CRITICALITY VIA FIRST ORDER DEVELOPMENT OF THE PERIOD CONSTANTS

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Abstract. In this work we study the criticality of some planar systems of polynomial differential equations having a center for various low degrees n. To this end, we present a method which is equivalent to the use of the first non-identically zero Melnikov function in the problem of limit cycles bifurcation, but adapted to the period function. We prove that the Taylor development of this first order function can be found from the linear terms of the corresponding period constants. Later, we consider families which are isochronous centers being perturbed inside the reversible centers class, and we prove our criticality results by finding the first order Taylor developments of the period constants with respect to the perturbation parameters. In particular, we obtain that at least 22 critical periods bifurcate for $n = 6$, 37 for $n = 8$, 57 for $n = 10$, 80 for $n = 12$, 106 for $n = 14$, and 136 for $n = 16$. Up to our knowledge, these values improve the best current lower bounds.

1. Introduction

Melnikov functions are widely used on the well-known problem of limit cycles bifurcation in planar systems of differential equations, in the line of 16th Hilbert Problem. In analogy to this question, some authors have proposed an equivalent approach for studying the number of oscillations of the period function of a center, also known as critical periods. Works such as [6, 15, 35] propose this technique to deal with the lower bounds on the number of critical periods by using the equivalent to the first order Melnikov function for the period.

Another question intimately related to the periodicity of a system is the isochronicity characterization. Huygens was the forerunner of isochronicity studies and aroused the interest on this line of research, see [3]. In the last 30 years many authors have studied the existence of differential equations with equilibrium points of center type that satisfy this isochronicity property, see for example [11, 22] and the interesting survey of Chavarriga and Sabatini [5].

To deal with the aforementioned problems we will start by introducing some preliminary concepts and classical results on these topics. This introductory part is based on that of our recent work [32], so many of the ideas here presented are directly extracted from that paper.

Let us consider a real polynomial system of differential equations in the plane with a nondegenerate center at the origin, this is the linear part at the equilibrium point having zero trace and positive determinant. It is a well known fact that, by a suitable change of coordinates and time rescaling, it can be written in the form

$$\begin{cases} \dot{x} = -y + X(x,y) =: P(x,y), \\ \dot{y} = x + Y(x,y) =: Q(x,y), \end{cases} \tag{1}$$

where X and Y are polynomials of degree $n \geq 2$ which start at least with quadratic monomials. We define the *period annulus* of a center as the largest neighborhood Ω of

2010 *Mathematics Subject Classification.* Primary: 34C23, 34C07.

Key words and phrases. Bifurcation of critical periods; Criticality; Period constants; Period function; Time-reversible centers.

the origin with the property that the orbit of every point in $\Omega \setminus \{(0,0)\}$ is a simple closed curve that encloses the origin, so these trajectories are periodic. Suppose the origin is a center for system (1) and that the number $\rho^* > 0$ is so small that the segment $\Sigma = \{(x,y) : 0 < x < \rho^*, y = 0\}$ of the x-axis lies wholly within the period annulus. For ρ satisfying $0 < \rho < \rho^*$, let $T(\rho)$ denote the least period of the trajectory through $(x,y) = (\rho, 0) \in \Sigma$. The function $T(\rho)$ is the *period function* of the center, which by the Implicit Function Theorem is real analytic. Moreover, we say that the center of system (1) is isochronous if its period function $T(\rho)$ is constant, which means that every periodic orbit in a neighborhood of the origin has the same period. We will recall in this work that the period function can be written as

$$T(\rho) = 2\pi\left(1 + \sum_{k=1}^{\infty} \widehat{T}_k \rho^k\right). \tag{2}$$

According to [1] only the coefficients with even subscript actually play a role, in the sense that if for a certain k we vanish $\widehat{T}_1, \ldots, \widehat{T}_{2k}$ this implies $\widehat{T}_{2k+1} = 0$. Therefore, it is convenient to denote $T_k := \widehat{T}_{2k}$, and we will use this notation from now on. We recall that the coefficients T_k are known as the *period constants* of the center, see for example [27]. In the next section we will see how to compute them. In the case that (1) depends on some parameters, the period constants are polynomials on them ([10]). A direct consequence of (2) is that, in the considered situation, system (1) has an isochronous center at the origin if and only if $T_k = 0$ for all $k \in \mathbb{N}$. This result is also justified by Shafer and Romanovski in [27]. This shows that the period constants play the same role when studying isochronicity as Lyapunov constants when characterizing centers. Moreover, every value $\rho > 0$ for which $T'(\rho) = 0$ is called a *critical period*, and if it is a simple zero, i.e. $T''(\rho) \neq 0$, we call it a *simple* or *hyperbolic critical period*. The number of simple critical periods provides the number of oscillations of the period function. For a family of vector fields having an equilibrium point of center type, we can say that it has criticality c if the maximum number of oscillations of the period function is not higher than c. In analogy to the local cyclicity finiteness conjecture in 16th Hilbert problem ([28]) we think that, in any class of planar polynomial vector fields of degree n having a center of type (1), the number of oscillations of the period function will be uniformly bounded by a function depending only on the degree n.

About the problem of the monotonicity of the period function (2), it is usually studied in polynomial center families ([7, 31, 33]). The uniqueness of critical periods is studied for example in [13] for a class of polynomial complex centers. Recently, this uniqueness problem has also been considered for some Hamiltonian and quadratic Loud families in [26, 33]. For the quadratic family we recommend the nice work done by Chicone and Jacobs in [8]. The study of critical periods for the classical quadratic Loud family was extended to some generalized Loud's centers, see [23]. For cubics, in particular for homogenenous cubics nonlinearities, we refer the reader to [19, 29]. For more information on the period function and the criticality problem we suggest the reading of [20] and [27].

This finiteness property should also be true if we restrict our attention to the well-known *time-reversible* polynomial vector fields class. The most common symmetry is the reversibility with respect to straight lines. As the linear part of system (1) is invariant with respect to any rotation, without loss of generality we can consider only differential systems which are invariant under the change $(x, y, t) \mapsto (x, -y, -t)$. This classic reversibility makes the system have a symmetry with respect to the straight line $y = 0$. This is the convention that we will use in all the results of this work. Let us denote by $\mathcal{C}(n)$ the criticality restricted to the degree n class; as the general *criticality problem* is very difficult, we will focus on the bifurcation of *local* critical periods near the origin in this *reversible* class. We will denote by $\mathcal{C}_\ell(n)$ the maximum number of (local) critical periods that can bifurcate from the origin of an n-th degree reversible planar system. Our aim is to find the highest possible lower bound of this number for different values of the degree n. This question is considered in analogy to the (local) cyclicity problem, whose purpose is to find the maximum number of limit cycles –these are zeros of the Poincaré map– that bifurcate from an equilibrium. Observe that the concept of hyperbolic critical period is also defined in analogy to a hyperbolic limit cycle, following the idea of having multiplicity one.

The main objective of this paper is to present a method which allows to obtain high numbers of (local) critical periods with less computational effort, and to apply it to some systems. The problem of finding the maximum number of local critical periods which can bifurcate from a plane vector field is completely solved only for the quadratic case $n = 2$, for which Chicone and Jacobs proved in [8] that $\mathcal{C}_\ell(2) = 2$. For cubic and quartic systems, we proved in [32] that 6 and 10 critical periods unfold, respectively. This work also proves the unfolding of $(n^2 + n - 2)/2$ critical periods for n between 5 and 9, and $(n^2 + n - 4)/2$ for n between 10 and 16. There are also a few works dealing with lower bounds for general families of degree n. One is given by Cima, Gasull, and da Silva in [9] proving that $\mathcal{C}_\ell(n) \geq 2\,[(n-2)/2]$, where $[\cdot]$ denotes the integer part. Another one is the bound that Gasull, Liu, and Yang propose in [15], which grows as $n^2/4$. Very recently, in 2020, Cen proves in [4] a lower bound of $(n^2 - 4)/2$ for even n and $(n^2 + 2n - 5)/2$ for odd n. The main bifurcation technique uses the development (2) and usually each local oscillation is obtained from a perturbative parameter.

When system (1) is considered in the reversible class it has $(n^2+3n-4)/2$ parameters. But, as usual in this kind of bifurcation mechanisms, at least one of the monomials in the perturbation terms X, Y in (1) can be rescaled to be one, so only $(n^2 + 3n - 6)/2$ parameters actually play some role in the bifurcation. Therefore, we think that this number will be the actual value for $\mathcal{C}_\ell(n)$. In fact, it is for $n = 2$ and it coincides with the best known lower bound for it for $n = 3$. We observe that, for small values of the degree n, the detailed lower bound values above together with the new ones in this work are very close to the conjectured value. We notice that we have selected the reversible class because it is the one having the highest number of parameters, so it is bound to be the best family to have the highest number of critical periods. Finally, we remark that we have only considered one period annulus.

