, \\ \sqrt{2}\frac{-1^{(n-1)/2}}{\pi n} & \text{if } n \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

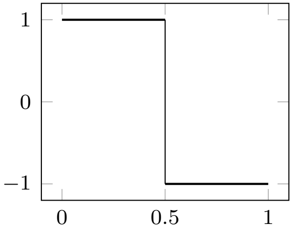


Daubechies wavelet (compact support)

No closed formula

$$h[n] = \begin{cases} 0.48296291314\ldots & \text{if } n = 0, \\ 0.83651630373\ldots & \text{if } n = 1, \\ 0.22414386804\ldots & \text{if } n = 2, \\ -0.12940952255\ldots & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Examples of mother wavelets



Haar wavelet (compact support)

$$\psi(x) := \mathbf{1}_{[0,\frac{1}{2})}(x) - \mathbf{1}_{[\frac{1}{2},1)}(x)$$

where $\mathbf{1}_{[a,b)}(x) = \begin{cases} 1 & \text{if } x \in [a,b), \\ 0 & \text{otherwise.} \end{cases}$

$$h[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is the unique Daubechies wavelet with an explicit formula.

Fixing and translating the wavelet

We will be focused on the Daubechies wavelets family. Each Daubechies wavelet minimize its support, $[1-p, p]$, constrained to the maximal number of vanishing moments, p:

$$\int_{1-p}^{p} x^k \psi(x)\, dx = 0 \text{ for } 0 \leq k < p.$$

Since our framework is $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, we transform a $\mathbb{R}$-function into a $\mathbb{S}^1$-function by setting $\psi_{j,n}^{\mathrm{PER}}$ as follows:

$$\psi_{j,n}^{\mathrm{PER}}(\theta) = \sum_{\iota\in\mathbb{Z}} \psi_{j,n}\overbrace{(\theta+\iota)}^{x\in\mathbb{R}\ :\ \mathrm{frac}(x)=\theta} = 2^{-j/2}\sum_{\iota\in\mathbb{Z}} \psi\left(\frac{\overbrace{(\theta+\iota)}^{x} - 2^j n}{2^j}\right).$$

$\psi_{j,n}^{\mathrm{PER}}$ are 1-periodic functions belonging to $\mathscr{L}^1(\mathbb{S}^1)$.

Fixing and translating the wavelet

It is known that an orthonormal basis of $\mathscr{L}^2(\mathbb{S}^1)$ is given by $\{1, \psi^{\mathrm{PER}}_{-j,n}$ with $j \geq 0$ and $n = 0, 1, \ldots, 2^j - 1\}$ provided that $\psi(x)$ is an orthonormal wavelet from a $\mathbb{R}$-MRA (see [HeWe]).

Hence, once ψ is given, we are (almost) ready to compute

$$\varphi \sim a_0 + \sum_{j=0}^{N} \sum_{n=0}^{2^j-1} d_{-j,n} \psi^{\mathrm{PER}}_{-j,n}(\theta).$$

Thus, we need to perform a *feasible strategy* to evaluate ψ^{PER} (and $\psi^{\mathrm{PER}}_{-j,n}$) at $\theta \in \mathbb{S}^1$.

[HeWe] Hernández, Eugenio and Weiss, Guido, *A first course on wavelets*, CRC Press, Boca Raton, FL, 1996, xx+489.

Computing regularities with wavelet coefficients

Theorem

Let $s \in \mathbb{R} \setminus \{0\}$ and let ψ be a mother Daubechies wavelet with more than $\max(s, 5/2 - s)$ vanishing moments. Then $f \in \mathscr{B}^s_{\infty,\infty}$ if and only if there exists $C > 0$ such that for all $j \leq 0$

$$\sup_{n\in\mathbb{Z}} |\langle f, \psi^{\mathrm{PER}}_{j,n}\rangle| \leq C2^{\tau j} \qquad \text{with} \qquad \tau = \begin{cases} s + \frac{1}{2} & \text{if } s > 0, \\ s - \frac{1}{2} & \text{if } s < 0, \end{cases}$$

In the case of Haar, [Tri02], there is an analogous result.

- **[Coh]** Cohen, Albert, *Numerical analysis of wavelet methods*, North-Holland, 2003, xviii+336.
- **[Tri01]** Triebel, Hans, *Theory of function spaces. III*, Birkhäuser Verlag, Basel, 2006, xii+426.
- **[Tri02]** Triebel, Hans, *Bases in function spaces, sampling, discrepancy, numerical integration*, European Mathematical Society, Zürich, 2010, x+296.

Computing regularities with wavelet coefficients

Corollary (Keller's Theorem)

The upper semicontinuous function $\lambda\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ whose graph is in φ, is in $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$ with $s \in (0,1]$ when $\varepsilon > 0$.

