UNIVERSITAT AUTÒNOMA DE BARCELONA
Departament de Matemàtiques

PhD THESIS

Numerical Computation of Invariant Objects with Wavelets

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Introduction

The main purpose of this Thesis is to give an interface between dynamical systems and analysis topics by means of the development of a software. Hence, the framework of this Thesis is Numerical Analysis. In particular, such developed interface focuses into obtain an analytical approximation of some invariant objects. From such approximation, and due to the difficulty to make calculations in an explicit way, we want to try to assess some properties of such invariant object. This is, roughly speaking, the main topic of the present dissertation.

In the following we will develop, in a not in-depth way, the motivation and the underlying problems. To this end, let us introduce the main subject of our study: a family of pinched skew products which are defined on the Cartesian product of $\mathbb{S}^1$ and $\mathbb{R}^+ = [0, \infty)$. They are of the type

$$\begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = \mathfrak{F}_{\sigma,\varepsilon}(\theta_n, x_n) = \begin{pmatrix} R_\omega(\theta_n) \\ F_{\sigma,\varepsilon}(\theta_n, x_n) \end{pmatrix}, \tag{1}$$

where $R_\omega(\theta) = \theta + \omega$, $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $\sigma, \varepsilon \in \mathbb{R}$. Precisely, in these classes of dynamical systems invariant sets with a strange geometry appear. From now on, we will call the invariant φ, which depends on σ and ε by construction. Now, let us motivate our study.

Obtaining analytical approximation of this "*weird*" φ the use of wavelets instead of "*Fourier approach*" naturally arises. Indeed, the Fourier techniques, which are widely studied and used, tries to get expansions as:

$$\varphi \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

which are not convenient for our case. Therefore, the aim of this Thesis is to describe an efficient algorithm for the semi-analytical computation of the invariant object using wavelets. However, we want to mention that the methodology developed strongly depends on the Lyapunov exponent on φ. Recall that the Lyapunov exponent must be understood as the mean growth rate of the distance between neighboring initial point trajectories. The approximation is based on the computation of

$$\left\langle \varphi, \psi_{-j,n}^{\mathrm{PER}} \right\rangle = \int_{\mathrm{supp}(\psi_{-j,n}^{\mathrm{PER}})} \varphi(\theta)\psi_{-j,n}^{\mathrm{PER}}(\theta)\, d\theta$$

since we want to approximate a function from $\mathscr{L}^2(\mathbb{S}^1)$ by a wavelet expansion of the type

$$\varphi \sim a_0 + \sum_{j=0}^{\infty} \sum_{n=0}^{2^j-1} \left\langle \varphi, \psi_{-j,n}^{\mathrm{PER}} \right\rangle \psi_{-j,n}^{\mathrm{PER}}.$$

Observe that the unknown is, precisely, $\langle \varphi, \psi_{-j,n} \rangle$ because

$$\psi_{j,n}^{\mathrm{PER}}(x) = \sum_{\ell\in\mathbb{Z}} \psi_{j,n}(x+\ell) = 2^{-j/2} \sum_{\ell\in\mathbb{Z}} \psi\left(\frac{(x+\ell)-2^j n}{2^j}\right)$$

and $\psi(x)$ is a given wavelet. Namely, $\psi(x)$ is a function such that its dyadic translations and its dilations by powers of two form an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$.

The aim for this exercise is twofold. From one side, to study bifurcations and, perhaps, to give information of the dynamics of the object. On the other side, to estimate the regularity of φ. Notice that the study of this regularity, depending on parameters, provides another point of view to the "*fractalization routes*" as it is described, for instance, in [Nis96, JT08]. Therefore we need to calculate as well as possible the coefficients $\left\langle \varphi, \psi_{-j,n}^{\mathrm{PER}} \right\rangle$. We will compact the notation using $\mathrm{D}^{\mathrm{PER}}$ as the vector, which can be infinite dimensional, of wavelet coefficients.