The method we propose to obtain lower bounds on the number of critical periods is based on the equivalence of the first Melnikov function for the period of the perturbation of an isochronous system and the linear developments with respect to the perturbation parameters of the period constants also near the same isochronous system, an idea already introduced in [18] for cyclicity and Lyapunov constants. This is our main technique and is presented in the following result.

Theorem 1. *Let $\lambda = (a_{20}, a_{11}, \ldots, b_{20}, b_{11}, \ldots) \in \mathbb{R}^{(n^2+3n-4)/2}$ be perturbative parameters such that the next polynomial perturbations of a system of differential equations in the plane of the form* (1),

$$\begin{cases} \dot{x} = -y + X_c(x,y) + \displaystyle\sum_{k+l=2}^{n} a_{kl}\, x^k y^l, \\ \dot{y} = x + Y_c(x,y) + \displaystyle\sum_{k+l=2}^{n} b_{kl}\, x^k y^l, \end{cases} \tag{3}$$

and

$$\begin{cases} \dot{x} = -y + X_c(x,y) + \varepsilon \sum\limits_{k+l=2}^{n} a_{kl}\, x^k y^l, \\ \dot{y} = x + Y_c(x,y) + \varepsilon \sum\limits_{k+l=2}^{n} b_{kl}\, x^k y^l, \end{cases} \tag{4}$$

have a center at the origin which is isochronous respectively when $\lambda = 0$ *and* $\varepsilon = 0$. *Let us denote by* $T_k^{(1)}(\lambda)$ *the first order truncation of the Taylor series, with respect to* λ, *of the period constants* $T_k(\lambda)$ *of* (3). *If we write the Taylor series in* ε *of the period function of system* (4) *as*

$$T(\rho,\lambda,\varepsilon) = 2\pi + \sum_{k=1}^{\infty} \mathcal{T}_k(\rho,\lambda)\varepsilon^k, \tag{5}$$

then, for ρ *small enough, the first averaging function* $\mathcal{T}_1(\rho,\lambda)$ *writes as*

$$\mathcal{T}_1(\rho,\lambda) = \sum_{k=1}^{N} T_k^{(1)}(\lambda)\left(1 + \sum_{j=1}^{\infty} \alpha_{kj0}\rho^j\right)\rho^{2k}, \tag{6}$$

with the Bautin ideal $\langle T_1,\ldots,T_N,\ldots\rangle = \langle T_1,\ldots,T_N\rangle$.

We notice that, by the isochronicity property of the unperturbed system, $T_k^{(1)}(0) = 0$. The utility of the above result in terms of finding a high number of critical periods lies in its following corollary.

Corollary 2. *Let us consider the* $m \times l$ *matrix* G_m *whose element in position* (i,j) *is the coefficient of the* j*th perturbative parameter in the first-order expression of the* i*th period constant of a perturbed system* (3), *so* G_m *is the matrix of coefficients of the first order truncation of the Taylor series of the first* m *period constants. If the rank of* G_m *is* N *then at least* $N-1$ *critical periods bifurcate from the origin of the center* (3) *or* (4).

Observe that the size of matrix G_m is determined by the number of considered period constants m and the number of perturbative parameters l. Using this technique we have been able to improve the lower bound of $\mathcal{C}_\ell(n)$ known so far for some even values of the degree n (see [32]), as the following theorem states.

Theorem 3. *The number of local critical periods in the family of polynomial time-reversible centers of degree* n *is at least* $\kappa(n)$, *this is* $\mathcal{C}_\ell(n) \geq \kappa(n)$, *where*

n	4	6	8	10	12	14	16
$\kappa(n)$	10	22	37	57	80	106	136

To the best of our knowledge, the highest lower bound for $\mathcal{C}_\ell(4)$ was achieved in [32] and is also 10. Observe that we do not improve this number, but we will see that here we obtain the same lower bound for the local criticality with a much simpler method both in conceptual and computational terms.

The essential tool for proving the above result is the local bifurcation of zeros of the first derivative of the period function (2). That is, for each degree n, finding the highest value for the multiplicity of a zero of T' and its unfolding in the corresponding reversible polynomial centers family. More concretely, this is done by perturbing inside the time-reversible class some isochronous centers with homogeneous polynomial nonlinearities. This approach is used together with the technique provided by Corollary 2.

This work is devoted to prove Theorems 1 and 3, and has the following structure. Section 2 introduces a few preliminaries on various topics related to the present work: reversible centers, isochronicity characterization techniques, computation of period constants, and inverse integrating factors. Section 3 provides the proofs of Theorem 1 and

Corollary 2, after giving an example which illustrates how Theorem 1 works in a particular family. Later, Section 4 characterizes some isochronous centers of 6th degree and general even degree n. Finally, these isochronous centers are used in Section 5 to show the bifurcation of critical periods which proves Theorem 3. We remark that all the computations have been done using the computer algebra system Maple ([21]).

2. Preliminaries

2.1. **Perturbed reversible families.** We introduce here how we write the families that we will use throughout the paper. Let us consider (1) in complex coordinates $(z,w) = (z,\overline{z}) = (x+\mathrm{i}\,y, x-\mathrm{i}\,y)$, which is written as

$$\begin{cases} \dot{z} = \mathrm{i}\,z + Z(z,w) =: \mathcal{Z}(z,w), \\ \dot{w} = -\,\mathrm{i}\,w + \overline{Z}(z,w) =: \overline{\mathcal{Z}}(z,w), \end{cases} \tag{7}$$

where Z is a polynomial starting with monomials of at least second degree and $\overline{Z}$ is its complex conjugate. Assume that this system has an isochronous center. Let us add a perturbation also starting with quadratic terms as in (3). Then in (z,w) coordinates it writes as

$$\begin{cases} \dot{z} = \mathcal{Z}(z,w) + \sum\limits_{l+m\geq 2}^{\nu} c_{lm} z^l w^m, \\ \dot{w} = \overline{\mathcal{Z}}(z,w) + \sum\limits_{l+m\geq 2}^{\nu} \overline{c}_{lm} z^m w^l, \end{cases} \tag{8}$$

where ν is the perturbation degree and $c_{lm} \in \mathbb{C}$ are perturbative parameters. In general, we will have perturbations such that $\nu = n$, this meaning that the perturbation degree is actually the system degree. However, in Section 5 we will consider some cases in which $\nu = n+1$, as we will justify later.

We are interested in reversible perturbations so that the center property is kept. It can be trivially proved that a perturbation of the form (8) is reversible if it satisfies $\overline{c}_{lm} = -c_{lm}$, or equivalently, it is purely imaginary and $c_{lm} = \mathrm{i}\,\varrho_{lm}$ for some $\varrho_{lm} \in \mathbb{R}$ (see [32]). Therefore, throughout this work we will deal with perturbed systems of the form

$$\begin{cases} \dot{z} = \mathcal{Z}(z,w) + \mathrm{i} \sum\limits_{l+m\geq 2}^{\nu} \varrho_{lm} z^l w^m \\ \dot{w} = \overline{\mathcal{Z}}(z,w) - \mathrm{i} \sum\limits_{l+m\geq 2}^{\nu} \varrho_{lm} z^m w^l, \end{cases} \tag{9}$$

with $\varrho_{lm} \in \mathbb{R}$, which still have a center at the origin despite the perturbation and being $\dot{z} = \mathcal{Z}(z,w)$ a planar polynomial system of degree n having an nondegenerate isochronous center at the origin.

2.2. **Isochronicity: linearizability, Lie bracket and commuting systems.** In this subsection we present three different methods which may help to check whether a center is isochronous or not. Actually, all three methods are equivalent in terms of characterizing the isochronicity of a system ([1, 5]).

We will start by justifying that the isochronicity property is equivalent to linearizability, and we will provide the linearization tools known as Darboux linearization. The observations and results introduced here are based on [27].

Let us consider the canonical linear center

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases} \tag{10}$$

which is trivially isochronous. Since isochronicity does not depend on the coordinates in use, without changing time, any system with a center which can be brought to (10) by means of an analytic change of coordinates must be isochronous. Such a change of coordinates is called a *linearization*, and in this case we say that the system is *linearizable*. From this observation the next result follows.