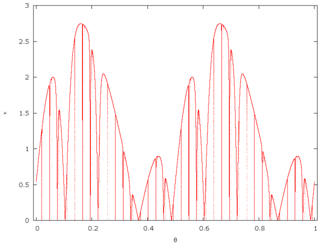
Lemma

The upper semicontinuous function $\lambda\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ whose graph is in φ, is in $\mathscr{B}^0_{\infty,\infty}(\mathbb{S}^1)$ when $\varepsilon = 0$.

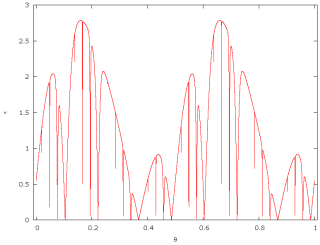
The above result justifies the use of Besov spaces instead of the Hölder ones because of the *regularity zero*.

Computing regularities with wavelet coefficients

We will use a tailored version of these results using the wavelet coefficients $d_{-j}[n]$'s.



A pinched φ of the System (2).



A quasi-pinched φ of the System (2).

To this end, we need to calculate the wavelet coefficients.

Outline

Computing coefficients using Fast Wavelet Transform

We know that given a function $f \in \mathscr{L}^2(\mathbb{R})$ and a MRA, then:

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle f, \psi_{j,n}\rangle\psi_{j,n}(x) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} d_j[n]\psi_{j,n}(x),$$

where $d_j[n] := \langle f, \psi_{j,n}\rangle$ denote the *wavelet coefficients*. But, we look for truncated wavelet approximations of f of the type:

$$f \sim \sum_{j=0}^{J}\sum_{n=0}^{2^j-1}\langle f, \psi_{-j,n}\rangle\psi_{-j,n} = \sum_{j=0}^{J}\sum_{n=0}^{2^j-1} d_{-j}[n]\psi_{-j,n}(x).$$

We use the Fast Wavelet Transform (FWT) to manage this problem.

Computing coefficients using Fast Wavelet Transform

To do so, we truncate $\mathrm{P}_{\mathcal{V}_{-J}}(f)$ to its finite dimensional version $\mathcal{V}_{-J}$ to get $f \sim \sum_{n=0}^{2^J-1} \langle f, \phi_{-J,n} \rangle \phi_{-J,n} = \sum_{n=0}^{2^J-1} a_{-J}[n]\phi_{-J,n}$ where $a_j[n] := \langle f, \phi_{j,n} \rangle$ denote the *scaling coefficients*. Therefore, using

$$\mathcal{V}_{-J} = \mathcal{V}_{-J+1} \oplus \mathcal{W}_{-J+1}:$$

$$\begin{aligned}
f &\sim \sum_{n=0}^{2^J-1} a_{-J}[n]\phi_{-J,n} \\
&= \sum_{n=0}^{2^{J-1}-1} \langle f, \phi_{-J+1,n} \rangle \phi_{-J+1,n} + \sum_{n=0}^{2^{J-1}-1} \langle f, \psi_{-J+1,n} \rangle \psi_{-J+1,n} \\
&= \sum_{n=0}^{2^{J-1}-1} a_{-J+1}[n]\phi_{-J+1,n} + \sum_{n=0}^{2^{J-1}-1} d_{-J+1}[n]\psi_{-J+1,n} \\
&= \ldots \text{ apply iteratively this decomposition } \ldots \\
&= a_0\phi_{0,0} + \sum_{j=0}^{J} \sum_{n=0}^{2^j-1} d_{-j}[n]\psi_{-j,n}(x).
\end{aligned}$$

Computing coefficients using Fast Wavelet Transform

Thus, a formula to compute the coefficients $a_{j+1}[n]$ and $d_{j+1}[n]$ from the coefficients $a_j[n]$ is needed. It is given by (see [Mal])

Mallat Theorem

Let $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ be an MRA. Then, the following recursive formulas hold.

- At the *decomposition*:

$$a_{j+1}[p] = \sum_{n\in\mathbb{N}} h[n-2p]a_j[n] \text{ and } d_{j+1}[p] = \sum_{n\in\mathbb{N}} g[n-2p]a_j[n].$$

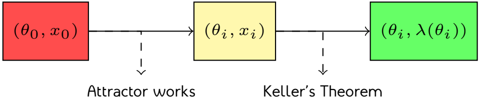
- At the *reconstruction*:

$$a_j[p] = \sum_{n\in\mathbb{N}} h[p-2n]a_{j+1}[n] + \sum_{n\in\mathbb{N}} g[p-2n]d_{j+1}[n].$$

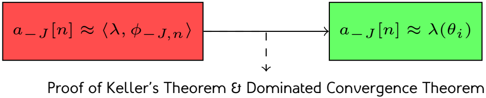
Using the FWT to compute wavelet coefficients

To compute an estimate of the Hölder exponent of the attractor, fixing $J = 30$ for the FWT, we will perform the following steps:

Step 0 Obtain a signal with



Step 1 Calculate $a_{-J}[n]$, where $0 \leq n \leq 2^J - 1$, by means of



Step 2 Compute, using the FWT, the coefficients

$$d_j[n] = \langle \lambda, \psi_{j,n} \rangle$$

where $0 \leq j \leq J$ and, for each j, $0 \leq n \leq 2^j - 1$.