Finally, we want to make a comment concerning to the wavelet coefficients $\mathrm{D}^{\mathrm{PER}}$ and the regularity pf φ. Leaving aside the classification of φ in terms of its regularity, we have used the regularity to decide, in some sense, the quality of the "*numerically obtained*" $\mathrm{D}^{\mathrm{PER}}$. Indeed, since, by construction, $\mathrm{D}^{\mathrm{PER}}$ can always restore φ then, how we can control if they are good enough? The answer is the regularity in those cases where the regularity is known.

An overview

In order to calculate the desired coefficients, let us make one step backwards. As it is described in Chapter 3 and 5 one of the main ingredients, besides of how to compute $\psi_{-j,n}^{\mathrm{PER}}(\theta)$, is the solution of a (non)-linear system of equations. Indeed, the Equation (1) can be understood as

$$\begin{array}{rccc} \mathfrak{F}_{\sigma,\varepsilon}: & \mathbb{S}^1\times\mathbb{R} & \longrightarrow & \mathbb{S}^1\times\mathbb{R} \\ & (\theta,x) & \longmapsto & (R_\omega(\theta), F_{\sigma,\varepsilon}(\theta,x)), \end{array}$$

and the invariant object, φ, is a solution of the invariance equation

$$\mathrm{Inv}(\theta) = F_{\sigma,\varepsilon}(\theta,\varphi(\theta)) - \varphi(R_\omega(\theta)) = 0.$$

An underlying subject related to the above equation is the Transfer Operator $\mathcal{M}$. Let us skip such concepts, which will be clearly explained in Chapter 3, and focus on something masked behind the invariance equation. Indeed, we turn our attention to

$$F_{\sigma,\varepsilon}(\theta,\varphi(\theta)) - \varphi(R_\omega(\theta)) = (F_{\sigma,\varepsilon}\circ\varphi)(\theta) - (\varphi\circ R_\omega)(\theta) = 0 \Leftrightarrow (F_{\sigma,\varepsilon}\circ\varphi)(\theta) = (\varphi\circ R_\omega)(\theta)$$

Working on this point of view, the invariance equation becomes similar to a cohomological equation. Precisely, the nature of the solutions of the cohomological equation, in terms of regularity, is widely studied in many cases. Moreover, it arises as a good starting point to understand the reducibility methods in a general context and systems depicted in [FR11, Jor01, HdlL06a, HdlL06b, HdlL07]. These are used to undertake actions avoiding the problematic regions, which depend on σ and ε.

We point out that the methods used along the memoir are slightly different from such techniques. Nevertheless, due to the "*simplicity*" of the environment space, $\mathbb{S}^1 \times \mathbb{R}$, the Lyapunov exponent is the link between the notion of reducibility and our point of view, as it can be seen in [JT08].

Moving to the regularity ideas, remark that the idea of the Transfer Operator described along this Thesis will have two meanings. As a first instance, the Transfer Operator must be understood as the projection on the second component of the system given by Equation (1) (see [AM08, Kel96]):

$$\mathbf{T}(\varphi)(\theta) = F_{\sigma,\varepsilon}(\theta, \varphi(R_\omega(\theta)).$$

On the other side, when one uses the reducibility language to the Transfer Operator, the Invariance equation becomes an operator:

$$\mathbf{T}(\varphi)(\theta) = \varphi(R_\omega(\theta)) - F_{\sigma,\varepsilon}(\theta, \varphi(\theta)).$$

Notice that, both are essentially the same and they are used to derive regularity properties (see e.g [dlLO99, HdlL06a, HdlL06b, HdlL07, Kel96]). The main difficulty in carrying out the regularity assessment with these techniques, depending on the parameters σ and ε, is to plug them into wavelets theory. This is, precisely, one of the main purposes of the developed software.