Theorem 4 ([27])**.** *The origin of system* (1) *is an isochronous center if and only if there is an analytic change of coordinates* $(x, y) \mapsto (x + \zeta(x, y), y + \eta(x, y))$ *that reduces* (1) *to the canonical linear center* (10)*.*

We notice that, in the above result, $\zeta(0, 0) = \eta(0, 0) = 0$ and $\nabla\zeta(0, 0) = \nabla\eta(0, 0) = 0$. This theorem tells us that the isochronicity of a planar analytic system is equivalent to its linearizability, so the linarizability of a system can be studied to prove its isochronicity. In this line, we present now one of the most efficient tools for checking linearizability, which is Darboux linearization. For $z, w \in \mathbb{C}$, a *Darboux linearization* of a polynomial system (7) is an analytic change of variables $(z, w) \mapsto (\chi(z, w), \xi(z, w))$ whose inverse linearizes (7) and is such that $\chi(z, w)$ and $\xi(z, w)$ are of the form

$$\begin{cases} \chi(z, w) = \prod_{j=0}^{m_1} f_j^{\alpha_j}(z, w) = z + \widetilde{\chi}(z, w), \\ \xi(z, w) = \prod_{j=0}^{m_2} g_j^{\beta_j}(z, w) = w + \widetilde{\xi}(z, w), \end{cases} \tag{11}$$

for some m_1, m_2, where $f_j, g_j \in \mathbb{C}[z, w]$, $\alpha_j, \beta_j \in \mathbb{C}$, and $\widetilde{\chi}(z, w)$ and $\widetilde{\xi}(z, w)$ begin with terms of order at least two. A system is *Darboux linearizable* if it admits a Darboux linearization. In our case, as the considered vector fields come from real systems the conjugacy relationship $\xi(z, w) = \overline{\chi}(z, w)$ holds, so throughout this paper we will only give the first component $\chi(z, w)$ of the provided linearizations.

We define the *Lie bracket* of two complex planar vector fields $\mathcal{Z}, \mathcal{U}$, corresponding to two real vector fields, as

$$[\mathcal{Z}, \mathcal{U}] = \frac{\partial \mathcal{Z}}{\partial z}\mathcal{U} + \frac{\partial \mathcal{Z}}{\partial w}\overline{\mathcal{U}} - \frac{\partial \mathcal{U}}{\partial z}\mathcal{Z} - \frac{\partial \mathcal{U}}{\partial w}\overline{\mathcal{Z}}. \tag{12}$$

This definition appears also in [13]. We notice that, as we have mentioned above, both vector fields $\mathcal{Z}$ and $\mathcal{U}$ are described only from their first components, because the second ones are obtained by complex conjugation. The first proof of the next geometrical equivalence was done by Sabatini in [30].

Theorem 5 ([1])**.** *Equation* (7) *has an isochronous center at the origin if and only if there exists* $\dot{z} = \mathcal{U}(z, w) = z + O(|z, w|^2)$ *such that* $[\mathcal{Z}, \mathcal{U}] = 0$.

Finally, we will deal with the utility of commuting systems (see [5]). Let us consider two systems of the form (1) and denote by $\phi(t, (x_0, y_0))$ and $\psi(s, (x_0, y_0))$ their respective solutions such that $\phi(0, (x_0, y_0)) = (x_0, y_0)$ and $\psi(0, (x_0, y_0)) = (x_0, y_0)$. Let τ_1, τ_2 be positive real numbers, and $S = [0, \tau_1] \times [0, \tau_2]$ be a rectangle, that will be called a *parametric rectangle*. We say that the local flows $\phi(t, (x_0, y_0))$ and $\psi(s, (x_0, y_0))$ *commute* if, for every parametric rectangle S such that both $\phi(t, \psi(s, (x_0, y_0)))$ and $\psi(s, \phi(t, (x_0, y_0)))$ exist whenever $(t, s) \in S$, one has

$$\phi(t, \psi(s, (x_0, y_0))) = \psi(s, \phi(t, (x_0, y_0))).$$

By a classical result, two local flows commute if and only if the Lie bracket (12) of their corresponding vector fields vanishes identically (see [2, 24]). In this case we say that the vector fields commute. It is then natural to think that commutativity can actually be used to characterize isochronous centers, a fact proved in [30] and stated in next theorem.

Theorem 6 ([5])**.** *The center at the origin of system* (1) *is isochronous if and only if there exists a second vector field defined in a neighbourhood of the origin which is transversal to* (1) *at nonsingular points and both commmute.*

2.3. **Period constants computation methods.** The methods presented here are those already introduced in our previous work [32]. We start this subsection by presenting the classical method to find period constants (see [27]). By performing the usual change to polar coordinates $(x, y) = (r\cos\varphi, r\sin\varphi)$, one can rewrite system (1) as

$$\begin{cases} \dot{r} = \sum\limits_{k=1}^{n-1} U_k(\varphi) r^{k+1}, \\ \dot{\varphi} = 1 + \sum\limits_{k=1}^{n-1} V_k(\varphi) r^k, \end{cases} \tag{13}$$

where $U_k(\varphi)$ and $V_k(\varphi)$ are homogeneous polynomials in $\sin\varphi$ and $\cos\varphi$ of degree $k+2$. Eliminating time and doing the Taylor series expansion in r we obtain

$$\frac{dr}{d\varphi} = \sum_{k=2}^{\infty} R_k(\varphi) r^k, \tag{14}$$

where $R_k(\varphi)$ are 2π-periodic functions of φ and the series is convergent for all φ and for all sufficiently small r. The initial value problem for (14) with the initial condition $(r, \varphi) = (\rho, 0)$ has a unique truncated solution

$$r(\varphi) = \rho + \sum_{j=2}^{M} A_j(\varphi) \rho^j, \tag{15}$$

up to some finite order $M \in \mathbb{N}$. Let us see how to find the coefficients $A_j(\varphi)$. By the chain rule, we have

$$\frac{dr}{d\varphi}\frac{d\varphi}{dt} - \frac{dr}{dt} = 0. \tag{16}$$

If we substitute (13) and (15) in (16), we obtain

$$\left(\sum_{j=2}^{M} A_j'(\varphi)\rho^j\right)\left(1 + \sum_{k=1}^{n-1} V_k(\varphi)\left(\rho + \sum_{j=2}^{M} A_j(\varphi)\rho^j\right)^k\right) - \sum_{k=1}^{n-1} U_k(\varphi)\left(\rho + \sum_{j=2}^{M} A_j(\varphi)\rho^j\right)^{k+1} = 0. \tag{17}$$

Now for j from 2 to M, we can extract the coefficient of ρ^j from the left hand side of (17) and equate it to zero, this is

$$A_j'(\varphi) - C_j(\varphi) = 0,$$

where $-C_j$ denotes the remaining part after A_j'. Observe that due to the structure of (17), for a certain j we have that $C_j(\varphi)$ can only contain terms $A_i(\varphi)$ for $i < j$. With a slight abuse of notation, this allows to constructively obtain the expressions for A_j as

$$A_j(\varphi) = \int_0^{\varphi} C_j(\theta)\, d\theta. \tag{18}$$

Let us now substitute the solution (15) into the second equation of (13), which yields a differential equation of the form

$$\frac{d\varphi}{dt} = 1 + \sum_{k=1}^{M+n-1} F_k(\varphi)\rho^k.$$

Rewriting this equation as

$$dt = \frac{d\varphi}{1 + \sum\limits_{k=1}^{M+n-1} F_k(\varphi)\rho^k} = \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi$$

and integrating from 0 to 2π yields

$$T(\rho) = \int_0^{T(\rho)} dt = \int_0^{2\pi} \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi = 2\pi + \sum_{k=1}^{\infty} \left(\int_0^{2\pi} \Psi_k(\varphi)\, d\varphi\right) \rho^k, \quad (19)$$

where the series converges for $0 \le \varphi \le 2\pi$ and sufficiently small $\rho \ge 0$. From (19) it follows that the least period of the trajectory of (1) passing through $(x, y) = (\rho, 0)$ for $\rho \ne 0$ is given by (2), which is the period function. Now we can directly see that the period constants $\widehat{T}_k$ are given by the expression

$$\widehat{T}_k = \frac{1}{2\pi} \int_0^{2\pi} \Psi_k(\varphi)\, d\varphi. \quad (20)$$

This is the classical method to compute the period constants.

Assume now that system (13) is an isochronous center and we add a perturbation which depends on some parameters $\lambda \in \mathbb{R}^d$ and such that the center property is kept, as we have assumed in (3). We can follow exactly the same procedure as before, and now we have that the period constants

$$\widehat{T}_k(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_k(\varphi, \lambda)\, d\varphi \quad (21)$$

are polynomials in the parameters λ (see [10]).

As we have mentioned above, this is the classical method to compute period constants. However, the integrals in (20) easily become too difficult to be explicitly obtained, so this technique is not useful in many cases for high degree polynomial vector fields. Here we present an equivalent approach which avoids integrals and reduces the problem to solving linear systems of equations. Our method is based on the ideas given in [1] and uses the Lie bracket and normal form theory.