Using the FWT to compute wavelet coefficients

Step 3 For $0 \leq j \leq J$, calculate

$$s_j = \log_2 \left(\sup_{0 \leq n \leq 2^j - 1} |d_j[n]| \right).$$

Step 4 Make a linear regression to estimate the slope τ of the graph of the pairs (j, s_j) with $j = 0, -1, -2, \ldots, -J$. Afterwards, use the *regularity* theorem to get s provided that the wavelet used had more than $\max(s, \frac{5}{2} - s)$ vanishing moments.

This algorithm gives an effective way of computing wavelet coefficients and regularities in a generic way.

Using the FWT to compute wavelet coefficients

Remark

1. **Step 3** and **4** justify why we need a hulking computation of wavelets coefficients. Indeed,

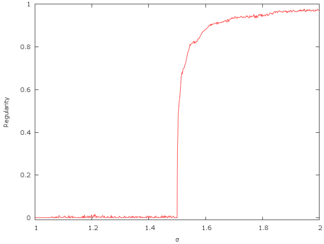
$$J \text{ samples} \Leftrightarrow 2^{J+1} \text{ coefficients.}$$

2. The points θ_i that give the attractor are, a priori, not equally spaced. This is solved by conjugating the attractor with a diffeomorphism of class $\mathcal{C}^2$ to a version of the attractor with points equally spaced and, also, sorting the signal to get the values $\lambda(\theta_i)$ in the right ordering. The conjugacy is not a problem since one can prove that the regularity of both attractors is the same using a result from [Tri03].

[Tri03] Triebel, Hans, *Theory of function spaces. II*, Birkhäuser Verlag, Basel, 1992, viii+370.

Using the FWT to compute wavelet coefficients

With these tricks, we get the following regularity graph for the one-parameter family of skew products, with $\varphi \neq 0$, given by the System (2):



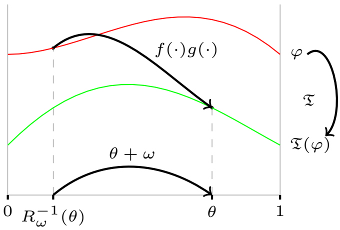
Regularity along $\varepsilon(\sigma)$.

The results are obtained by using a sample of 2^{30} points, a transient $N_0 = 10^5$ and the Daubechies Wavelet with 16 vanishing moments. We can detect in a *correct way* the regularity leap in "$\mathcal{O}(N)$".

The extremely complicate geometry of φ provokes a lack of precision in the computed regularities with $\sigma \gtrapprox 1.5$.

Computing coefficients using the Invariance Equation

The functional version of the aforesaid systems can be studied using the iteration of the *Transfer Operator*:



Let $\mathscr{P}$ be the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$. Define $\mathfrak{T}(\varphi)(\theta)$ as:

$$\varphi \mapsto f_\sigma(\varphi(R_\omega^{-1}(\theta))) \cdot g_\varepsilon(R_\omega^{-1}(\theta)).$$

The graph of a function $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$ is invariant for the System (2) if and only if φ is a fixed *point* of $\mathfrak{T}$. That is:

$$f_\sigma(\varphi(R_\omega^{-1}(\theta))) \cdot g_\varepsilon(R_\omega^{-1}(\theta)) = \mathfrak{T}(\varphi)(\theta) = \varphi(\theta).$$

Which is the *Invariance Equation*: $f_\sigma(\varphi(\theta)) \cdot g_\varepsilon(\theta) = \varphi(R_\omega(\theta))$.

Computing coefficients using the Invariance Equation

To solve the above functional equation we write the attractor as

$$\varphi(\theta) = \phi_{0,0} + \sum_{j=0}^{J} \sum_{n=0}^{2^j-1} d_{-j}[n]\psi_{-j,n}^{\mathrm{PER}}(\theta) = d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta)$$

where the coefficients d_0 and d_ℓ are the unknowns. Setting $\ell = \ell(j,n) = 2^j + n$, we have collected them in a vector $\mathrm{D}^{\mathrm{PER}}$:

$$\mathrm{D}^{\mathrm{PER}} := (\phi_{0,0}, d_0[0], \ldots, d_{-J}[2^J - 1]) = (d_0, d_1, \ldots, d_\ell).$$

As usual we plug this expression into the Invariance Equation:

$$d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta)) = f_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta)\right) \cdot g_\varepsilon(\theta).$$

Computing coefficients using the Invariance Equation

To compute it, we discretize the variable θ into N dyadic points $\theta_i = \frac{i}{N} \in \mathbb{S}^1$ for $i = 0, 1, \ldots, N-1$ and we impose that the Invariance Equation is verified on such θ_i:

$$\underbrace{d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i)) - f_\sigma \left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i) \right) \cdot g_\varepsilon(\theta_i)}_{\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})_i} = 0.$$

Thus, we get a non-linear system of N equations with N unknowns. To work and compute with $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})$, we need to define the following $N \times N$ matrices:

- Ψ whose columns are $\psi_\ell^{\mathrm{PER}}(\theta_i)$,
- Ψ_R whose columns are $\psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i))$.