Zoom in

In Chapter 5 we describe the vector $\mathrm{D}^{\mathrm{PER}}$ using the invariance equation and the Transfer Operator, we find. To do so, we deal with the Newton's method. In such situation, we solve the following (non)-linear system

$$\left(\tilde{\Psi}_N^{\mathrm{PER}} - \Delta_N \Psi_N^{\mathrm{PER}}\right)(\Theta)\mathrm{D}_N^{\mathrm{PER}} = -\mathrm{Inv}(\Theta),\ \Theta \in \mathbb{S}^1\overbrace{\times \cdots \times}^{N}\mathbb{S}^1$$

for a certain N-variable functions $\Delta = \frac{\partial F_{\sigma,\varepsilon}(\theta,\varphi(\theta))}{\partial x}$ and Ψ^{PER}, related to $F_{\sigma,\varepsilon}$ and the wavelets ψ^{PER} respectively. Hence, in order to find the solution, the Newton-Kantorovich hypothesis must be verified. Having said that, we warn that we will not go further in this topic because we have applied the idea of if Newton's method converges, converges.

However, we can go a little bit in-depth in the above equation if we restrict ourselves to the Haar's case. Indeed, see Chapter 5 for an exhaustive explanation, our equation becomes

$$\left(\tilde{\Psi}_N^{\mathrm{PER}} - \Delta_N \Psi_N^{\mathrm{PER}}\right)(\Theta)\mathrm{D}_N^{\mathrm{PER}} = \left(\mathsf{Id} - \Delta_N P^\top\right)(\Theta)\mathrm{D}_N^{\mathrm{PER}} = -\mathrm{Inv}(\Theta),$$

using a permutation matrix given by

$$P_{i,j} = \begin{cases} 1 & \text{if } (j + \lfloor 2^j \omega \rfloor) \mod N, \\ 0 & \text{otherwise.} \end{cases}$$

The idea behind this change of variables is the preconditioning techniques for linear systems. Observe that $P^\top$ is a matrix whose image is the same point θ translated a certain quantity $\widetilde{\omega}$. Moreover, the permutation matrix $P^\top$, when j tends to∞, becomes *the* rotation $R_\omega(\theta)$. Indeed, for all $x \in \mathbb{R}$ it follows that $x - 1 \leq \lfloor x \rfloor \leq x$. Using such inequality it can be shown that $\lim_{j\to\infty} \frac{\lfloor 2^j \omega \rfloor}{2^j} = \omega$.That is, $P_\infty^\top = R_\omega(\theta)$. Namely, we have translated the Transfer Operator to another one more understandable. Indeed, we can find the vector $\mathrm{D}^{\mathrm{PER}}$ iteratively solving

$$\mathbf{T}_{\sigma,\varepsilon}(\mathrm{D}_k^{\mathrm{PER}}) = (\mathsf{Id} - \Delta_{\sigma,\varepsilon} \circ R_\omega)(\Theta)\mathrm{D}_k^{\mathrm{PER}} = -\mathrm{Inv}(\Theta), \tag{2}$$

with a given initial seed $\mathrm{D}_0^{\mathrm{PER}}$. Hence, the "*contraction – invertible properties*" of $\mathbf{T}_{\sigma,\varepsilon}$ will decide the convergence towards $\mathrm{D}^{\mathrm{PER}}$. Taking $\mathbf{T}_{\sigma,\varepsilon}$ as an (infinite) matrix will help us. Actually, $\mathbf{T}_{\sigma,\varepsilon}$ as a matrix will be a contractive matrix if its spectral radius, $\varsigma(\mathbf{T}_{\sigma,\varepsilon})$, is less than one. Such condition is equivalent to demand $\varsigma(\Delta_{\sigma,\varepsilon} \circ R_\omega) < 1$, and this is equivalent to have the Lyapunov exponent of the system given by Equation (1) less than zero (see Chapter 5 for more details). In view of that, it arises three "*natural*" situations:

(a) If the Lyapunov exponent is less than zero, then the operator is invertible and Equation (2) converges to $\mathrm{D}^{\mathrm{PER}}$ hopefully.

(b) If the Lyapunov exponent is close to zero, then the operator nearby not invertible and we have kernel. Hence, other techniques must be applied.

(c) If the Lyapunov exponent is positive, then system given by Equation (1) has a repellor and we do not consider such case.