We will consider system (7) in complex coordinates. In this case, Z and $\overline{Z}$ do not need to be polynomials, but can be convergent series which start at least with quadratic terms. For the sake of simplicity, we will deal with

$$\dot{z} = \mathrm{i}\, z + Z(z, w) = \mathcal{Z}(z, w) \quad (22)$$

instead of (7) and using $w = \bar{z}$, taking into account that the second component is the complex conjugate of the first one. By applying near the identity changes of variables, as the spirit of normal form transformations, system (22) can be simplified to

$$\dot{z} = \mathrm{i}\, z + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\, \beta_{2j+1}) z (zw)^j + O_{2N+3}(z, w),$$

where $N \in \mathbb{N}$ is arbitrary and $\alpha_{2j+1}, \beta_{2j+1} \in \mathbb{R}$. The above normal form can be expressed in polar coordinates as follows,

$$\begin{cases} \dot{r} = \sum\limits_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O_{2N+3}(r), \\ \dot{\varphi} = 1 + \sum\limits_{j=1}^{N} \beta_{2j+1} r^{2j} + O_{2N+2}(r). \end{cases} \quad (23)$$

As we are considering system (1), which has a center at the origin, the normal form of system (23) becomes

$$\begin{cases} \dot{r} = r^{2N+3}\mathcal{R}(r,\varphi), \\ \dot{\varphi} = 1 + \beta_3 r^2 + \beta_5 r^4 + \cdots + \beta_{2N+1} r^{2N} + r^{2N}\Theta(r,\varphi), \end{cases}$$

for any $N \in \mathbb{N}$, where $\beta_3, \beta_5, \ldots, \beta_{2N+1} \in \mathbb{R}$ and the functions $\mathcal{R}(r,\varphi)$ and $\Theta(r,\varphi)$ are analytical in r and 2π periodic in φ.

It can be proved (see [1]) that coefficients β_{2j+1} are equivalent to the aforementioned period constants, in the sense that a center is isochronous if and only if $\beta_{2j+1} = 0$ for all $j \geq 1$.

We can take advantage of Theorem 5 in order to propose an alternative constructive method to find the first N period constants of a system –the reader is referred to our previous work [32] for more details about it. The benefit of this approach is that it reduces the problem of finding period constants to the resolution of linear systems of equations, instead of dealing with integrals which can become cumbersome or even unsolvable. We have checked that this new approach allows us to go further in the computation of period constants than the classical previously explained method. This technique has been computationally implemented with Maple ([21]) and used to calculate all the necessary period constants throughout this paper.

2.4. **Inverse integrating factor.** To end this preliminaries section we will recall the notion of inverse integrating factor. Let U be an open subset of $\mathbb{R}^2$. A class $\mathcal{C}^1(U)$ function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of system (1) if V verifies the partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V \tag{24}$$

in U. The name *inverse integrating factor* arises from the fact that if $V(x,y)$ satisfies (24) then its reciprocal $1/V(x,y)$ is an integrating factor of (1). This implies that the system can be transformed into an integrable system by means of being multiplied by $1/V(x,y)$. For more information the reader is referred to [12].

3. Proof of Theorem 1 and Corollary 2

Consider a privileged perturbative parameter ε such that the perturbed system is written as (4). Considering its period function (5), we can express the power series of $\mathcal{T}_1(\rho,\lambda)$ with respect to ρ and rewrite (5) as

$$T(\rho,\lambda,\varepsilon) = 2\pi + \left(\sum_{j=1}^{\infty} \theta_j(\lambda)\rho^j\right)\varepsilon + \sum_{k=2}^{\infty} \mathcal{T}_k(\rho,\lambda)\varepsilon^k, \tag{25}$$

for some functions $\theta_j(\lambda)$. This idea is equivalent to the Melnikov method when studying limit cycles. Theorem 1 presented in Section 1 states that the first order coefficients in $\mathcal{T}_1(\rho,\lambda)$ from (25), these are functions $\theta_j(\lambda)$, are exactly the first order truncation of the Taylor series of the period constants in (21) with respect to λ. This is inspired by [18], where the authors prove the equivalence between the first order truncation of the Lyapunov constants and the first Melnikov function for limit cycles.

Before the proof of Theorem 1 and its Corollary 2, we will start by illustrating the equivalence between both methods with a particular example. Consider the next polynomial system with homogeneous nonlinearities of degree 6 written in polar coordinates as

$$\begin{cases} \dfrac{dr}{dt} = r^6 (\sin\varphi + 2\sin 3\varphi) =: r^6 U(\varphi), \\ \dfrac{d\varphi}{dt} = 1 - \frac{5}{3} r^5 (3\cos\varphi + 2\cos 3\varphi) =: 1 + r^5 V(\varphi), \end{cases} \tag{26}$$

which has the form (13). It can be shown that this system has a reversible isochronous center at the origin by using that it has a rational first integral, written in Cartesian coordinates as

$$H(x,y) = \frac{(x^2+y^2)^5}{1 - \frac{50}{3}x^5 - \frac{20}{3}x^3y^2 + 10xy^4},$$

and applying Proposition 7. Here, the time-reversibility condition is moved to the invariance with respect to the change $(r,\varphi,t) \mapsto (r,-\varphi,-t)$. First we consider a change of variables $\hat{r} := r^5$ to simplify notation, then $d\hat{r}/dt = (d\hat{r}/dr)\cdot(dr/dt) = 5r^4 dr/dt$. Therefore, system (26) becomes

$$\begin{cases} \dfrac{d\hat{r}}{dt} = 5\hat{r}^2\, U(\varphi), \\ \dfrac{d\varphi}{dt} = 1 + \hat{r}\, V(\varphi). \end{cases}$$

Now we add a time-reversible polynomial perturbation with parameters $\lambda = (\lambda_1,\dots,\lambda_7) \in \mathbb{R}^7$ also corresponding to homogeneous nonlinearities of degree 6, and having the form

$$\begin{cases} \dfrac{d\hat{r}}{dt} = 5\hat{r}^2 \left(U(\varphi) + \widetilde{U}(\varphi,\lambda) \right), \\ \dfrac{d\varphi}{dt} = 1 + \hat{r} \left(V(\varphi) + \widetilde{V}(\varphi,\lambda) \right), \end{cases} \tag{27}$$

where

$$\begin{aligned} \widetilde{U}(\varphi,\lambda) :=& \lambda_1 \sin\varphi + \lambda_2 \sin 3\varphi + \lambda_3 \sin 5\varphi + \lambda_4 \sin 7\varphi, \\ \widetilde{V}(\varphi,\lambda) :=& -(5\lambda_1 - \lambda_5)\cos\varphi - \frac{1}{3}(5\lambda_2 - 3\lambda_6)\cos 3\varphi + \lambda_7 \cos 5\varphi + \lambda_1 \cos 7\varphi. \end{aligned}$$

Let us propose a truncated solution up to order 4 as in (15), this is

$$\hat{r} = \rho + A_2(\varphi)\rho^2 + A_3(\varphi)\rho^3 + A_4(\varphi)\rho^4.$$

By using (17) and (18), we obtain that $A_2(\varphi) = A_3(\varphi) = A_4(\varphi) = 0$. Now applying formula (20) to first order terms, we finally write the linear parts with respect to λ of the first and second period constants as

$$\begin{aligned} T_1^{(1)} =& -\frac{5}{2}\lambda_5 - \frac{5}{3}\lambda_6, \\ T_2^{(1)} =& \frac{625}{27}\lambda_3 - \frac{1000}{63}\lambda_4 - \frac{3125}{6}\lambda_5 - \frac{3250}{9}\lambda_6 + \frac{625}{27}\lambda_7. \end{aligned} \tag{28}$$

To exemplify the second method we will consider system (27) with a privileged perturbative parameter ε, this is

$$\begin{cases} \dfrac{d\hat{r}}{dt} = 5\hat{r}^2 \left(U(\varphi) + \varepsilon\widetilde{U}(\varphi,\lambda) \right), \\ \dfrac{d\varphi}{dt} = 1 + \hat{r} \left(V(\varphi) + \varepsilon\widetilde{V}(\varphi,\lambda) \right). \end{cases}$$

As we explained in Section 2, in this case we can express the period function as a power series in ε (see equation (25)), so

$$T(\rho,\lambda,\varepsilon)=2\pi+\mathcal{T}_1(\rho,\lambda)\varepsilon+\sum_{k=2}^{\infty}\mathcal{T}_k(\rho,\lambda)\varepsilon^k,$$

and then $\mathcal{T}_1(\rho,\lambda)=\sum_{j=1}^{\infty}\theta_j(\lambda)\rho^j$. Finally, after performing the calculations we check that the two first nonzero coefficients $\theta_j(\lambda)$ are the linear parts of period constants obtained in (28).

Followingly we present the proofs of Theorem 1 and Corollary 2.