The matrix Ψ (and Ψ_R)

A *generic* matrix Ψ (and Ψ_R) has this shape:

1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$
1	$\psi^{\text{PER}}_{0,0}$	$\psi^{\text{PER}}_{1,0}$	$\psi^{\text{PER}}_{1,1}$	$\psi^{\text{PER}}_{2,0}$	$\psi^{\text{PER}}_{2,1}$	$\psi^{\text{PER}}_{2,2}$	$\psi^{\text{PER}}_{2,3}$	$\psi^{\text{PER}}_{3,0}$	$\psi^{\text{PER}}_{3,1}$	$\psi^{\text{PER}}_{3,2}$	$\psi^{\text{PER}}_{3,3}$	$\psi^{\text{PER}}_{3,4}$	$\psi^{\text{PER}}_{3,5}$	$\psi^{\text{PER}}_{3,6}$	$\psi^{\text{PER}}_{3,7}$

For Ψ_R, the rows are given by $R_\omega(\theta_i) = \theta_i + \omega \pmod 1$.

Computing coefficients using the Invariance Equation

Each component of the vector of $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})$ is

$$\underbrace{\overbrace{d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i))}^{i-\text{th component of } \Psi_R \mathrm{D}^{\mathrm{PER}}} - \overbrace{f_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i)}^{\mathrm{B}}}_{\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})_i}.$$

Defining B as the i-th component of the N-dimensional vector $\wp$, i.e $[\wp]_i = f_\sigma\left([\Psi \mathrm{D}^{\mathrm{PER}}]_i\right) \cdot g_\varepsilon(\theta_i)$, we rewrite $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})$ as:

Algebraic expression of $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})$

$$\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}) = \Psi_R \mathrm{D}^{\mathrm{PER}} - \wp.$$

Solving $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}) = 0$

We will use the Newton's Method to find $\mathrm{D}^{\mathrm{PER}}_{\star)}$ such that $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{\star)}) = 0$. That is, given a seed $\mathrm{D}^{\mathrm{PER}}_{0)}$ and a tolerance `tol`:

$$\text{Newton's Method} := \begin{cases} \text{find } \mathrm{D}^{\mathrm{PER}}_{\star)} \text{ with } |\mathrm{D}^{\mathrm{PER}}_{\star)} - \mathrm{D}^{\mathrm{PER}}_{n)}| < \texttt{tol}, \\ \text{solving } \mathbf{JF}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{n)})(\mathrm{X}) = -\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{n)}), \end{cases}$$

for the unknown $\mathrm{X} = \mathrm{D}^{\mathrm{PER}}_{n+1)} - \mathrm{D}^{\mathrm{PER}}_{n)}$.

To compute the Jacobian matrix, we need $\frac{\partial \mathbf{F}_{\sigma,\varepsilon}}{\partial d_\ell}$. To do so, recall that $\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}})_i$ is equal, for each θ_i, to

$$d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i)) - f_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i).$$

Deriving the Jacobian matrix $\mathbf{JF}_{\sigma,\varepsilon}$

$$d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i)) - f_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i).$$

Then, each entry of the Jacobian matrix, $(\mathbf{JF}_{\sigma,\varepsilon})_{i,\ell} = (\frac{\partial \mathbf{F}_{\sigma,\varepsilon}}{\partial d_\ell})_{i,\ell}$, is

$$\mathbf{JF}_{i,\ell} = \begin{cases} 1 - f'_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i) & \text{if } \ell = 0, \\ \psi_\ell^{\mathrm{PER}}(R_\omega(\theta_i)) - f'_\sigma\left(d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta_i)\right) \cdot g_\varepsilon(\theta_i) \cdot \psi_\ell^{\mathrm{PER}}(\theta_i) & \text{if } \ell \neq 1. \end{cases}$$

In the same way as before, define the following $N \times N$ matrix:

- $\Delta_{\sigma,\varepsilon}$ whose entries are the vector $\frac{\partial F_{\sigma,\varepsilon}}{\partial x} = f'_\sigma([\Psi \mathrm{D}^{\mathrm{PER}}]_i) g_\varepsilon(\theta_i)$.

