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Numerical Approximation of MHD Equations for Real Gases

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Abstract. We consider the MHD equations for real gases described by a Van der Waals equation of state. We present an explicit calculation of the spectral decomposition of the Jacobian of the fluxes and we propose a characteristic-based upwind numerical scheme to approximate the solution of the system of equations in the one dimensional case. We show a numerical example where we observe wave dynamics significantly stronger than the one obtained for the ideal MHD case.

1. Introduction

The magnetohydrodynamics (MHD) system of equations for real gases can be expressed as

$$\begin{aligned} \rho_t + \nabla(\rho\mathbf{v}) &= 0 \\ (\rho\mathbf{v})_t + \nabla\Big(\rho\mathbf{v}\mathbf{v}^T + (P + \frac{1}{2}\mathbf{B}^2)I - \mathbf{B}\mathbf{B}^T\Big) &= 0 \\ \mathbf{B}_t - \nabla\times(\mathbf{v}\times\mathbf{B}) &= 0 \\ E_t + \nabla\Big((E+P^*)\mathbf{v} - (\mathbf{v}\times\mathbf{B})\times\mathbf{B}\Big) &= 0 \end{aligned}$$

where ρ, $\mathbf{v}$, $\mathbf{B}$ and E denote the mass density, the velocity field, the magnetic field and the total energy respectively. The energy is expressed as $E = \frac{1}{2}\rho q^2 + \frac{1}{2}B^2 + \rho\varepsilon$ where q^2 and B^2 are the squares of the magnitudes of the velocity field and the magnetic field respectively and ε the specific internal energy. $P^* = P + \frac{1}{2}B^2$ is the total pressure and $P = P(\rho, \varepsilon)$ the hydrodynamic pressure defined through a real gas equation of state (EOS).

The study of wave dynamics in real gases under severe regimes like the ones encountered in astrophysical scenarios is a field of increasing interest. The deviation of real gases from the ideal gas case is significant and therefore a more general analytic expression of the EOS permitting the development of specific features is necessary. Van der Waals EOS is a powerful and versatile mathematical model allowing strong complex wave dynamics including thermodynamic phase change (Landau & Lifschitz 1987; Thompson 1971; Menikof & Plohr 1989). The behavior of shock waves in real gases described by the Euler equations ruled by a Van der Waals EOS (Thompson 1971; Thompson & Lambrakis 1973) represents an initial step for the analysis of the wave dynamics arising in real plasmas.

In order to explore the complex dynamics of MHD equations for real gases we consider a numerical scheme that is designed considering full information of the wave structure of the system through the spectral decomposition of the Jacobians of the fluxes. We propose a complete system of eigenvectors and the corresponding eigenvalues of the Jacobian for the MHD fluxes in terms of the thermodynamic magnitudes of the Van der Waals EOS. We then design a characteristic-based numerical scheme following a similar approach as the one proposed by Serna (2009) for ideal MHD. We perform computations for a one dimensional shock tube problem showing a significantly stronger wave dynamics than the one obtained for the ideal MHD case.

2. Local characteristic approach for Van der Waals plasmas

We consider the hyperbolic system of equations for the MHD case in divergent form in one dimension

$$\partial_t\mathbf{u} + \partial_x\mathbf{f}(\mathbf{u}) = 0 \tag{1}$$

where $\mathbf{u}$ is the vector of conserved variables

$$\mathbf{u} = (\rho,\ \rho u,\ \rho v,\ \rho w,\ B_y,\ B_z,\ E)^T \tag{2}$$

and $\mathbf{f}(\mathbf{u})$ the flux vector represented as

$$\begin{aligned} \mathbf{f}(\mathbf{u}) = (&\rho u,\ \rho u^2 + P^* - B_x^2,\ \rho uv - B_xB_y,\ \rho uw - B_xB_z,\ uB_y - vB_x, \\ &uB_z - wB_x,\ u(E+P^*) - B_x(uB_x + vB_y + wB_z))^T \end{aligned} \tag{3}$$

where u, v, w represent the velocity field components and B_x, B_y, B_z the magnetic field ones. We assume B_x constant.

The pressure is defined from the expression of the Van der Waals EOS

$$P = \frac{R}{C_V}(\varepsilon + \eta_a\rho)\frac{\rho}{1-\eta_b\rho} - \eta_a\rho^2 \tag{4}$$

where R is the gas constant, C_V is the specific heat at constant volume and $\eta_a > 0$ and $\eta_b > 0$ are positive constants accounting for the intermolecular forces and the molecule size respectively.

Hyperbolicity of a system of the form (1) of dimension m implies that the diagonalization of the Jacobian of the flux decouples the original hyperbolic system in m scalar conservation laws defining the so-called characteristic fields and the corresponding characteristic fluxes.

The eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{u})$ are denoted as $\lambda_1(\mathbf{u}), \cdots, \lambda_m(\mathbf{u})$ counting each one as many times as its multiplicity. The complete system of right and left eigenvectors are defined as $R = \{\mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u})\}$ and $L = \{\mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u})\}$ diagonalizing $\mathbf{f}'(\mathbf{u})$ such that $\mathbf{r}_i \cdot \mathbf{l}_j = \delta_{ij}$ and

$$L(\mathbf{u})\,\mathbf{f}'(\mathbf{u})\,R(\mathbf{u}) = \Lambda = \mathrm{diag}(\lambda_1(\mathbf{u}), \cdots, \lambda_m(\mathbf{u})) \tag{5}$$

Next we propose the spectral decomposition of the Van der Waals MHD equations. Let us define $(b_x, b_y, b_z) = (B_x, B_y, B_z)/\sqrt{\rho}$ and $b^2 = b_x^2 + b_y^2 + b_z^2$. The general expression of the square of the acoustic sound speed is given as

$$a^2 = P_\rho + \frac{PP_\varepsilon}{\rho^2} \tag{6}$$