Proof of Theorem 1. Consider the series expansions of the perturbative parameters λ in terms of a privileged parameter ε,

$$\lambda_l(\varepsilon)=\sum_{j=0}^{\infty}\lambda_{jl}\varepsilon^j, \tag{29}$$

we have that the period function writes

$$T(\rho,\lambda)=\sum_{k=1}^{N}T_k(\lambda)\rho^{2k}\left(1+\sum_{j=1}^{\infty}\alpha_{kj}(\lambda)\rho^j\right),$$

with α_{kj} vanishing at zero in the variables λ. We can now consider the power series expansion in ε of the period function

$$T(\rho,\varepsilon)=\sum_{k=1}^{\infty}\tau_k(\rho)\varepsilon^k=\sum_{k=1}^{\infty}\frac{1}{k!}\left(\left.\frac{\partial^k T(\rho,\varepsilon)}{\partial\varepsilon^k}\right|_{\varepsilon=0}\right)\varepsilon^k.$$

Notice that the series representation of the period function is only local, but the Global Bifurcation Lemma, see [8], implies that the coefficients

$$\tau_k(\rho)=\frac{1}{k!}\left.\frac{\partial^k T(\rho,\varepsilon)}{\partial\varepsilon^k}\right|_{\varepsilon=0}$$

are defined and analytic in the period annulus of the center.

Considering the power series expansions (29), we have that for each k

$$T_k(\lambda(\varepsilon))=\sum_{m=1}^{\infty}T_k^{(m)}(\lambda(\varepsilon))\varepsilon^m,$$

and

$$\alpha_{kj}(\lambda(\varepsilon))=\sum_{i=0}^{\infty}\alpha_{kji}\varepsilon^i.$$

Rearranging the series for ε and ρ small enough it follows that

$$T(\rho,\varepsilon)=\sum_{k=1}^{N}\sum_{m=1}^{\infty}T_k^{(m)}\varepsilon^m\left(1+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\alpha_{kji}\rho^j\varepsilon^i\right)\rho^{2k}.$$

Hence, choosing the coefficient of ε in the equation above –this is $m=1$ and $i=0$–, for ρ small we have the expression

$$\mathcal{T}_1(\rho)=\tau_1(\rho)=\sum_{k=1}^{N}T_k^{(1)}\left(1+\sum_{j=1}^{\infty}\alpha_{kj0}\rho^j\right)\rho^{2k},$$

where all $T_k^{(m)}$ depend on λ and, consequently, the first order truncation of $T_k^{(1)}$ are linear combinations of the original parameters λ in the statement. $\square$

Proof of Corollary 2. If the rank of G_m is N, one can rearrange the terms of the linear parts $T_k^{(1)}$ from expression (6) in Theorem 1 according to the linear relationship between the parameters, and by applying Weierstrass Preparation Theorem (see [34]), this implies that $N-1$ critical periods can bifurcate from the origin and the statement follows. □

4. Isochronicity of some even degree systems

In this section we will present some results about the isochronicity of some even degree systems. As we have already mentioned, the studied polynomial systems have homogeneous nonlinearities of degree n. We will consider systems with even n, and the reason is as follows. It is a well-known fact that, given a parametric family of systems, its period constants are polynomials whose variables are the parameters of the system and having a particular structure based on their weight and quasi-degree (for more details see for instance [10, 14]). It can be checked that this structure implies that, when the nonlinearities are homogeneous of degree n, some of the corresponding period constants are identically zero. When n is even and $k = i(n-1)$, for $i \in \mathbb{N}$, we obtain $T_k \not\equiv 0$, while when n is odd this property holds for $k = i(n-1)/2$. Therefore, the computational effort is lower using only homogeneous nonlinearities when the objective is to get systems having at the origin a point with the highest multiplicity value for the period function. Clearly, for even degrees we can go further with less computations and this allows us to obtain higher criticality. This fact was already observed in the analogous problem of studying cyclicity using Lyapunov constants –for example, Giné took advantage of it in [16, 17].

We will start with the following proposition that characterizes a class of systems of even degree n, whose proof is a generalization of a reasoning inspired by reading [5].

Proposition 7. *Let $n > 1$ be a natural number and $p(x,y)$ a homogeneous polynomial of degree $n-1$ such that $p(x,-y) \equiv p(x,y)$. The system*

$$\begin{cases} \dot{x} = -y + X_n(x,y), \\ \dot{y} = x + Y_n(x,y), \end{cases} \tag{30}$$

with $X_n(x,y)$ and $Y_n(x,y)$ homogeneous polynomials of degree n, associated to the first integral

$$H(x,y) = \frac{(x^2+y^2)^{n-1}}{1+p(x,y)}, \tag{31}$$

has a time-reversible (with respect to the x-axis) isochronous center at the origin.

Proof. System (30) has a center at the origin because the first integral (31) is well defined and, moreover, it is time-reversible since also the first integral is so. To see the isochronicity let us first write the first integral (31) in polar coordinates,

$$H(r,\varphi) = \frac{r^{2(n-1)}}{1+r^{n-1}\Phi(\varphi)}, \tag{32}$$

where $\Phi(\varphi)$ is a trigonometric polynomial in φ.

Due to the reversible linear plus homogeneous structure and the parity of the polynomials being n even, $\Phi(\varphi) = \sum_{k=1}^{n/2} a_k \cos((2k-1)\varphi)$. Here we have used the well-known fact that $\cos(m\varphi) = f_m(\cos\varphi)$ and $\sin((m+1)\varphi) = g_m(\cos\varphi)\sin\varphi$, where f_m and g_m are the mth degree Chebyshev polynomials of the first and second kind, respectively (see [25] for more information on this topic).

Let us see that this function $\Phi(\varphi)$ is actually directly related to the expression of system (30) in polar coordinates. As (32) is a first integral, it satisfies $\frac{\partial H}{\partial r}\dot{r} + \frac{\partial H}{\partial \varphi}\dot{\varphi} = 0$, so

$$\frac{\dot{r}}{\dot{\varphi}} = -\frac{\frac{\partial H}{\partial \varphi}}{\frac{\partial H}{\partial r}} = \frac{r^n \Phi'(\varphi)}{(n-1)\left(2 + r^{n-1}\Phi(\varphi)\right)}.$$

Therefore, system (30) is written in polar coordinates as

$$\begin{cases} \dot{r} = r^n \frac{\Phi'(\varphi)}{2(n-1)}, \\ \dot{\varphi} = 1 + r^{n-1}\frac{\Phi(\varphi)}{2}. \end{cases} \tag{33}$$

From the level curve $H(r,\varphi) = 1/h$, where h is an arbitrary nonzero real number, we obtain $hr^{2(n-1)} = 1 + r^{n-1}\Phi(\varphi)$, and solving this second degree equation in r^{n-1} we get

$$r^{n-1} = \frac{\Phi(\varphi) \pm \sqrt{\Phi^2(\varphi) + 4h}}{2h}. \tag{34}$$

From the second differential equation in (33) and using (34), we obtain that the period function of the system is

$$T(r) = \int_0^{2\pi} \frac{d\varphi}{1 + r^{n-1}\frac{\Phi(\varphi)}{2}} = \int_0^{2\pi} \left(1 \pm \frac{\Phi(\varphi)}{\sqrt{\Phi^2(\varphi) + 4h}}\right) d\varphi = 2\pi \pm \int_0^{2\pi} \frac{\Phi(\varphi)}{\sqrt{\Phi^2(\varphi) + 4h}} d\varphi.$$

Finally, as $\Phi(\varphi)$ is a sum of terms of the form $\cos((2k-1)\varphi)$, it is easy to see that the last integral is zero by making the change $\theta = \varphi + \pi$ and using the periodicity of $\Phi(\varphi)$. Therefore, the period function is constant and the statement follows. $\square$

The next results prove the isochronicity of some sixth-degree polynomial systems, mainly by finding linearizations of them.

Proposition 8. *The time-reversible system (with respect to the x-axis) with polynomial homogeneous nonlinearities of sixth degree*

$$\begin{cases} \dot{x} = -y + \frac{32}{3}x^5 y + \frac{80}{9}x^3 y^3 - \frac{2}{3}x y^5, \\ \dot{y} = x - \frac{80}{9}x^6 - \frac{8}{3}x^4 y^2 + \frac{55}{9}x^2 y^4 + y^6, \end{cases} \tag{35}$$

has an isochronous center at the origin.