Compact version of $\mathbf{JF}_{\sigma,\varepsilon} \Rightarrow \Psi$ and Ψ_R computed once

In view of that, we can rephrase $\mathbf{JF}_{\sigma,\varepsilon}$ as $\Psi_R - \Delta_{\sigma,\varepsilon}\Psi$. That is, at each Newton iterate we have to solve

$$-\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{n)}) = \mathbf{JF}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{n)})(\mathrm{X}) = (\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathrm{X} = b.$$

The seed and the linear system from Newton's method

Using the Trapezoidal rule

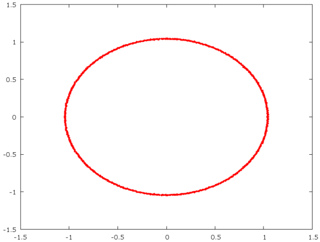
$$d_\ell = \int_{\mathbb{S}^1} \psi_\ell^{\mathrm{PER}} \varphi \, d\theta \approx \frac{1}{N} \sum_{i=0}^{N-1} \psi_\ell^{\mathrm{PER}}(\theta_i) \varphi(\theta_i),$$

one has

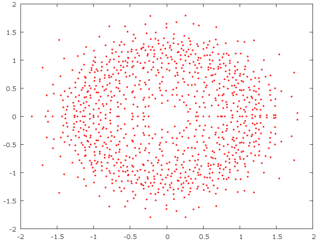
$$\mathrm{D}_{0)}^{\mathrm{PER}} := \Psi^\top \big(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1})\big)^\top .$$

We have to solve (many times) the system $(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathrm{X} = b$.

The linear system ($N \times N$) is big and difficult to solve *naively*:



Eigenvalues for a non-pinched case.



Eigenvalues for a quasi-pinched case.

When the matrix Ψ generates Ψ_R

An example of Haar matrix Ψ (which is orthogonal) is:

$$\Psi = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix}.$$

It is defined by taking $t = i - ns$, where $s = 2^{J-j}$, and

$$\psi_{j,n}(i/N) = \begin{cases} \frac{1}{\sqrt{N}} 2^{-j/2} & \text{for } 0 \leq t < s/2, \\ -\frac{1}{\sqrt{N}} 2^{-j/2} & \text{for } s/2 \leq t < s, \\ 0 & \text{if } t \geq 0. \end{cases}$$

Lemma

Set $r = \lfloor \omega N \rfloor$ and let $P = (p_{i,j})$ be the permutation matrix such that $p_{i,j} = 1$ if and only if $j = i + r \pmod{N}$. Then,

$$\boxed{\Psi_R = P\Psi} \quad \Rightarrow \Psi\Psi_R^\top = P^\top \text{ and } \Psi_R\Psi_R^\top = \mathrm{Id}.$$

Using Haar to solve the Invariance Equation

We have to solve (many times) the system $(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\mathrm{X} = b$. Recall that a *right precondition strategy* is to *solve firstly* $A\mathbf{P}y = b$ and, *after*, calculate $\mathbf{P}^{-1}x = y$ to get the solution x.

In the case of Haar, $\mathrm{X} = \Psi_R^\top y$, the *initial* system becomes $(\Psi_R - \Delta_{\sigma,\varepsilon}\Psi)\Psi_R^\top y = (\mathrm{Id} - \Delta_{\sigma,\varepsilon}P^\top)y = b$. And the matrix *is*:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon \\
f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & f'_\sigma g_\varepsilon & 0 & 0 & 1
\end{pmatrix}
$$

By performing Gauss Method formally on the system we obtain an explicit recurrence that solves the system in $\mathcal{O}(N)$ time.

A bootstrap on efficiency

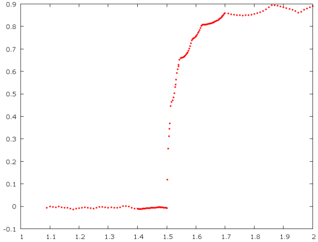
The previous change of variables suggest that we should do this change permanently and always work with the *rotated wavelet coefficients* defined as $\boxed{c = \Psi_R \mathrm{D}^{\mathrm{PER}}}$

Simplifying consequences

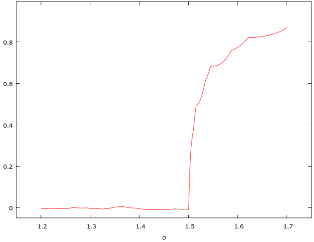
1. Since $\mathrm{D}^{\mathrm{PER}} = \Psi_R^\top c$, then $\Psi \mathrm{D}^{\mathrm{PER}} = \Psi \Psi_R^\top c = P^\top c$. (reconstruction)
2. $\left[\Psi_R \mathrm{D}^{\mathrm{PER}}\right]_i - f\left(\left[\Psi \mathrm{D}^{\mathrm{PER}}\right]_i\right) \cdot g(\theta_i) = 0$, is equivalent to $c_i - f\left(\left[P^\top c\right]_i\right) \cdot g(\theta_i) = 0$. (evaluation of the Invariance Equation)
3. Since $\mathrm{D}^{\mathrm{PER}}_{0)} := \Psi^\top(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1}))^\top$ and $\Psi_R \Psi^\top = (\Psi \Psi_R^\top)^\top = (P^\top)^\top = P$ then define $c_{0)} := P(\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_{N-1}))^\top$. (rotated seed)

Using Haar to compute wavelet coefficients

Despite of the huge linear system to solve, as in FWT case, we can detect the pinched point in "$\mathcal{O}(N)$ time". Indeed, the system is huge, because we are solving a $N \times N$ system of equations. But, for $N = 2^{26}$ each Newton iterate takes less than 10 secs.



