Analysis and numerical approximation of viscosity solutions with shocks: application to the plasma equation

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Abstract.
We consider a new class of Hamilton-Jacobi equations arising from the convective part of general Fokker-Planck equations ruled by a non-negative diffusion function that depends on the unknown and on the gradient of the unknown. The new class of Hamilton-Jacobi equations represents the propagation of fronts with speed that is a nonlinear function of the signal. The equations contain a nonstandard Hamiltonian that allows the presence of shocks in the solution and these shocks propagate with nonlinear velocity. We focus on the one-dimensional plasma equation as an example of the general Fokker-Planck equations having the features we are interested in analyzing. We explore features of the solution of the corresponding Hamilton-Jacobi plasma equation and propose a suitable fifth order finite difference numerical scheme that approximates the solution in a consistent way with respect to the solution of the associated Fokker-Planck equation. We present numerical results performed under different initial data with compact support.

Keywords: Fokker-Planck equation, Hamilton-Jacobi equations, plasma equation, numerical schemes
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INTRODUCTION

Classical Fokker-Planck equations represent a wide class of anisotropic diffusion equations that model transport by diffusion of a physical magnitude in a continuum medium ([1, 2, 3, 11]). Solutions of these equations contain diffusion fronts that propagate with finite speed. A Fokker-Planck equation can be written in conservation form as

$$u_t = \operatorname{div}\Big(g(u,|\nabla u|)\nabla u\Big) \tag{1}$$

where $g(u,p)$ is a non-negative real function defined for $u \geq 0$.

Diffusion equations of the form (1) might allow fronts (jump discontinuities) in their solutions and we are interested in analyzing the terms responsible of their formation. In order to analyze the convective part of the Fokker-Planck equation we expand spatial derivatives of the flux equation (1) as

$$u_t = \frac{\partial g}{\partial u}|\nabla u|^2 + g(u,|\nabla u|)\Delta u + \frac{\partial g}{\partial p}\frac{L(\nabla u)}{|\nabla u|} \tag{2}$$

where

$$L(\nabla u) = u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} \tag{3}$$

in two spatial dimensions.

As it is proven in [9], diffusion fronts propagate according to a hyperbolic conservation law and their dynamics is ruled by the convective part of (1) which is the first term in (2). The Hamilton-Jacobi equation associated to this term is

$$u_t = \frac{\partial g}{\partial u}|\nabla u|^2 \tag{4}$$

which results from removing diffusions terms in (2).

In the following sections we consider the plasma equation as an example of a Fokker-Planck equation where diffusion fronts are generated. We show evidence of the presence of jumps in the solution of the plasma equation as a consequence of the formation of shocks in the solution of the associated Hamilton-Jacobi plasma equation. We present numerical results supporting our study.

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