Proof. The system has a center due to the fact that it is time-reversible with respect to the x-axis, since it remains invariant under the change $(x, y, t) \mapsto (x, -y, -t)$. The statement follows just checking that the system has a Darboux linearization (in complex coordinates) of the form (11),

$$\chi(z, w) = z\, \chi_1^{-1/5}\, \chi_2^{4/5}\, \chi_3^{1/10}\, \chi_4^{-3/10},$$

with

$$\chi_1(z,w) = 1 - \frac{5}{144}z^5 - \frac{35}{36}z^4w - \frac{55}{8}z^3w^2 - \frac{35}{36}z^2w^3 - \frac{5}{144}zw^4,$$
$$\chi_2(z,w) = 1 - \frac{5}{144}z^4w - \frac{35}{36}z^3w^2 - \frac{55}{8}z^2w^3 - \frac{35}{36}zw^4 - \frac{5}{144}w^5,$$
$$\chi_3(z,w) = 1 - \frac{40}{27}z^4w - \frac{40}{9}z^3w^2 - \frac{40}{9}z^2w^3 - \frac{40}{27}zw^4,$$
$$\begin{aligned}\chi_4(z,w) = 1 &+ \frac{125}{7776}z^{12}w^3 + \frac{2375}{2592}z^{11}w^4 + \frac{12875}{648}z^{10}w^5 + \frac{128375}{648}z^9w^6 + \frac{1081625}{1296}z^8w^7\\ &+ \frac{1081625}{1296}z^7w^8 + \frac{128375}{648}z^6w^9 + \frac{12875}{648}z^5w^{10} + \frac{2375}{2592}z^4w^{11} + \frac{125}{7776}z^3w^{12}\\ &+ \frac{25}{72}z^8w^2 + \frac{325}{36}z^7w^3 + \frac{3575}{72}z^6w^4 - \frac{2125}{18}z^5w^5 + \frac{3575}{72}z^4w^6 + \frac{325}{36}z^3w^7\\ &+ \frac{25}{72}z^2w^8 - \frac{5}{3}z^4w - \frac{35}{3}z^3w^2 - \frac{35}{3}z^2w^3 - \frac{5}{3}zw^4.\end{aligned}$$

□

Proposition 9. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \in \mathbb{R}[x,y]$ be nonidentically zero homogeneous polynomials with degrees $5, 5,$ and 10, respectively, such that $\mathcal{H}_i(x,-y) \equiv \mathcal{H}_i(x,y)$, for $i = 1,2,3$. A time-reversible polynomial system (with respect to the x-axis) of degree $n = 6$ of the form (1) having an isochronous center at the origin with an inverse integrating factor of the form $V(x,y) = (x^2+y^2)\,U_1(x,y)\,U_2(x,y)$, being $U_1(x,y) = 1 + \mathcal{H}_1(x,y)$ and $U_2(x,y) = 1 + \mathcal{H}_2(x,y) + \mathcal{H}_3(x,y)$, writes as*

$$\begin{cases} \dot{x} = -y + \frac{6}{5}x^5y - \frac{4}{5}x^3y^3, \\ \dot{y} = x - x^6 + \frac{21}{5}x^4y^2 + \frac{16}{5}x^2y^4, \end{cases} \tag{36}$$

or

$$\begin{cases} \dot{x} = -y + \frac{6}{5}x^5y - \frac{6}{5}xy^5, \\ \dot{y} = x - x^6 + \frac{6}{5}x^4y^2 + 3x^2y^4 + \frac{4}{5}y^6. \end{cases} \tag{37}$$

Proof. We notice that $U_1(x,y) = 0$ and $U_2(x,y) = 0$ are two algebraic invariant curves which, as well as the inverse integrating factor, are invariant with respect to the change $(x,y) \mapsto (x,-y)$. Due to the reversibility, the considered systems take the form

$$\begin{cases} \dot{x} = -y + p_1x^5y + p_2x^3y^3 + p_3xy^5 =: P(x,y), \\ \dot{y} = x + q_1x^6 + q_2x^4y^2 + q_3x^2y^4 + q_4y^6 =: Q(x,y), \end{cases} \tag{38}$$

and the invariant curves write as

$$U_1(x,y) = 1 + a_1x^5 + a_2x^3y^2 + a_3xy^4,$$
$$U_2(x,y) = 1 + b_1x^5 + b_2x^3y^2 + b_3xy^4 + c_1x^{10} + c_2x^8y^2 + c_3x^6y^4 + c_4x^4y^6 + c_5x^2y^8 + c_6y^{10}.$$

From the statement is also clear that $P, Q, U_1, U_2 \in \mathbb{R}[x,y]$.

As V is actually an inverse integrating factor of system (38), the relation (24) must be satisfied. Now equating the corresponding coefficients we obtain a system of polynomial equations, which can be solved by means of a computer algebra system. Among the obtained solutions are only interested in those which satisfy that $U_1(x,y) \neq 0$, $U_2(x,y) \neq 0$, and $r' \neq 0$, were r is the radial component in the usual polar coordinates. The latter condition is imposed in order to avoid trivial cases, as the fact that $r' = 0$ implies that the system can be rescaled to the canonical linear center (10).

The next step is to test those solutions and check if they could correspond to isochronous centers by computing some period constants. We must reject those which give period constants that cannot be vanished at the same time, since this means that they are not isochronous. Finally, we have only two solutions which are candidates to be isochronous, and correspond to systems (36) and (37). To prove the isochronicity of such systems we will propose a linearization in complex coordinates and a transversal commuting system for each of them, and then apply Theorems 4 and 6.

The functions U_1, U_2 for systems (36) and (37) that we have obtained are respectively

$$\begin{aligned} U_1^A(x,y) &= 1 - \frac{4}{3}x^5 - \frac{4}{3}x^3y^2, \\ U_2^A(x,y) &= 1 - 2x^5 + x^{10} + x^8y^2, \end{aligned}$$

and

$$\begin{aligned} U_1^B(x,y) &= 1 - 2x^5 - 4x^3y^2 - 2xy^4, \\ U_2^B(x,y) &= 1 - 2x^5 - 2x^3y^2 + x^{10} + 3x^8y^2 + 3x^6y^4 + x^4y^6. \end{aligned}$$

The corresponding (complex) linearizations are $\chi^A(z,w) = z\,\chi_1^A\,\chi_2^A$ with

$$\begin{aligned} \chi_1^A(z,w) =& 1 - \frac{1}{6}zw^4 - \frac{1}{2}z^2w^3 - \frac{1}{2}z^3w^2 - \frac{1}{6}z^4w, \\ \chi_2^A(z,w) =& 1 + \frac{1}{4}w^5 + \frac{7}{16}zw^4 - \frac{1}{4}z^2w^3 - \frac{7}{8}z^3w^2 - \frac{1}{2}z^4w - \frac{1}{16}z^5, \end{aligned}$$

and $\chi^B(z,w) = z\,\chi_1^B\,\chi_2^B$ with

$$\begin{aligned} \chi_1^B(z,w) =& 1 - z^3w^2 - z^2w^3, \\ \chi_2^B(z,w) =& 1 - \frac{1}{2}z^4w - \frac{5}{4}z^3w^2 + \frac{3}{4}zw^4. \end{aligned}$$

For the sake of completeness in the isochronicity characterization we have also found the (real) transversal commuting systems

$$\begin{cases} \dot{x} = x\,(1 - x^5 + x^3y^2)\,U_1^A(x,y), \\ \dot{y} = y\,(1 - 6x^5 - 4x^3y^2)\,U_1^A(x,y), \end{cases}$$

and

$$\begin{cases} \dot{x} = x\,(1 - x^5 + 2x^3y^2 + 3xy^4)\,U_1^B(x,y), \\ \dot{y} = y\,(1 - 6x^5 - 8x^3y^2 - 2xy^4)\,U_1^B(x,y), \end{cases}$$

associated to (36) and (37), respectively. We notice that the second functions U_2^A and U_2^B do not appear in the above transversal systems. $\square$

5. Critical periods

In this section we will apply Corollary 2 to obtain lower bounds on the number of critical periods for some polynomial systems to prove Theorem 3. Before that, we will introduce a notation that will be useful throughout the section.

Consider a system (3) and let $\mathbf{r}_\ell = (r_1, \ldots, r_\ell)$ be the sequence of ranks of the matrices obtained from the first order truncated Taylor series of the first ℓ ordered period constants with respect to the parameters λ, being $r_k = \operatorname{Rank} G_k$ and the matrix G_k as defined in Corollary 2 from the coefficients of the linear homogeneous polynomials $T_1^{(1)}(\lambda), \ldots, T_k^{(1)}(\lambda)$. In the case that a consecutive subsequence of length m of ranks takes a constant value $\widetilde{r}$ ($r_k = r_{k+1} = \cdots = r_{k+m-1} = \widetilde{r}$ for some $k, m \in \mathbb{N}$) we will substitute the whole subsequence $r_k, r_{k+1}, \ldots, r_{k+m-1}$ by $\widetilde{r}_m$.