Regularity along $\varepsilon(\sigma)$.



Zoom around 1.5 the pinched point.

Because Haar it is not a basis of $\mathscr{B}^s_{\infty,\infty}$ (for $s > 0$), we need other Daubechies wavelets.

Using Daubechies to solve the Invariance Equation

We have to solve $(\Psi_R - \mathit{\Delta}_{\sigma,\varepsilon}\Psi)\mathrm{X} = b$, where $b = -\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D}^{\mathrm{PER}}_{n)})$. Applying $\mathrm{X} = \Psi_R^\top y$ does not work because $\Psi_R \neq P\Psi$. However, recall that *left precondition strategy* is to solve $\mathbf{P}Ax = \mathbf{P}b$. We will use $\Psi_R^\top = \mathbf{P}$ because $\Psi_R^\top(\Psi_R - \mathit{\Delta}_{\sigma,\varepsilon}\Psi) \simeq \mathrm{Id} - \Psi_R^\top \mathit{\Delta}_{\sigma,\varepsilon}\Psi$.

To do so, since $N \times N$ is huge, we will compute *massively* $\psi^{\mathrm{PER}}_{j,n}(\theta_i)$. *Massively* because for each $\theta_i = \frac{i}{N}$, $j = 0, \ldots, J$ and n (also for $R_\omega(\theta_i)$):

$$\psi^{\mathrm{PER}}_{j,n}(\theta_i) = 2^{-j/2} \sum_{\iota \in \mathbb{Z}} \psi\left(\frac{(\theta_i + \iota) - 2^j n}{2^j}\right).$$

To calculate it, set u to be a $2p-1$ dimensional vector whose entries are $u_i(\theta) = (-1)^{1-\mathrm{floor}(2\theta)} h[i + 1 - \mathrm{floor}(2\theta)]$ for $i = 0, \ldots,\ 2p-2$. Also, define two matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ in terms of $h[n]$.

Daubechies – Lagarias on the circle

We have adapted the $\mathbb{R}$-Daubechies – Lagarias algorithm to $\mathbb{S}^1$ to evaluate Daubechies wavelets with $p > 1$ vanishing moments.

Wavelet point – *long* row calculator (p vanishing moments)

Because of the compact support of ψ it follows that,

- taking $\Lambda_\theta \subset [\operatorname{ceil}(1-p-\theta), \operatorname{floor}(p-1-\theta)]$,

$$\psi^{\text{PER}}(\theta) = \sum_{\iota \in \Lambda_\theta} \lim_{k\to\infty} u(\theta+\iota)' \left(\prod_{i \in \operatorname{dyad}(\operatorname{frac}(2\theta+\iota),k)} \mathbf{M}_i \right) \frac{1}{2p-1} 1^\top.$$

For $\psi^{\text{PER}}_{j,n}(\theta)$ define $t = \operatorname{floor}(2^{-j}\theta)$, $\alpha = \operatorname{frac}(2^{-j}\theta)$ and $\tilde{\alpha} = \operatorname{ceil}(\alpha)$. To save computational efforts:

- $\aleph_\theta \subset [\max(0, 2^{-j}\iota + t + \tilde{\alpha} - p), \min(2^{-j}-1, 2^{-j}\iota + t + p - 1)]$,
- $\Lambda_\theta = \left[\operatorname{ceil}\left(\frac{1-p}{2^{-j}} - \theta\right), \operatorname{floor}\left(\frac{p-1}{2^{-j}} - \theta\right)\right]$.

[Daub] Daubechies, Ingrid, *Ten lectures on wavelets* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1992, xx+357.

[Vid] Vidakovic, Brani, *Statistical modeling by wavelets* John Wiley & Sons, Inc., New York, 1999, xiv+382.

Daubechies – Lagarias on the circle (on practice)

As a toy example, consider the following matrix Ψ where each row is a $\frac{i}{16} \in \mathbb{S}^1$, where $i = 0, \ldots, 15$ ($J = 4 \Rightarrow N = 2^4 = 16$).

$$
\begin{vmatrix}
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}} \\
1 & \psi_{0,0}^{\mathrm{PER}} & \psi_{1,0}^{\mathrm{PER}} & \psi_{1,1}^{\mathrm{PER}} & \psi_{2,0}^{\mathrm{PER}} & \psi_{2,1}^{\mathrm{PER}} & \psi_{2,2}^{\mathrm{PER}} & \psi_{2,3}^{\mathrm{PER}} & \psi_{3,0}^{\mathrm{PER}} & \psi_{3,1}^{\mathrm{PER}} & \psi_{3,2}^{\mathrm{PER}} & \psi_{3,3}^{\mathrm{PER}} & \psi_{3,4}^{\mathrm{PER}} & \psi_{3,5}^{\mathrm{PER}} & \psi_{3,6}^{\mathrm{PER}} & \psi_{3,7}^{\mathrm{PER}}
\end{vmatrix}
$$

Daubechies – Lagarias on the circle (on practice)

But, Ψ verifies relations and properties (and Ψ_R also).