In other words, what will control in which of the above cases we are will be the parameters σ and ε. Notice that it is linked to the question whether those parameters, specially ε, "*control*" the regularity properties of φ or the regularity is inherent to φ.

Zoom out

From a general point of view, wavelet coefficients determine the regularity of a function in the same way as in the Fourier world. Hence, in our case, $\mathrm{D}^{\mathrm{PER}}$ will decide if φ is on a certain regularity space. Let us denote this space by $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, where $s \in \mathbb{R}$. Roughly speaking, $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$ are the generalization of Hölder spaces (under certain constraints) for all real value of s (see e.g [Tri83, Tri06, Tri10]). This range (even negative!) seems a bit weird; however there are systems where φ it is not a graph of a (usual) function, as it can be guessed in Figure 1. On the contrary, an invariant object with a strange geometry at a first glance may become a nice object after some manipulations (see e.g [AM08, Jor14]). Thus, all range of s must be allowed and other regularity spaces may be considered.

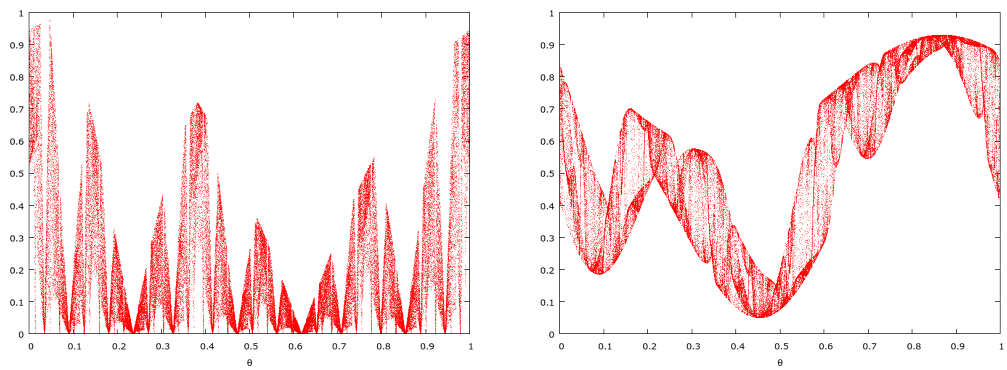

Figure 1: On the left picture it is shown an attractor *with* area [AM08]. On the right picture it is displayed the Nishikawa-Kaneko model with $\sigma = 3.0$ and $\varepsilon = 0.18$ [Nis96].

However, we will have two important and related drawbacks. The first one is linked to the three "*natural*" situations described at the end of the last paragraph. For certain values of σ and ε the Lyapunov exponent is relatively close to zero. In these cases, the vector $\mathrm{D}^{\mathrm{PER}}$ is very hard to calculate, in CPU time consuming. We use standard continuation techniques on the way of solving such problem. The other disadvantage is that "*a priori*" Equation (2) does not knows, in some sense, if $\mathrm{D}^{\mathrm{PER}}_k$ will have some nice vanishing property. That is, using this methodology is mandatory to converge towards $\mathrm{D}^{\mathrm{PER}}$ to be able to classify φ in one of these spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$.

Organization and contributions of this Thesis

This dissertation is divided into two parts. The first one (Chapters 1, 2and 3) is devoted to recall and introduce all the theory and the methodology which is used into the second part (Chapters 4, 5 and 6). In Chapter 1 a self-contained and general introduction to wavelets it is done. However, since we will use Daubechies wavelets with $p \geq 1$ vanishing moments, we will focus in such wavelets (in $\mathbb{R}$ or $\mathbb{S}^1$). Actually, along this thesis we have performed the translation from the $\mathbb{R}$-Daubechies wavelets language to $\mathbb{S}^1$. Following the generic notation, in the literature we have call it ψ^{PER}. Also, in this first Chapter a crash course on the notion of regularity, in terms of the functional spaces $\mathscr{B}^s_{2,2}(\mathbb{S}^1)$ and $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, and the relationship between them and the wavelets coefficients, $\left\langle f, \psi^{\mathrm{PER}}_{-j,n} \right\rangle$, is done.