5.1. **4th degree systems.** Let us consider the following systems with quartic homogeneous nonlinearities,

$$(\dot{x},\dot{y}) = \left(-y+(a+4b)x^3y+axy^3, x+(a+4b)x^2y^2+ay^4\right), \tag{39}$$

$$(\dot{x},\dot{y}) = \left(-y-7x^3y+5xy^3, x+3x^4-10x^2y^2-y^4\right), \tag{40}$$

$$(\dot{x},\dot{y}) = \left(-y-4x^3y+2xy^3, x+3x^4-7x^2y^2-4y^4\right), \tag{41}$$

$$(\dot{x},\dot{y}) = \left(-y+4x^3y+10xy^3, x-5x^2y^2+y^4\right), \tag{42}$$

$$(\dot{x},\dot{y}) = \left(-y-(4a+2b)x^3y-(4a-4b)xy^3, x+ax^4+(2a-5b)x^2y^2-(a-b)y^4\right), \tag{43}$$

$$(\dot{x},\dot{y}) = \left(-y+x^3y+xy^3, x\right), \tag{44}$$

$$\begin{aligned}(\dot{x},\dot{y}) = \big(&-y+100(a+3)^2x^3y+4(5a-81)(5a-9)xy^3, x-75(a+3)^2x^4\\ &-10(a+3)(5a-201)x^2y^2+(5a-9)^2y^4\big),\end{aligned} \tag{45}$$

for $a,b\in\mathbb{R}$. All these systems are reversible with respect to the x-axis, as they are invariant under the change $(x,y,t)\mapsto(x,-y,-t)$. The isochronicity of these systems is studied in [5], where the authors make an attempt to characterize all the isochronous centers of a linear center perturbed with homogeneous polynomials of degree 4. They conclude that the first 6 systems are all the possibilities, but they do not manage to prove the isochronicity of the latter. The above ordered list of systems corresponds to the ones in [5] labeled as $\mathrm{H4}_i$, for $i=1,\ldots,7$. Notice that we have rescaled the systems for the sake of simplicity and switched their symmetry so that they are reversible with respect to the x-axis as in the rest of this paper.

Let us observe that we are presenting a technique for the perturbation of isochronous centers and the isochronicity of (45) has not been proved. Despite this, if it was not isochronous the method would be valid anyway, since we could ask for the vanishing only of the first k period constants for a certain k and the approach would work anyway if we are not dealing with higher period constants.

In the following proposition we give lower bounds for the criticality of systems (39)–(45).

Proposition 10. *For each system* (39)–(45)*, let us consider a quartic perturbation inside the reversible class which starts with quadratic terms as in* (9)*. Then the perturbation of systems* (40), (41), *and* (42) *unfold at least* 10 *critical periods, while the perturbation of systems* (39), (44), (43), *and* (45) *unfold at least* $7,8,9,$ *and* 9 *local critical periods, respectively.*

Proof. Firstly, for each system we will find the linear part with respect to the perturbative parameters of the first 20 period constants perturbed in the reversible polynomial class detailed in the statement. Secondly, we will evaluate the corresponding sequence of ranks $\mathbf{r}_{20}$. Finally, the statement will follow applying Corollary 2. We notice that each lower bound will be the maximum achieved rank minus 1.

Straightforward computations show that for systems (39), (44), (43) we have

$$\mathbf{r}_{20} = (1,2,3,4,5,6,7_2,8_{12}),$$
$$\mathbf{r}_{20} = (1,2,3,4,5,6_2,7,8_3,9_9),$$
$$\mathbf{r}_{20} = (1,2,3,4,5,6,7,8,9,10_{11}),$$

respectively, so at least $7,8,$ and 9 local critical periods bifurcate from the origin, respectively. For system (45) we obtain the same sequence as for (43) and, consequently, the same number of local critical periods. For all three systems (40), (41), and (42) we have obtained the best result for these families with homogeneous nonlinearities because the sequence of ranks is

$$\mathbf{r}_{20} = (1,2,3,4,5,6,7,8,9,10_2,11_9).$$

Hence, at least 10 local critical periods bifurcate from the origin and the proof is finished. □

Notice that we have computed a few extra period constants to check that, in some sense, the sequence of ranks stabilizes and that no extra oscillation of the period function will easily appear by applying this first order bifurcation mechanism. We remark that the 10 local critical periods obtained above prove the part of Theorem 3 corresponding to degree 4.

5.2. **6th degree systems.** In this subsection we will study lower bounds for the local criticality of the 6th degree isochronous centers from Section 4 using the tools provided by Theorem 1 and Corollary 2, in a similar way to the previous subsection. The result is as follows.

Proposition 11. *For each system* (35)–(37)*, let us consider a sextic perturbation inside the reversible class which starts with quadratic terms as in* (9)*. Then the perturbation of system* (35) *unfolds at least* 22 *critical periods, while the perturbation of systems* (36) *and* (37) *unfolds at least* 20 *local critical periods.*

Proof. The proof follows analogously as we have done in Proposition 10. The difference is only the corresponding sequences of ranks. The described perturbation provides

$$\mathbf{r}_{35} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17_2, 18, 19_2, 20_2, 21, 22_2, 23_9),$$

for (35) and

$$\mathbf{r}_{35} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17_2, 18, 19_2, 20_3, 21_{11}),$$

for both systems (36) and (37). Consequently, the respective lower bounds are the ones detailed in the statement. □

According to our previous work [32], the highest achieved lower bound for $\mathcal{C}_\ell(6)$ is 20. Notice that in Proposition 11 we have obtained the same lower bound with systems (36) and (37) but with a more efficient technique and, moreover, we have improved it with system (35). Actually, the fact that we obtain at least 22 local critical periods for system (35) proves $\mathcal{C}_\ell(6) \geq 22$ in Theorem 3.

5.3. n**th degree systems.** In this subsection we will study the bifurcation of local critical periods for n-th degree isochronous systems, provided by Proposition 7, for several values of n. As we have already mentioned, systems with homogeneous nonlinearities and even degree will usually have higher criticality than those with odd degree, so we will take advantage of this fact to also study odd degree systems by perturbing systems of even degree $n-1$ with an odd nth degree perturbation.

Let us start with the following genericity criticality result for 4th and 6th degrees.

Proposition 12. *Isochronous systems* (30) *of degrees* $n = 4$ *and* $n = 6$ *with a first integral of the form* (31)*, when they are perturbed in the class of reversible polynomials of degree* n*, generically unfold 9 and 21 local critical periods, respectively.*

Proof. For the case $n = 4$ we have a first integral

$$H_4(x, y) = \frac{(x^2 + y^2)^3}{1 + ax^3 + bxy^2},$$

with $a, b \in \mathbb{R}$, and the corresponding reversible isochronous system is

$$\begin{cases} \dot{x} = -y - \left(a - \frac{1}{3}b\right) x^3 y - \frac{2}{3}bxy^3, \\ \dot{y} = x + \frac{1}{2}ax^4 - \left(\frac{1}{2}a - \frac{5}{6}b\right) x^2y^2 - \frac{1}{6}by^4. \end{cases}$$

Now if we change to complex coordinates and add a quartic reversible perturbation as in (9), we can find the first order developments of the first 10 period constants and compute their determinant with respect to the perturbative parameters $\varrho_{02}, \varrho_{03}, \varrho_{04}, \varrho_{11}, \varrho_{12}, \varrho_{13}, \varrho_{20}, \varrho_{21}, \varrho_{22}, \varrho_{30}$, which after being rescaled via a multiplicative constant is

$$\begin{gathered}(-3045a^5-17535a^4b-19362a^3b^2-5166a^2b^3+1975ab^4+125b^5)(-42735a^5\\ -126049a^4b-6974a^3b^2+35766a^2b^3+6909ab^4+475b^5)(-295507521a^7\\ -165909573a^6b+517786803a^5b^2+19400559a^4b^3-132219763a^3b^4\\ -14086623a^2b^5+4613697ab^6+320885b^7)(a-b)^6(3a+b)^7.\end{gathered}$$

This determinant is nonzero except for a set of null measure. Therefore, generically we obtain rank 10 which means 9 local critical periods by using Corollary 2.

For the case $n=6$, the first integral is

$$H_6(x,y)=\frac{(x^2+y^2)^5}{1+ax^5+bx^3y^2+cxy^4},$$

with $a,b,c\in\mathbb{R}$, and the corresponding system is

$$\begin{cases}\dot{x}=-y-\left(a-\frac{1}{5}b\right)x^5y-\left(\frac{4}{5}b-\frac{2}{5}c\right)x^3y^3-\frac{3}{5}cxy^5,\\ \dot{y}=x+\frac{1}{2}ax^6-\left(\frac{1}{2}a-\frac{7}{10}b\right)x^4y^2-\left(\frac{3}{10}b-\frac{9}{10}c\right)x^2y^4-\frac{1}{10}cy^6.\end{cases}$$

Analogously to the quartic case, we find the first order developments of the first 22 period constants of this system after being perturbed and compute their determinant with respect to 22 perturbative parameters. The resulting determinant, which is a polynomial of degree 92 in (a,b,c), has such a long expression to be written here. We conclude that the rank is generically 22 and the finishes using again Corollary 2. □

We have also dealt with systems of higher even degrees $n=8,10,12,14$, and 16, as the following proposition states.