The matrix is not necessarily sparse for $j \leq j_0$ — The matrix is sparse for $j > j_0$

$\times -1$

θ_0	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	$\psi^{\mathrm{PER}}_{3,0}$	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,7}$
θ_1	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	$\psi^{\mathrm{PER}}_{3,0}$	0	0	0	0	0	0	0
θ_2	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	$\psi^{\mathrm{PER}}_{3,0}$	$\psi^{\mathrm{PER}}_{3,1}$	0	0	0	0	0	0
θ_3	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	$\psi^{\mathrm{PER}}_{3,1}$	0	0	0	0	0	0
θ_4	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	$\psi^{\mathrm{PER}}_{3,1}$	$\psi^{\mathrm{PER}}_{3,2}$	0	0	0	0	0
θ_5	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	$\psi^{\mathrm{PER}}_{3,2}$	0	0	0	0	0
θ_6	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	$\psi^{\mathrm{PER}}_{3,2}$	$\psi^{\mathrm{PER}}_{3,3}$	0	0	0	0
θ_7	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	$\psi^{\mathrm{PER}}_{3,3}$	0	0	0	0
θ_8	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	$\psi^{\mathrm{PER}}_{3,3}$	$\psi^{\mathrm{PER}}_{3,4}$	0	0	0
θ_9	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	$\psi^{\mathrm{PER}}_{3,4}$	0	0	0
θ_{10}	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	$\psi^{\mathrm{PER}}_{3,4}$	$\psi^{\mathrm{PER}}_{3,5}$	0	0
θ_{11}	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,5}$	0	0
θ_{12}	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,5}$	$\psi^{\mathrm{PER}}_{3,6}$	0
θ_{13}	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,6}$	0
θ_{14}	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,6}$	$\psi^{\mathrm{PER}}_{3,7}$
θ_{15}	1	$\psi^{\mathrm{PER}}_{0,0}$	$\psi^{\mathrm{PER}}_{1,0}$	$\psi^{\mathrm{PER}}_{1,1}$	$\psi^{\mathrm{PER}}_{2,0}$	$\psi^{\mathrm{PER}}_{2,1}$	$\psi^{\mathrm{PER}}_{2,2}$	$\psi^{\mathrm{PER}}_{2,3}$	0	0	0	0	0	0	0	$\psi^{\mathrm{PER}}_{3,7}$

Daubechies – Lagarias on the circle (on practice)

As a consequence, Ψ has a *stairway* structure (and Ψ_R also).

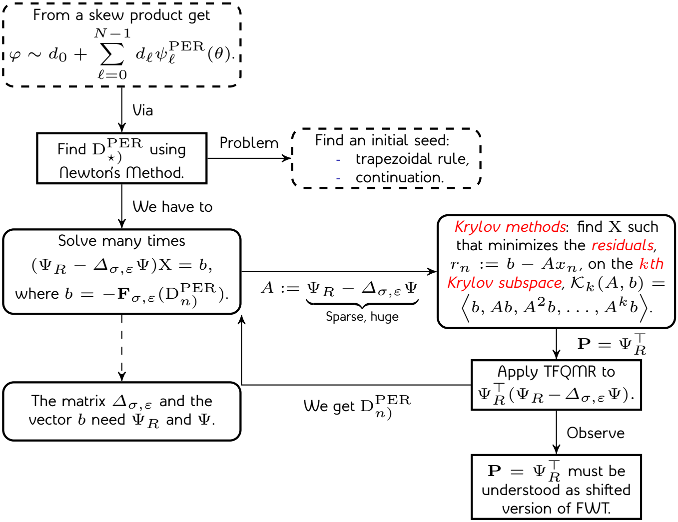
What we store for $j \leq j_0$ | What we calculate and store for $j > j_0$

What we calculate for $j \leq j_0$

1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$	$\psi^{PER}_{3,0}$	$\psi^{PER}_{3,1}$	$\psi^{PER}_{3,2}$	$\psi^{PER}_{3,3}$	$\psi^{PER}_{3,4}$	$\psi^{PER}_{3,5}$	$\psi^{PER}_{3,6}$	$\psi^{PER}_{3,7}$
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$	$\psi^{PER}_{3,0}$	$\psi^{PER}_{3,1}$	$\psi^{PER}_{3,2}$	$\psi^{PER}_{3,3}$	$\psi^{PER}_{3,4}$	$\psi^{PER}_{3,5}$	$\psi^{PER}_{3,6}$	$\psi^{PER}_{3,7}$
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$	$\psi^{PER}_{3,0}$	$\psi^{PER}_{3,1}$	$\psi^{PER}_{3,2}$	$\psi^{PER}_{3,3}$	$\psi^{PER}_{3,4}$	$\psi^{PER}_{3,5}$	$\psi^{PER}_{3,6}$	$\psi^{PER}_{3,7}$
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								
1	$\psi^{PER}_{0,0}$	$\psi^{PER}_{1,0}$	$\psi^{PER}_{1,1}$	$\psi^{PER}_{2,0}$	$\psi^{PER}_{2,1}$	$\psi^{PER}_{2,2}$	$\psi^{PER}_{2,3}$								