Precisely, the main topic of Chapter 2 is the calculation of the wavelets coefficients using two different techniques; namely, Fast Wavelet Transform and the solution of (non)-linear systems of equations. In a more concrete terms, we have performed an algorithm to calculate, in an efficient and fast way, ψ^{PER} on a (really big!) mesh of points of $\mathbb{S}^1$. The method is based on the Daubechies–Lagarias algorithm (see [DL91, DL92]). Such computation, which is a key point of this disquisition, will be the main part of the Chapter 2.

Finally, the last chapters of this first part are devoted to give a short compilation of the theoretical framework where this dissertation is dealt. In Section 3.2 it is shown the machinery and also we try to characterize the mechanisms involved in the geometric properties of a particular family of quasi periodically forced skew products on the cylinder. As a matter of fact, combining them with ideas from [AM08, Har12] it is possible to extend the results of [Sta97, Sta99] to a more weird class of functions. Also we derive "*theoretically*" the regularity, in terms of $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, for φ of the Keller-GOPY model. Despite of this two remarkable facts, such effort is to justify the use of the software in other cases of SNA's, as those ones in [AM08, Nis96], among the study of them in terms of regularity.

Moving to the second part of this Thesis, two different exercises are done. The first one, following [dlLP02], is the development of an algorithm to estimate regularities, in terms of $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, for $s \in \mathbb{R}$. Such algorithm is the main goal of Chapter 4, and it is used in many situations, but not only with the Fast Wavelet Transform. Indeed, in Chapter 5 we perform the same kind of regularity assessment with a different methodology to obtain $\left\langle \varphi, \psi^{\mathrm{PER}}_{-j,n} \right\rangle$.

Certainly, in the last Chapter(s), we have focused to solve the Invariance Equation for several situations and dynamical systems. The solvers, which are iterative, are the main contribution of such Chapter(s). Due to the good properties of the Daubechies wavelet family, such as compact support or vanishing moments among others (see e.g [Dau92, HW96, Tri06]), we have derived two iterative strategies to find $\left\langle \varphi, \psi^{\mathrm{PER}}_{-j,n} \right\rangle$. Both of them are based in the same argument but, due to the simplicity of the Haar wavelet ($p = 1$), the first strategy give us a close to explicit method calculate the Haar wavelet coefficients. As mentioned, beyond the "*numerical approximation of the invariant objects*", we have estimated the regularity, in terms of $\mathscr{B}^s_{2,2}(\mathbb{S}^1)$ and $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, of a certain models of skew products using the Haar wavelet. Both goals have been repeated with other Daubechies wavelet. Nevertheless, we would like to remark that, when $p > 1$, the core of the iterative method is the aforementioned massive evaluation of ψ^{PER} and, also, the left conditioned discretization of the Transfer Operator. As an extra point, a numerical exploration of the Lyapunov exponent of a particular instance of the kind of systems in [AM08] it is done.

Loose ends of this Thesis

It is worth pointing out that in each of the chapters there are some open problems and questions which we summarize at the end as a (possible) future work. Concretely, the above strategies open the door too study the operator in terms of Newton-Kantorovich theorem. This would guarantee the stability of the iterative method. In this way, there are some open questions such as which is the limit operator, when one uses a precondition technique, and how the convergence of the iterative method is affected by the Lyapunov exponent. Of course, for a well suited norm, Newton-Kantorovich theorem must may be used to detect, "*a priori*", the lack of regularity of φ. Moving to the continuation methodology, since we have convergence, it seems useful to find, in a more general context, such strips of convergence towards the desired case of σ and ε using the operator framework.

Also, the developed software it might be updated to work on high dimensions and some qualitative-quantitative properties of φ. For example, a modification of it can produce rigorous numerical estimation of φ's traits, such as the Hausdorff Dimension or its length. Moreover, theoretical bounds on the aforesaid traits could be done using the Haar wavelet because of its simplicity. Nonetheless, as it is described in Chapter 5 and 6, the bug of the precision must be completely understood.

Contents