Proposition 13. *There exist isochronous reversible systems of degrees* $n=8,10,12,14$, *and* 16 *having a first integral of the form* (31) *which unfold at least* 37, 57, 80, 106, *and* 136 *local critical periods under a polynomial reversible perturbation of degree* n*, respectively.*

Proof. Here we will consider perturbations of the form (9) being $\nu=n$, this is, both the isochronous system and the perturbation having the same degree n.

Due to Proposition 7, all the chosen systems have an isochronous reversible center at the origin, so we can follow the same idea and notation as in the proofs of Propositions 10 and 11. Hence, by evaluating the sequence of ranks $\mathbf{r}_\ell$ for a high enough number of period constants and applying Corollary 2, we deduce the lower bound for the criticality values detailed in the statement. We will only list the first integrals, the systems and the sequence of ranks.

For the case $n=8$, we propose a first integral

$$H_8(x,y)=\frac{(x^2+y^2)^7}{1+x^7+2x^5y^2+3x^3y^4+4xy^6},$$

and the corresponding system

$$\begin{cases}\dot{x}=-y-\frac{5}{7}x^7y-\frac{6}{7}x^5y^3-\frac{3}{7}x^3y^5-\frac{16}{7}xy^7,\\ \dot{y}=x+\frac{1}{2}x^8+\frac{11}{14}x^6y^2+\frac{23}{14}x^4y^4+\frac{43}{14}x^2y^6-\frac{2}{7}y^8.\end{cases}$$

In this case we have

$$\mathbf{r}_{64}=(1,2,3,\ldots,36_3,37_4,38_{13}).$$

In the case $n = 10$ the first integral and system are

$$H_{10}(x,y) = \frac{(x^2+y^2)^9}{1+8x^9+90x^7y^2+\frac{6}{7}x^5y^4+5x^3y^6-54xy^8}$$

and

$$\begin{cases} \dot{x} = -y + 2x^9y - \frac{1676}{21}x^7y^3 + x^5y^5 - \frac{82}{3}x^3y^7 + 30xy^9, \\ \dot{y} = x + 4x^{10} + 51x^8y^2 - \frac{722}{21}x^6y^4 + \frac{55}{14}x^4y^6 - \frac{311}{6}x^2y^8 + 3y^{10}. \end{cases} \tag{46}$$

The first 100 period constants of this system provide the following sequence of ranks

$$\mathbf{r}_{100} = (1,2,3,\ldots,56_4,57_5,58_{16}).$$

For $n = 12$ we take the first integral

$$H_{12}(x,y) = \frac{(x^2+y^2)^{11}}{1+4x^{11}+99x^9y^2+\frac{1023}{2}x^7y^4+\frac{3047}{24}x^5y^6+\frac{770}{3}x^3y^8+44xy^{10}}$$

and the system

$$\begin{cases} \dot{x} = -y + 5x^{11}y + 3x^9y^3 - \frac{3071}{8}x^7y^5 + x^5y^7 - \frac{430}{3}x^3y^9 - 24xy^{11}, \\ \dot{y} = x + 2x^{12} + \frac{113}{2}x^{10}y^2 + \frac{1233}{4}x^8y^4 - \frac{3103}{48}x^6y^6 + \frac{3085}{16}x^4y^8 + 7x^2y^{10} - 2y^{12}, \end{cases} \tag{47}$$

which has

$$\mathbf{r}_{140} = (1,2,3,\ldots,79_6,80_5,81_{21}).$$

For $n = 14$ the first integral and the corresponding system are

$$H_{14}(x,y) = \frac{(x^2+y^2)^{13}}{1+10x^{13}+221x^{11}y^2+\frac{2691}{2}x^9y^4-3x^7y^6-x^5y^8-\frac{13}{8}x^3y^{10}}$$

and

$$\begin{cases} \dot{x} = -y + 7x^{13}y + 3x^{11}y^3 - \frac{29619}{26}x^9y^5 + 2x^7y^7 + \frac{7}{104}x^5y^9 + x^3y^{11}, \\ \dot{y} = x + 5x^{14} + \frac{245}{2}x^{12}y^2 + \frac{3145}{4}x^{10}y^4 - \frac{24333}{52}x^8y^6 - \frac{259}{208}x^4y^{10} + \frac{3}{16}x^2y^{12}. \end{cases} \tag{48}$$

The linear parts of the period constants of the above system provides the following sequence of ranks:

$$\mathbf{r}_{200} = (1,2,3,\ldots,105_6,106_7,107_{39}).$$

Finally, for degree $n = 16$ we propose the first integral

$$H_{16}(x,y) = \frac{(x^2+y^2)^{15}}{1-2x^{15}+45x^{13}y^2+\frac{735}{2}x^{11}y^4+\frac{3215}{2}x^9y^6-\frac{40}{11}x^7y^8-2x^5y^{10}-\frac{5}{3}x^3y^{12}}$$

and system

$$\begin{cases} \dot{x} = -y + 5x^{15}y + 7x^{13}y^3 + 3x^{11}y^5 - \frac{42470}{33}x^9y^7 + 2x^7y^9 + \frac{2}{3}x^5y^{11} + x^3y^{13}, \\ \dot{y} = x - x^{16} + \frac{53}{2}x^{14}y^2 + \frac{853}{4}x^{12}y^4 + \frac{1981}{2}x^{10}y^6 - \frac{64025}{132}x^8y^8 + -\frac{9}{11}x^6y^{10} - \frac{7}{6}x^4y^{12} + \frac{1}{6}x^2y^{14}. \end{cases} \tag{49}$$

The corresponding sequence of ranks for the linear parts of its period constants is

$$\mathbf{r}_{260} = (1,2,3,\ldots,135_7,136_8,137_{44}).$$

$\square$

The above result provides the proof of all the cases for even $n \geq 8$ from Theorem 3. We have not gone further in the degree because we have reached the computational limit of our computing machines. Inside the considered family having a first integral of the form (31), with the values found in this subsection for $n = 4, 6, 8, 10$ we have provided a good lower bound $\mathcal{C}_\ell(n) \geq (n^2 + 2n - 6)/2$, but the ones for $n = 12, 14, 16$ are lower than expected. Therefore, this general family is not good enough to get the conjectured value for $\mathcal{C}_\ell(n)$ in the introduction, although they are the best values obtained so far.

Finally, we will present a last result concerning systems with odd degrees.

Proposition 14. *There exist reversible isochronous systems of degrees* $n = 10, 12, 14,$ *and* 16 *having a first integral of the form* (31) *which unfold at least* $66, 91, 119,$ *and* 151 *critical periods, respectively, under a reversible perturbation of odd degree* $\nu = n + 1$.

Proof. Here we consider the even degree n reversible isochronous systems (46), (47), (48), (49) in (9) but perturbed with reversible odd degree $\nu = n + 1$. The proof follows similarly to the previous results, so we only indicate the respective sequences of ranks for a high enough number of period constants in order to get the lower bounds written in the statement:

$$\begin{aligned}
\mathbf{r}_{120} &= (1, 2, 3, \ldots, 65_9, 66_9, 67_{17}),\\
\mathbf{r}_{170} &= (1, 2, 3, \ldots, 90_{11}, 91_{11}, 92_{22}),\\
\mathbf{r}_{230} &= (1, 2, 3, \ldots, 118_{13}, 119_{13}, 120_{29}),\\
\mathbf{r}_{300} &= (1, 2, 3, \ldots, 150_{15}, 151_{15}, 152_{38}).
\end{aligned}$$

□

This technique of using an even degree system with an odd degree perturbation to obtain higher criticality was already introduced in [15], and has resulted in a higher criticality than directly perturbing all our best candidates with homogeneous nonlinearities of odd degree. It is worth noticing that we have also tested this approach with odd degrees $5, 7,$ and 9, but we have not presented them here because they do not improve the local criticality we already obtained in [32]. The bounds we obtain for degrees $11, 13, 15,$ and 17 in Proposition 14 are better than those from our previous work [32] but do not improve the ones from [4]. However, we have explained them anyway because it is interesting to illustrate how this method works and its efficiency.

Acknowledgments

This work has been realized thanks to Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER), PID2019-104658GB-I00, and FPU16/04317, the Generalitat de Catalunya - AGAUR grant 2017SGR1617, and the European Community grant Dynamics-H2020-MSCA-RISE-2017-777911.

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