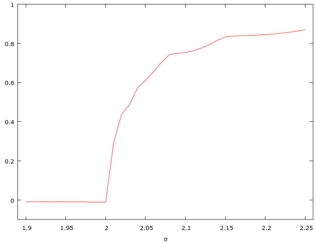
With these relations we can calculate and store Ψ and Ψ_R in a fast and feasible way.
For example $2^{24} \times 2^{24}$ spents about 9h.
Because of $\Psi_R - \Delta_{\sigma,\varepsilon}\Psi$ they are only computed once.

Using Daubechies to compute wavelet coefficients

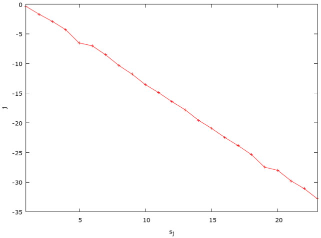


Using Daubechies to compute wavelet coefficients

With these tools we get the following regularity graph of the Keller-GOPY attractor. The results are obtained by using a sample of 2^{24} points in $\mathbb{S}^1$ and the Daubechies Wavelet with 10 vanishing moments.



The detection of the regularity leap for another parameterization.



How we compute the regularity of a particular instance of φ.

As before, we can detect the pinched point in “in $\mathcal{O}(N)$ time” and with less iterates than Haar.

Conclusions

Our aim was the study of the use of wavelets in the numerical computation of invariant objects framework. That is, give a *generic way* to get $\varphi \sim d_0 + \sum_{\ell=0}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta)$. For us, φ is a SNA.

Theoretical point of view

1. Due to the geometry and topology of φ, we have introduced and justified the use of $\mathscr{B}^s_{\infty,\infty}$ in the SNA's framework.
2. Under "Keller's assumptions", we have *classified* $\varphi \in \mathscr{B}^s_{\infty,\infty}$ and related the wavelet coefficients of φ, $\mathrm{D}^{\mathrm{PER}}$, with such classification. Moreover, such relationship it can be used, for example, when facing the fractalization route.
3. Due to the volume of calculations involved, we have introduced and justified the use of Newton's Method, Krylov methods and the FWT to calculate $\mathrm{D}^{\mathrm{PER}}$ in our framework.

Conclusions

Theoretical point of view

5. Focusing on the use of Newton's Method, we have related the use of the Trapezoidal rule with the initial seed $\mathrm{D}_{0)}^{\mathrm{PER}}$.
6. Moreover, in the Haar's case we have related λ_φ with the convergence of Newton's Method and, also, find an explicit solution of the linear system, via a permutation matrix P (and a precondition strategy).
7. Focusing on the use of the FWT, we have shown a *generic* conjugacy between two skew products. Also, we have justified that the regularity of both attractors is the same.
8. Focusing on the *initial seed* of the FWT, we have proved that we can take the orbit of a point as $a_{-J}[n]$.

Conclusions

Algorithmic point of view

1. To work and compute, we have expressed the Invariance Equation as "matrix$\times$vector". Using the same idea (and the same goals), we have *compacted* the Jacobian matrix $\mathbf{JF}_{\sigma,\varepsilon} = \Psi_R - \Delta_{\sigma,\varepsilon}\Psi$.
2. To work and compute with Ψ and Ψ_R, we have *rephrased* the Daubechies – Lagarias algorithm from $\mathbb{R}$ to $\mathbb{S}^1$. Using it and the inherited properties of the Daubechies wavelets, we have derived properties of Ψ and Ψ_R.
3. Moreover, we have found *good* precondition strategies to solve the system in a feasible way. As a consequence, we can go fast and deep. In particular, when $\psi(x)$ is the Haar wavelet, we have performed a strategy to get *the* explicit solution.
4. Focusing in the FWT performance, we have sorted a big signal of the attractor φ faster than "fast sorting algorithms" using Birkhoff's Ergodic Theorem.

Conclusions

From the Theoretical and Algorithmic conclusions:

Computational point of view

1. We have *rephrased* the Daubechies – Lagarias algorithm on a PC. Also, we have generated an independent software to work and compute with Ψ and Ψ_R on a (really big!) mesh of points of $\mathbb{S}^1$. The core of such software, besides the calculations involved, is the definition of a *particular* data structure for Ψ and Ψ_R.
2. Using the above point, we have performed a *modular* software to obtain $\mathrm{D}^{\mathrm{PER}}$ in "$\mathcal{O}(N)$ time" for a generic skew products on the cylinder (with an irrational rotation in the base). Its output, besides $\mathrm{D}^{\mathrm{PER}}$, is an estimate of the regularity of φ.